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A General Family of Fibonacci-Type Squences

Suriya Nanhongkai[1] & Utsanee Leerawat[1]

[1] Department of Mathematics, Facuty of Science, Kasetsart University, Bangkok, Thailand

Correspondence: Utsanee Leerawat, Department of Mathematics, Facuty of Science, Kasetsart University, Bangkok, Thailand. E-mail: fsciutl@ku.ac.th

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Abstract

In this work, we introduce a further generalization of the Fibonacci-type sequence, namely, generalized Fibonacci-type sequence. We also provide the general solution of nonhomogeneous generalized Fibonacci-type sequence, which can be expressed in terms of the Fibonacci-type numbers.

Keywords: Fibonacci, sequences

1. Introduction

The Fibonacci sequence $\{F_n\}_{n=0}^{\infty}$ is a sequence of numbers, starting with the integer pair 0 and 1, where the value of each element is calculated as the sum of two preceding it. That is $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$.

The recurrence relation $G_n = G_{n-1} + G_{n-2} + \sum_{i=0}^{k} \alpha_i n^i$ with initial conditions $G_0 = G_1 = 1$ was introduced by Peter R.J. Asveld (Asveld, 1987). The main result of (Asveld, 1987) consists of an expression of G_n in terms of Fibonacci number F_n and F_{n-1} and in the parameters $\alpha_0, \alpha_1, \ldots, \alpha_k$.

Later several authors like G. B. Djordjević and H. M. Srivastava introduced generalization of the Fibonacci numbers (see Djordjević & Srivastava, 2005 and 2006).

The Betanacci sequence, $\{B_n\}_{n=0}^{\infty}$, is defined recursively by

$$B_n = B_{n-1} + 2B_{n-2} \text{ for all } n \geq 2,$$

with initial conditions $B_0 = B_1 = 1$.

The general solution of $\{B_n\}_{n=0}^{\infty}$ is

$$B_n = \frac{2^{n+1} + (-1)^n}{3}.$$

In (Ratanavongsawad, 2009), K. Ratanavongsawad gave a generalization of the Fibonacci and Betanacci sequences, $\{T_n\}_{n=0}^{\infty}$, defined as follows:

$$T_n = T_{n-1} + 2^r T_{n-2} \text{ for all } n \geq 2 \text{ and non-negative integer } r,$$

with initial conditions $T_0 = T_1 = 1$.

For any non-negative integers r, k and any real numbers $\alpha_0, \alpha_1, \ldots, \alpha_k$, the recurrence relation

$$R_n = R_{n-1} + 2^r R_{n-2} + \sum_{i=0}^{k} \alpha_i n^i \text{ for all } n \geq 2,$$

with initial conditions $R_0 = R_1 = 1$.

The main result of (Ratanavongsawad, 2009) consists of an expression for R_n in terms of Beta-Fibonacci numbers.

In this work, we now introduce a further generalization of the Fibonacci-type sequence and then present the general solution in term of Fibonacci-type numbers.

2. A General Family of Fibonacci-Type Sequences

Definition 2.1. Let α be a positive integer such that $\alpha \geq 2$.

Define a sequence $\{S_n\}_{n=0}^{\infty}$ as follows:

$$S_n = (\alpha - 1)S_{n-1} + \alpha S_{n-2} \text{ for all } n \geq 2,$$

with initial conditions $S_0 = S_1 = 1$.

We call $\{S_n\}_{n=0}^{\infty}$ a *generalized Fibonacci-type sequence*.

Note that if $\alpha = 2$, then the sequence $\{S_n\}_{n=0}^{\infty}$ is reduced to the Betanacci sequence $\{B_n\}_{n=0}^{\infty}$.

The general term of the generalized Fibonacci-type sequence is

$$S_n = (\tfrac{\alpha-1}{\alpha+1})(-1)^n + (\tfrac{2}{\alpha+1})\alpha^n$$

for all non-negative integer n.

Theorem 2.2. Let α be a positive integer such that $\alpha \geq 2$.

For any non-negative integer k and any real numbers $a_0, a_1, \ldots, a_k$, a nonhomogeneous generalized Fibonacci-type sequence $\{G_n\}_{n=0}^{\infty}$ is defined recursively by

$$G_n = (\alpha - 1)G_{n-1} + \alpha G_{n-2} + \sum_{i=0}^{k} a_i n^i, \tag{1}$$

for all $n \geq 2$, with initial conditions $G_0 = G_1 = 1$.

Then the solution of (1) can be express as

$$G_n = (1 - D_k)S_n + E_k \textstyle\sum_{i=1}^{n}(-1)^i\alpha^{n-i} + \sum_{j=0}^{k} p_j(n)a_j,$$

where

(i) D_k is a linear combination of $a_0, a_1, \ldots, a_k$

(ii) E_k is a linear combination of $a_1, a_2, \ldots, a_k$ and

(iii) $p_j(n)$ is a polynomial of degree j for $j = 0, 1, \ldots, k$.

Proof

The solution $G_n^{(h)}$ of the homogeneous recurrence relation corresponding to (1) is

$$G_n^{(h)} = c_1(-1)^n + c_2\alpha^n$$

Next, to find the particular solution of (1), we set $G_n^{(p)} = \sum_{i=0}^{k} A_i n^i$.

We substitute $G_n^{(p)} = \sum_{i=0}^{k} A_i n^i$ in (1), we get

$$\textstyle\sum_{i=0}^{k} A_i n^i = (\alpha - 1)\sum_{i=0}^{k} A_i(n-1)^i + \alpha\sum_{i=0}^{k} A_i(n-2)^i + \sum_{i=0}^{k} a_i n^i.$$

Thus, for each $i = 0, 1, ..., k-1$, we have

$$2(\alpha-1)A_i + \sum_{m=i+1}^{k} C_{im}A_m + a_i = 0 \tag{2}$$

where $C_{im} = (-1)^{m-i}\binom{m}{i}(\alpha(1+2^{m-i})-1)$ for $i \le m$, and

$$2(\alpha-1)A_k + a_k = 0 \tag{3}$$

From (2) and (3), A_i is a linear combination of $a_i, a_{i+1}, \ldots, a_k$, for $i = 0, 1, \ldots, k$.

Hence $A_i = \sum_{j=i}^{k} d_{ij}a_j$,

where

$$d_{ij} = \begin{cases} -\frac{1}{2(\alpha-1)} & \text{for } i = j, \\ -\frac{1}{2(\alpha-1)}\sum_{m=i+1}^{j} C_{im}d_{mj} & \text{for } i < j \end{cases}.$$

Therefore the particular solution $G_n^{(p)}$ of (1) is

$$\begin{aligned} G_n^{(p)} &= \sum_{i=0}^{k}(\sum_{j=i}^{k} d_{ij}a_j)n^i \\ &= \sum_{j=0}^{k}(\sum_{i=0}^{j} d_{ij}n^i)a_j. \end{aligned}$$

Finally, the recurrence relation (1) has the solution

$$\begin{aligned} G_n &= G_n^{(h)} + G_n^{(p)} \\ &= c_1(-1)^n + c_2\alpha^n + \sum_{j=0}^{k}(\sum_{i=0}^{j} d_{ij}n^i)a_j. \end{aligned}$$

The initial conditions: $G_0 = G_1 = 1$, give

$$\begin{aligned} c_1 &= \frac{\alpha-1}{\alpha+1}(1-D_k) + \frac{1}{\alpha+1}E_k \qquad \textit{and} \\ c_2 &= \frac{2}{\alpha+1}(1-D_k) - \frac{1}{\alpha+1}E_k, \end{aligned}$$

where

$$\begin{aligned} D_k &= \sum_{j=0}^{k} d_{0j}a_j \qquad \textit{and} \\ E_k &= \sum_{j=1}^{k}\sum_{i=1}^{j} d_{ij}a_j, \end{aligned}$$

Since $S_n = (\frac{\alpha-1}{\alpha+1})(-1)^n + (\frac{2}{\alpha+1})\alpha^n$, G_n can be written as

$$G_n = (1 - D_k)S_n + E_k \sum_{i=1}^{n}(-1)^i\alpha^{n-i} + \sum_{j=0}^{k} p_j(n)a_j,$$

where $p_j(n) = \sum_{i=0}^{j} d_{ij}n^i$.

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