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Solve Special Case of Some Guran Problems

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Abstract

Throughout this paper, all topological groups are assumed to be topological differential manifolds and algebraically free, our aim in this paper is to prove the open problems number (7) and (8). Which are introduced by (Guran, I, 1998). In many cases of spaces and under a suitable conditions. therefore, we denote by *I(X)* and *I(Y)* to be a free topological groups over a topological spaces *X* and *Y* respectively where *X* and *Y* are assumed to be a non- empty sub manifolds Which are also a closed sub sets, and *P* is a classes of topological spaces, as a regular, normal, Tychonoff, lindelöf, separable connected, compact and Zero- dimensional space, and we have tried to use a hereditary properties and others of these spaces, so we can prove the open problems in these cases and we have many results showed in this paper.

Keywords: topological groups, free topological groups.

2010 mathematics subject classification: 22A05, 54H11.

1. Introduction

1.1 Introducing the Problem

The notion of free topological group *F(X)* over a topological space *X* was introduced by A. A. Markov. He proved that for any Tychonoff space *X* a free topological group *F(X)* exists and is unique, algebraically free and Tychonoff.

In this paper, the main idea which we use to prove the open problems number (7) and (8). Depend on a topological and hereditary properties of these spaces, and it also Depends on some properties of topological groups. Moreover, we replaced in this paper the free Topological inverse semi groups by the free Topological groups. Using these properties, we can determine some classes of topological spaces which satisfy the property in the texts of these open problems.

1.2 What is the Question?

The remained question is: can one prove or deny the statement:

Problem (7): Is *I(X)* zero-dimensional for zero-dimensional *X* ?

Problem (8): Describe classes *P* of Topological spaces having the following property

If *I(X)* is homeomorphic to *I(Y)* and $X \in P$ then $Y \in P$.

Where *I(X)* and *I(Y)* are a free Topological inverse semi groups over the topological spaces *X* and *Y* respectively.

1.3 What We Are Proving in This Paper?

In This paper, we prove that the open problems number (7) and (8) remain valid if the free Topological inverse semi groups replaced by the free Topological groups and all of these are assumed to be a topological differential manifolds.

2. Method

Firstly, we introduce some notions and terminology.

2.1 Definition

An n- dimensional topological manifold *M* is a topological space so that:

1. *M* is Hausdorff.

2. *M* is second-countable.

3. *M* is locally Euclidean of dimension n, i.e. for every $m \in M$, there exists a Triple $\{\varphi, U, V\}$, called a chart (around m), where *U* is an open neighborhood of *m* in *M*, *V* an open subset of $\mathbb{R}^n$, and $\varphi : U \to V$ a homeomorphism.

See (Lee, 2011, p 39; Engelking & Sieklak, 1986, p 225).

2.2 Definition

Let *M* be a manifold of dimensional n. A sub set $N \subseteq M$ is a sub manifold of dimensional k if for every $m \in N$ there is an neighborhood $U \subseteq M$ of *m* and an n-chart $\varphi : U \to V$ such that $m \in \varphi^{-1}(V \cap R^k) = N \cap U$

See (Baker, 2002, p 186).

2.3 Theorem

Let *M* be a connected manifold and $N \subseteq M$ be a non- empty sub manifold Which is also a closed sub set. If $\dim N = \dim M$ then $N = M$.

See (Baker, 2002, p 237).

2.4 Definition

A property *S* is said to be:

1. A topological property, if the following condition holds:

A space *X* has *S*, and *X* is homeomorphic to *Y*, then *Y* has *S*.

2. *S* is said to be a continuous invariant if the following condition holds:

A space *X* has *S*, and $f : X \to Y$ is a continuous surjection, then *Y* has *S*

3. *S* is said to be a hereditary if whenever *X* has *S*, every subspace of *X* has *S*.

See (Murdeshwar, 1990, p 131-132).

2.5 Definition

A topological group *G* is a group *G* with a (Hausdorff) topology such that the product mapping of $G \times G$ into *G* is jointly continuous and the inverse mapping of *G* onto itself associating x^{-1} with an arbitrary $x \in G$ is continuous.

See (Lin, 2013, p 2 ; Marris, 2003, p 545).

2.6 Definition

Let *X* be a subspace of a topological group. *G* Assume that

(1) The set *X* generates *G* algebraically, that is $< X >= G$

(2) Each continuous mapping $f : X \to H$ to a topological group *H* extends to a continuous homomorphism $\hat{f} : \mathrm{G} \to \mathrm{H}$

Then G is called the Markov free topological group on *X* and is denoted by $F(X)$.

See (Lin, 2013, p 2).

2.7 Corollary

If G is a topological group which is T_1, then *G* is completely regular and thus regular.

See (Marris, 2003, p 547)

3. Results

Let P be a classes of topological spaces. Then we have the following results in many cases of spaces:

3.1 Theorem

If *X* is a zero-dimensional closed sub manifold of classes *P*, then the free Topological group *I(X)* is also zero-dimensional.

Proof:

If *X* is a zero-dimensional, then *X* has basis $\{(A_\alpha \cap X)\}_{\alpha \in I}$ consisting of Clopen sub sets, where $\{A_\alpha\}_{\alpha \in I}$, is a basis of *I(X)*, and we know from (Engelking, et al., 1986; Murdeshwar, 1990) that the sub set $A_\alpha \cap X$ is an open (closed) in the sub space *X* if and only if A_α is an open (closed) set in the space *I(X)*. In addition, *X* is closed and also *X* is open, since every element of *X* is contained in an open sub set of *I(X)* contained in *X*. Hence, the members of the basis $\{A_\alpha\}_{\alpha\in I}$ are Clopen sets in *I(X)*, so *I(X)* is a zero-dimensional.

3.2 Theorem

Let *I(X)* is a connected and Tychonoff space of classes *P*. If *I(X)* is homeomorphic to *I(Y)*, then $X, Y \in P$.

Proof:

Since *I(X)* is a connected space and *X* is closed as Well as open sub set in *I(X)* by Theorem (3.1), so $X = I(X)$ by (Baker, 2002), then $X \in P$.

But *I(X)* is also a Tychonoff space, so *I(X)* is a T_1 -space, and because T_1 -space and connected space both have a topological property by (Engelking & Sieklak, 1986), then so is *I(Y)* (being *I(X)* is homeomorphic to *I(Y)*); and we know by the properties of topological groups (Marris, 2003), that $I(Y)$ is a completely regular space, so *I(Y)* is a Tychonoff space. Similarly, Y is Clopen set in *I(Y)*, then $Y = I(Y)$. Hence $Y \in P$.

3.1 Remark

Theorem (3.2) remains valid if *P* is a classes of $T_i\,(i = 0,1,2,3)$. It follows directly of topological and hereditary properties of these spaces.

3.3 Theorem

If *I(X)* is homeomorphic to *I(Y)* and *I(X)* is a separable and Tychonoff space of classes *P* , then $X, Y \in P$.

Proof:

By theorem (3.1), *X* is an open sub set in a separable space *I(X)*, and we know by (Murdeshwar, 1990), that the separability is open – hereditary then *X* is separable and Tychonoff space (because it is a hereditary property), thus $X \in P$. However, $f : I(X) \to I(Y)$ is continuous surjection (being f is homeomorphism), where *I(X)* is separable, then so is *I(Y)*, and because the space *I(Y)* is T_1 -space and is completely regular then *I(Y)* is a Tychonoff space. Similarly, it is also easy to show that *Y* is separable and a Tychonoff space (it follows from hereditary properties). Thus *Y* belongs to the classes *P*.

3.4 Theorem

Let *I(X)* be a separable and compact space of classes *P*. If *I(X)* is homeomorphic to *I(Y)*, then $X, Y \in P$.

Proof:

Since *X* is a closed sub set in *I(X)*, then it is a compact, but X is also open in *I(X)* by theorem (3.1), so *X* is a separable. Hence $X \in P$. Otherwise, $f : I(X) \to I(Y)$ is continuous surjection, where *I(X)* is a separable and compact space, then so is *I(Y)* and *Y* because *Y* is open as well as closed sub set in *I(Y)*. Thus $Y \in P$.

3.5 Theorem

If *I(X)* is homeomorphic to *I(Y)* and *I(X)* is a lindelöf and compact space of classes *P*, then $X, Y \in P$.

Proof:

X is a closed sub set in the lindelöf space *I(X)*. Therefore, *X* is itself lindelöf by (Engelking, et al.,1986; Murdeshwar, 1990). Since *I(X)* is also compact, *X* is compact. Thus $X \in P$. Furthermore, $f : I(X) \to I(Y)$ is continuous surjection (being *I(X)* is homeomorphic to *I(Y)*). Therefore, *I(Y)* is a lindelöf and compact space. Hence, *Y* is lindelöf and is compact, too. It comes easily from closed- hereditary. Thus $Y \in P$.

3.6 Theorem

Let *I(X)* be a compact and normal space of classes P. If *I(X)* is homeomorphic to *I(Y)*, then $X, Y \in P$.

Proof:

If *X* is a closed sub manifold in *I(X)*, then *X* is compact and normal by closed-hereditary. Hence $X \in P$. Suppose $f : I(X) \to I(Y)$ is homeomorphism between the spaces $I(X)$ and *I(Y)*. Then f is continuous surjection. Hence, *I(Y)* is compact since *I(X)* is. But also *I(Y)* is a completely regular space (being *I(Y)* is a topological group). Hence, *I(Y)* is a regular space, and because the regular and compact space is normal by (Murdeshwar, 1990).Thus *I(Y)* is normal. So the closed sub manifold Y belongs to the classes *P*, it follows easily from closed- hereditary by (Murdeshwar, 1990).

3.2 Remark

Theorem (3.6) is true for regular and compact spaces of classes *P*.

3.7 Theorem

Let *I(X)* be a lindelöf and normal space of classes *P*. If *I(X)* is homeomorphic to *I(Y)*, then $X, Y \in P$.

Proof:

If X is closed in *I(X)*, then *X* is a lindelöf and normal space (according to the closed- hereditary property). Thus $X \in P$. Let $f : I(X) \to I(Y)$ be a homeomorphism between the spaces *I(X)* and *I(Y)*. If *I(X)* is a lindelöf space, it is clear that *I(Y)* is a lindelöf, too. Since *I(Y)* is a topological group. Therefore, *I(Y)* is also completely regular by (Marris, 2003), so *I(Y)* is regular. Hence, *I(Y)* is a normal space by (Murdeshwar, 1990). By using closed- hereditary, it follows easily that the closed sub manifold *Y* is also a lindelöf and normal space. Thus $Y \in P$.

3.3 Remark

Theorem (3.7) remains valid if *P* is a classes of lindelöf and regular spaces.

4. Conclusions

We can determine some classes of topological spaces which gives the answer to the open problems number (7) and (8) in (Guran, I, 1998), under a suitable conditions we provide in this paper.

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