Journal of Mathematics Research; Vol. 8, No. 3; June 2016
ISSN 1916-9795 E-ISSN 1916-9809
Published by Canadian Center of Science and Education

Bernoulli Algebra on Common Fractions and Generalized Oscillations

Alexander P. Buslaev[1,2] & Alexander G. Tatashev[2,1]

[1] Moscow Automobile and Road State Technical University, Moscow, Russia

[2] Moscow Technical University of Communications and Informatics, Moscow, Russia

Correspondence: Alexander P. Buslaev. E-mail: apal2006@yandex.ru

Received: March 28, 2016 Accepted: April 13, 2016 Online Published: May 18, 2016

doi:10.5539/jmr.v8n3p82 URL: http://dx.doi.org/10.5539/jmr.v8n3p82

Abstract

Bernoulli algebra on set of proper common fractions with fixed denominator has been introduced and investigated. This algebra is one of most important components of a discrete dynamical system called logistic bipendulum.

Keywords: unary algebra, Bernoulli shift, number theory, dynamical system

1. Introduction

A dynamical system was introduced and investigated in (Kozlov, Buslaev, & Tatashev, 2015). This system is called a bipendulum. The bipendulum consists of two cells V_0 and V_1. A channel connects the cells. There are two particles P_0 and P_1. Each particle is in one of the cell at every instant. Two numbers $a^{(0)}$ and $a^{(1)}$ are given. These numbers belong to the segment $[0, 1]$. The binary representation of the number $a^{(i)}$

$$a^{(i)} = 0.a_1^{(i)}a_2^{(i)}\dots,\ i = 0, 1,$$

is called the plan of the particle P_i, $i = 0, 1$. If there are no delays, then the particle P_i is located in the cell $a_T^{(i)}$ at the instant T, $i = 0, 1$; $T = 1, 2, \dots$ The plan implementation can be delayed. If one of the particles is located in the cell V_0 and, according to its plan, this particle has to come to the cell V_1 at next instant, and the other particle is in the cell V_1 and plans to come to the cell V_0, then only one particle moves in accordance with a given rule. The choice is realized in accordance with priorities (a dynamical system) or equiprobably (a Markov process). The average (in a certain sense) number of delays is interesting for investigations. Suppose that the plans of particles are periodical fractions

$$a^{(i)} = 0.a_1^{(i)}\dots a_l^{(i)}(a_{l+1}, \dots, a_{l+m}),$$

i.e. proper common fractions. Shift onto a position to the left is equivalent to applying of the Bernoulli operation to a proper fraction. If we apply the Bernoulli shift to a fraction, we multiply the fraction by 2 and exclude the integer part. An algebra on proper fractions k/N, $k = 0, 1, \dots, N-1$, with respect to the Bernoulli shift has been introduced in (Kozlov, Buslaev, & Tatashev, 2015), and is called a Bernoulli algebra. The set of the algebra elements can be divided into disjoint subalgebras. It is found in (Kozlov, Buslaev, & Tatashev, 2015) that, if the plans are determined by elements of the same subalgebra, then the system comes to the state of synergy, i.e., there are no delays in the present instant and in the future. If plans of particles are given by elements of different orbits, then the tape velocities depend on orbits containing these elements. Bernoulli algebras are investigated in this paper.

The aim of this paper is to represent set of proper common fractions as Bernoulli algebras. These algebras are sets of logistic plans of bipendulums which was introduced earlier and are discrete dynamical systems with two positions and two pendulums. We discuss connection with Markov chains, graph theory, binary positional representational of numbers, and theory of functions. We have proved theorems about Bernoulli algebras.

2. Algebra of a Markov Process

In this section, we introduce a classification of elements of an algebra. This classification is similar to a known classification of Markov chain states.

Suppose algebra G is a set of elements together with a unary operation ω. We shall give definitions that allow to classify elements of the algebra. This classification is similar to the classification of Markov chain states (Gantmacher, 2004), (Borovkov, 1986). Suppose $x \in G$. The sequence $T(x) = \omega(x), \omega(\omega(x)), \dots$ is called the trajectory of an element x. We write $x \to y$ if $y \in T(x)$. Elements x and y, $x \neq y$ are called *communicating* with each other if $y \in T(x)$ and $x \in T(y)$, i.e., $x \to y$ and $y \to x$. The element x is called *inessential* if $x \notin T(y)$ for $\forall\ y \in T(x)$. The other elements are *essential.* The set of all essential elements can be divided into disjoint sets such that any two elements of the same set communicate with each other, and any two elements of different sets do not communicate with each other. These sets are called *classes of communicating essential elements* or *orbits*. An element x is called *absorbing* if $\omega(x) = x$. If a class of communicating essential elements contains $d \geq 1$ elements, then this class is called *a periodical class of communicating essential elements* or *an orbit with period d*.

3. Bernoulli Algebra on Common Fractions

In Section 3, we give definition of Bernoulli algebra. We introduce the concept of connected components of algebras. This concept is similar to concept of connected components in graph theory.

Consider the set of proper common fractions with denominator N

$$G_N = \left\{ \frac{0}{N}, \frac{1}{N}, \frac{2}{N}, \dots, \frac{N-1}{N} \right\}$$

with a unary operation, called the *Bernoulli shift,* (Schuster, 1984),

$$FB(x) = 2x - [2x],$$

where $[2x]$ is the integral part of $2x$.

Suppose W_N is the algebra on the set G_N with the operation FB, Figure 1. The number i/N $(0 \leq i \leq N-1)$ generates a subalgebra. Denote by $W_N(i)$ the subalgebra generated by the element i/N, $i = 0, 1, \dots, N-1$.

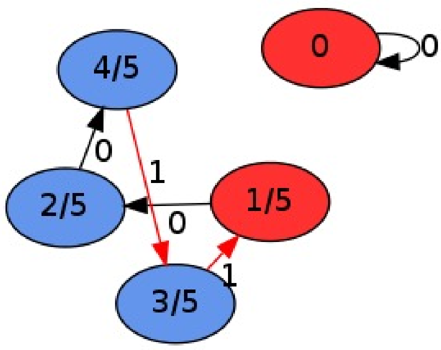

Figure 1. Algebra $W_5, 5 = 1[1] + 1[4]$

If i/N and j/N communicate with each other, then $W_N(j) = W_N(i)$, $i, j = 1, 2, \dots, N-1$. If $j/N \notin W_N(i)$ and $i/N \notin W_N(j)$, then $W_N(i)$ and $W_N(j)$ are disjoint subalgebras. Identifying subalgebras $W_N(i)$ and $W_N(j)$ with each other for $W_N(i) = W_N(j)$, we divide the algebra W_N into k_N disjoint subalgebras

$$W_N(i_1), W_N(i_2), \dots, W_N(i_{k_N}).$$

Each of these subalgebras contains a class of communicating essential elements and a set of inessential elements. This set of inessential elements can be empty. These subalgebras are called *connected components of the algebra.* We shall introduce the concept of the algebra W_N. graph. The graph of the algebra W_N is directed graph (Harary, 1969) which contains N vertices $v_0, v_2, \dots, v_{N-1}$. The graph contains the arc (v_i, v_j), where v_i is the tail of the arc and v_j is the head of the arc if and only if $FB(i/N) = j/N$. A connected component of the algebra W_N graph corresponds to a connected component of this algebra.

4. Binary Representation and Trajectory of the Element in the Bernoulli Algebra

In Section 4, we consider binary representation and trajectories of elements in Bernoulli algebra. We give geometric interpretation of Bernoulli algebras in the form of graphs.

Consider a common fraction k/N, $N \geq 1$, $0 \leq k < N$. The following theorem allows to find the binary representation of this fraction

$$\frac{k}{N} = \sum_{i=1}^{\infty} a_i \cdot 2^{-i},$$

where a_i is equal to 0 or 1, $i = 1, 2, \dots$

Theorem 1. *Suppose*

$$b_0 = k/N,\ b_i = FB(b_{i-1}),\ i = 1, 2, \dots$$

Then $a_i = 0$ if $b_i \geq b_{i-1}$, and $a_i = 1$ if $b_i < b_{i-1}$.

Proof. The fractional part of the number $2^{i-1}\frac{k}{N}$ is equal to that value if we apply $i-1$ times the operation $FB(\cdot)$, i.e., this value equals to b_{i-1},

$$b_{i-1} = \sum_{j=i}^{\infty} a_j \cdot 2^{i-j-1},\ i = 1, 2, \dots$$

This can be proved by induction on i. If $a_i = 0$, then $b_{i-1} < 0.5$ and, therefore, $b_i = 2b_{i-1} \geq b_{i-1}$. If $a_i = 1$, then $b_{i-1} > 0.5$, and, therefore, $b_i = 2b_{i-1} - 1 = b_{i-1} - (1 - b_{i-1}) < b_{i-1}$. Theorem 1 has been proved.

Suppose numbers l $(l \geq 1)$ and m $(m \geq 1)$ are such that $b_l = b_{l+m}$, and there are no numbers i and j that $i, j < l + m$ and $b_i = b_j$. Then $a_i = a_{i+m}$ for any $i \geq l$. Hence the binary representation of the common fraction k/N contains an aperiodic part with length $l - 1$ and a repeating part with length m. We write

$$\frac{k}{N} = 0.a_1 \dots a_{l-1}(a_l \dots a_{l+m-1}).$$

We have formulated an approach to find the binary representation of a common fraction. This approach is similar to the approach of converting the decimal representation of a common fraction to the p-ary representation, (Broido & Ilyina, 2006). It is easy to reconstruct the trajectory of the element k/N of the algebra W_N if we know the binary representation of the number k/N, Figure 2.

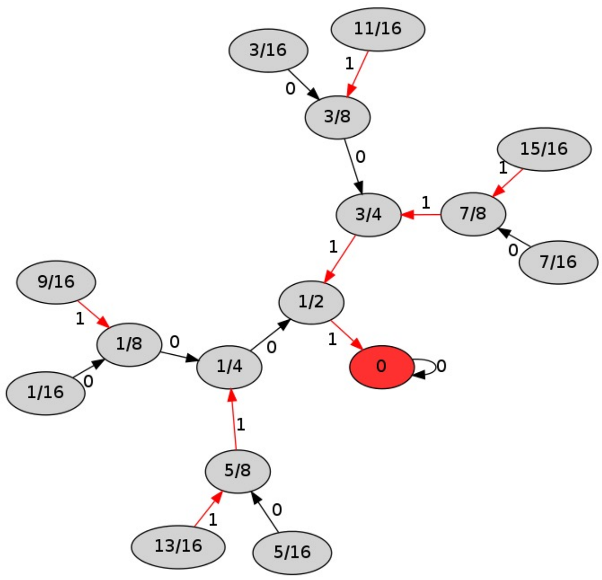

Figure 2. Algebra W_{16}, $16 = (8 \to 4 \to 2 \to 1 \to 1)$

5. Variation of the Binary Representation

In Section 5, we consider functional representation of plans. We give definition of variation which is one of most important characteristics of function. We also give definition of variation for binary representation of periodical fractions. This definition is equivalent to the general definition.

Suppose the binary representation of the number k/N is

$$\frac{k}{N} = 0.a_1 \dots a_l(a_{l+1} \dots a_{l+m}).$$

The *variation* of this representation is defined as

$$V(k/N) = \frac{1}{m} \sum_{i=l+1}^{l+m} |a_{i+1} - a_i|,$$

where the addition, in indexes, is meant modulo m. The limit

$$Var(a) = \lim_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k} |a_{i+1} - a_i|$$

is called the *variation of the binary representation* of a real number $a \in [0, 1)$

$$a = 0, a_1 a_2 \dots a_k \dots$$

if this limit exists. It is evident, that $V(k/N) = Var(k/N)$.

It is easy to give an example of a real number for that the variation of its binary representation is not defined. Suppose $a_1 = 1$, $a_j = 0$, $2^i \le j < 2^i + 2^{i-1}$, $a_j = 1$, $2^i + 2^{i-1} \le j < 2^i + 2^i$. Then the variation of the binary representation of the number a is not defined. Some number theory problems, related to binary representation of numbers, are considered in (Uteshev, Cherkasov, Shaposhnikov, 2001).

6. Markov Chain Interpretation of Algebra W_N

In Section 6, we give Markov chain interpretation of Bernoulli algebras.

A Markov chain, (Gantmacher, 2004), (Borovkov, 1986), corresponds to the algebra W_N. The chain contains N states $S_0, S_1, \dots, S_{N-1}$. The state S_i corresponds to the element i/N of the algebra W_N, $i = 0, 1, \dots, N-1$. The behavior of this chain is deterministic and depends only on the initial state. If at the instant $t-1$ the system is in the state S_i, and

$$FB\left(\frac{i}{N}\right) = \frac{j}{N},$$

then, at the instant t, the chain will be at the state S_j, $i, j = 0, 1, \dots, N-1$, with probability 1. Therefore, if

$$b_0 = \frac{k}{N},\ b_i = FB(b_{i-1}),\ i = 1, 2, \dots,$$

and the chain is in the state S_k, $k = 0, 1, \dots, N-1$, at the initial instant $t = 0$, then, at the instant t, the change will be in the state corresponding to the value b_t, $t = 1, 2, \dots$ Denote by p_{ij} the probability of the chain comes from the state S_i to the state S_j for a step, $i, j = 1, \dots, N$. Suppose P is the transition matrix of the chain

$$P = \begin{pmatrix} p_{00} & p_{01} & \cdots & p_{0,N-1} \\ \cdots & \cdots & \cdots & \cdots \\ p_{N-1,0} & p_{N-1,1} & \cdots & p_{N-1,N-1} \end{pmatrix}.$$

On the one hand, the matrix P is a stochastic matrix, and each element of this matrix is equal to 0 or 1. On the other hand, each row of the matrix P contains exactly one 1. If N is odd, then the matrix P is a permutation matrix. Since

$$FB\left(\frac{0}{N}\right) = \frac{0}{N} = 0,$$

it follows that the state S_0 is absorbing, i.e., $p_{00} = 1$. If the state S_k is inessential, and the chain comes from this state to the absorbing state S_0 for l steps, then the binary representation of the number k/N contains just l positive digits

$$\frac{k}{N} = 0.a_1 \dots a_l = 0, a_1 \dots a_l(0).$$

If the initial state S_k belongs to a periodic class of communicating states, and this class contains $d \ge 1$ states, then the repeating part of the number k/N binary representation contains d digits

$$\frac{k}{N} = 0.(a_1 \dots a_d).$$

If the initial state S_k is inessential and the chain comes from this state after l steps to a class of communicating states with period d, then the binary representation of the number k/N is

$$\frac{k}{N} = 0.a_1 \dots a_l(a_{l+1} \dots a_{l+d}),$$

i.e., the aperiodic part of the binary representation contains l digits, and the length of the repeating part equals d.

7. Binary Representation of Common Fractions and Division of the Algebra W_N into Classes

In Section 7, we consider connection between Bernoulli algebras, binary representation, and Markov chains.

Consider examples. Suppose $N = 8$, Figure 3. There is an absorbing element $0/8 = 0.(0)$ and inessential elements

$$\frac{1}{8} = 0.001(0),\ \frac{2}{8} = \frac{1}{4} = 0.01(0),\ \frac{3}{8} = 0.011(0),$$

$$\frac{4}{8} = 0.1(0),\ \frac{5}{8} = 0.101(0)\ \frac{6}{8} = 0.11(0),\ \frac{7}{8} = 0.111(0).$$

Values of the variation functions are equal to

$$V\left(\frac{1}{8}\right) = 0,\ V\left(\frac{2}{8}\right) = 0,\ V\left(\frac{3}{8}\right) = 0, V\left(\frac{4}{8}\right) = 0,$$

$$V\left(\frac{5}{8}\right) = 0,\ V\left(\frac{6}{8}\right) = 0,\ V\left(\frac{7}{8}\right) = 0,$$

The transition matrix of the Markov chain of the algebra W_8 have the form

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

The state S_0 is absorbing. The other states are inessential, Figure 3.

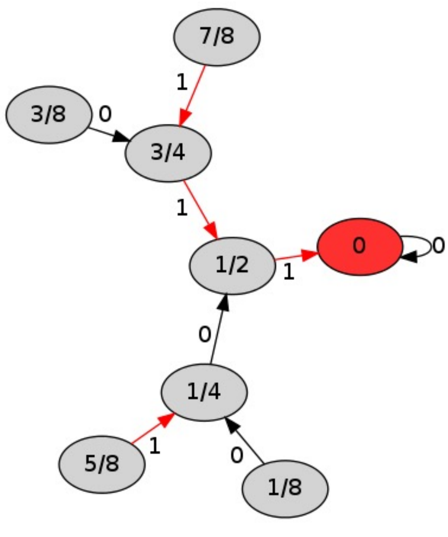

Figure 3. Algebra $W_8, 8 = (4 \to 2 \to 1 \to 1)$

8. Algebras W_N and W_{2N}

In Section 8, we prove a theorem which allow to find the form of algebra W_{2N} if the form of algebra W_N is known.

Let us compare algebras W_N and W_{2N} with each other.

Theorem 2. *Algebras W_N and W_{2N} contain the same number of connected components and, for any d, the same number of orbits with period d. The algebra W_{2N} contains a subalgebra isomorphic to the algebra W_N. This subalgebra consists* *of elements* $2i/2N$, $i = 0, 1, \dots, N-1$. *The algebra* W_{2N} *also contains* N *inessential elements* $x = 2i + 1/2N$, $i = 0, 1, \dots, N-1$, *such that* $FB(x) = y$ *for an element* y *of the subalgebra isomorphic to algebra* W_N.

Proof. The subalgebra W_{2N}, with elements $2i/2N$, $i = 0, 1, \dots, N-1$, is isomorphic to the algebra W_N as the values of the elements $2i/2N$ of the algebra W_{2N} and the element i/N of the algebra W_N, $i = 0, 1, \dots, N-1$, are the same. The elements $2i + 1/2N$, $i = 0, 1, \dots, N-1$, are inessential as, applying the Bernoulli shift, we obtain an element which belongs to the subalgebra isomorphic to the algebra W_N. Theorem 2 has been proved.

9. Structure of the Algebra W_N for an Odd N

In Section 9, we prove theorems about Bernoulli algebras in the case of odd values of N.

9.1. Preliminary Definitions and Results

Suppose the canonical representation of the number N, $N \geq 2$, has the form

$$N = p_1^{s_1} \dots p_l^{s_l}, \tag{1}$$

where $2 < p_1 < \dots < p_l$ are prime numbers, $s_1, \dots, s_l$ are natural numbers. We write $N = \overline{p}^{\overline{s}}$, where $\overline{p} = (p_1, \dots, p_l)$, $\overline{s} = (s_1, \dots, s_l)$. Denote by $E(N)$ the Euler's function (Vinogradov, 1972),

$$E(N) = \left(p_1^{s_1} - p_1^{s_1-1}\right)\dots\left(p_l^{s_l} - p_l^{s_l-1}\right).$$

Euler's theorem on numbers, (Vinogradov, 1972). *The value of* $E(N)$ *is equal to number of positive integers less than* N *that are coprime to* N.

This function can be also represented as

$$E(N) = N\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\dots\left(1 - \frac{1}{p_l}\right).$$

Euler's theorem on divisibility, (Vinogradov, 1972). *Suppose* k *and* N *are coprime natural numbers. Then* N *is a divisor of* $k^{E(N)} - 1$.

Fermat's little theorem is a special case of Euler's theorem on divisibility.

Fermat's little theorem, (Vinogradov, 1972). *If* k *is prime, then* k *is a divisor of* $N = 2^k - 2$.

Denote by $LCM(a_1, \dots, a_L)$ the least common multiple of numbers $a_1, \dots, a_l$, and N has the form (1). *Generalized Euler's function* is the function $e(N)$, $e(1) = 1$,

$$e(N) = LCM(p_1^{s_1-1}(p_1 - 1), \dots, p_l^{s_l-1}(p_l - 1)).$$

It is clear that any divisor of $e(N)$ is a divisor of $E(N)$.

The generalized Euler's divisibility theorem, (Vinogradov, 1972). *Let* k *and* N *be coprime positive integers. Then* N *is a divisor of* $k^{E(N)} - 1$.

According to this theorem, if k and N are coprime then there exists a positive number l such that N is a divisor of $k^l - 1$. *The smallest positive number* l *such that* N *is a divisor of* $k^l - 1$ *is called the order of* k *modulo* N, *(Vinogradov, 1972). Denote by* $m(N, k)$ *the order of* k *modulo* N.

The generalized Euler's order theorem, (Vinogradov, 1972). *Let* k *and* N *be coprime positive integers. Then* $m(N, k)$ *is a divisor of* $e(N)$.

Denote by $m(N)$ the order of 2 modulo N, i.e., $m(N) = m(N, 2)$. The number $m(N)$ is the smallest number m such that N, is a divisor of $2^m - 1$. The elements of the algebra W_N can be divided into two classes. They are *the class of noncancelable fractions and the class cancelable fractions*

$$W_N = W_N^{copr[ime]} + W_N^{canc[elable]}.$$

Consider the set of all elements k/N of the algebra W_N such that k/N is a noncancelable fraction. The number of these elements is equal to the number of positive integers less than N and coprime to N, i.e., the function $e(N)$ is equal to Euler's function. Suppose

$$\delta(k, E(N)) = \begin{cases} \frac{E(N)}{m(N)}, \ k = m(N), \\ 0, \ k \neq m(N). \end{cases} \tag{2}$$

In accordance with section 2 an orbit of period d is a set of elements of the algebra W_N

$$\frac{k_1}{N}, \frac{k_2}{N}, \dots, \frac{k_d}{N},$$

$0 \le k_i \le N-1$, $i = 1, \dots, d$ such that

$$\frac{k_{i+1}}{N} = FB\left(\frac{k_i}{N}\right) \neq \frac{k_1}{N},\ i = 1, \dots, d-1,$$

$$\frac{k_1}{N} = FB\left(\frac{k_d}{N}\right).$$

Denote by $O(N, k)$ *the number of orbits with period* k $(k \ge 2)$.

9.2. Simple Case

Suppose $N = p^s$, where $p \ge 3$ is prime, s is a natural number;

$$E(N) = p^s - p^{s-1}.$$

Theorem 3. *The following formula is true*

$$O(N, k) = \sum_{r=1}^{s} \delta(k, \varphi(p^r)) = \sum_{r=1}^{s} \delta(m(p^r), E(p^r)).$$

The proof is based on Lemma 1.

Lemma 1. *Suppose* $r \le s$, k/p^r *is an element of the algebra* W_{p^r} *such that it is noncancelable fraction. Then this element belongs to an orbit of period* $m(p^r)$. *The set of cancelable fractions of the algebra* W_{p^r} *is a subalgebra isomorphic to the algebra* $W_{p^{r-1}}$.

Proof. Suppose the number a satisfies the equation

$$2^m \cdot \frac{k}{N} = \frac{k}{N} + a.$$

The number a is natural if and only if the number N is the divisor of $2^m - 1$. We take into account that k and N are comprime, and $k < N$. Applying to a noncancelable element the Bernoulli shift operation $m(N)$ times, we obtain the same element. If the Bernoulli shift is applied to this element less than $m(N)$ times, then the same element cannot be obtained. It is obviously that the subalgebra of cancelable fractions of the algebra W_{p^r} and the algebra $W_{p^{r-1}}$ are isomorphic to each other. Lemma 1 has been proved.

The graphs of algebras W_{13}, W_{23}, W_{27} are shown in Figures 4 – 6.

9.3. Common Case

Let N be as (1).

Lemma 2. *Any element* k/N *of the algebra* W_N *belongs to an orbit, and, if* $k/N \in W_N^{copr}$, *i.e., the element* k/N *is noncancelable fraction, then the period of this orbit equals* $m(N)$.

Proof. Apply the Bernoulli shift operation to the element k/N i times. We obtain the same element if and only if the equation

$$2^i \cdot \frac{k}{N} = \frac{k}{N} + a, \tag{3}$$

contains natural number a. If $i = e(N)$ in (3), then the number a is natural. Take into account, that N is coprime to 2. Hence there exists an orbit of period not greater than $e(N)$, such that this orbit contains the element k/N. If the fraction k/N is a nocancelable fraction, then $i = m(N, 2) = m(N)$ is the smallest value of i such that the number a is a natural number. From this Lemma 2 follows.

Lemma 3. *Let* N *be an odd natural number. Two elements* $a \in W_N^{copr}$, $b \in W_N^{canc}$ *of the algebra* W_N *cannot belong the same orbit, i.e., if one of these elements is a noncancelable fraction, then the other element is a cancelable fraction.*

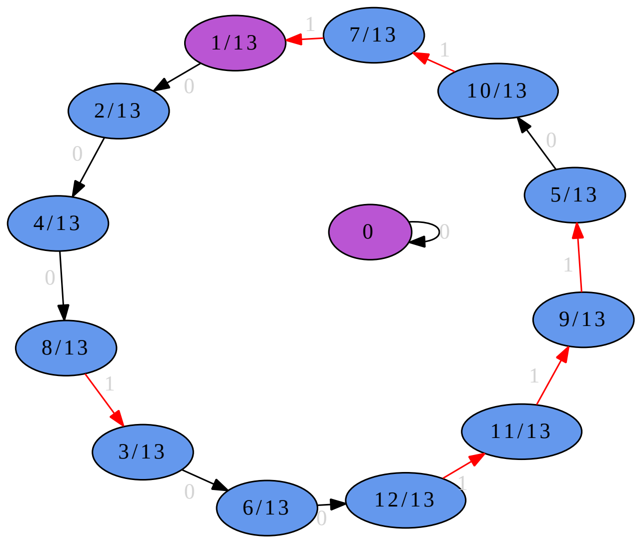

Figure 4. Algebra $W_{13}, 13 = 1[1] + 1[12]$

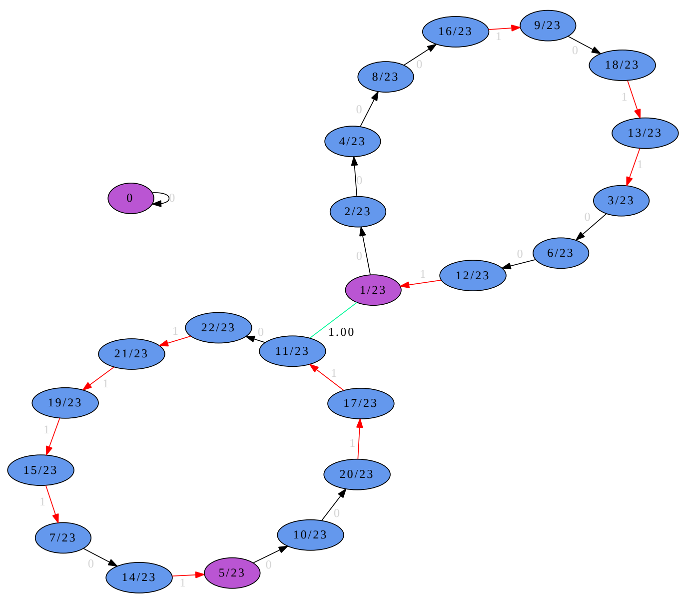

Figure 5. Algebra $W_{23}, 23 = 1[1] + 2[11]$

Proof. If an element is cancelable fraction, and we apply the Bernoulli shift operation to this element any times, we can obtain no noncancelable fraction.

Denote by $\delta_{d,N}$ the number of orbits with period d in the subalgebra W_N^{copr}. In accordance with Lemmas 2 and 3,

$$\delta_{m(p^r),p^r} = \frac{E(p^r)}{m(p^r)}.$$

If $d \neq m(p^r)$, then $\delta_{d,p^r} = 0$.

Suppose $R = R(N)$ is the set of vectors $r = (r_1, \ldots, r_l)$ with integer nonnegative numbers, $0 \leq r_1 \leq s_1, \ldots, 0 \leq r_l \leq s_l$, and at least one of numbers $r_1, \ldots, r_l$ is positive.

Theorem 4. *Suppose $O(N, k)$ is the number of orbits with period k ($k \geq 2$) in algebra W_N. Then*

$$O(N, k) = \sum_{\overline{r} \in R} \delta(k, \overline{p}^{\overline{r}}),$$

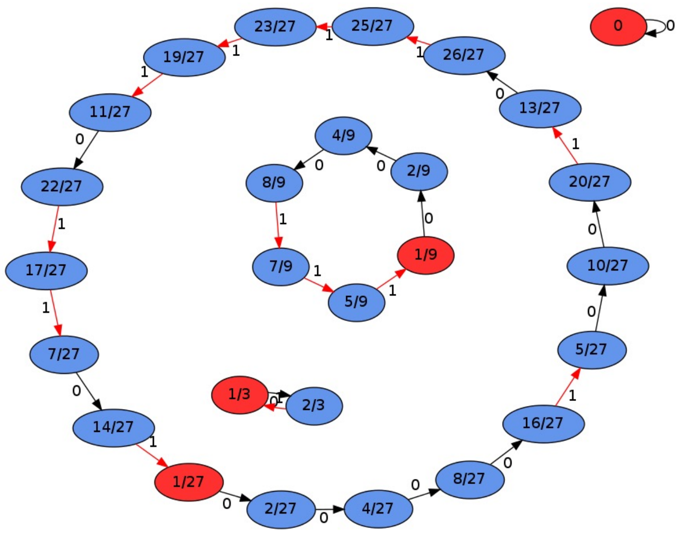

Figure 6. Algebra W_{27}, $27 = 1[1] + 1[2] + 1[6] + 1[18]$, (Kozlov, Buslaev, & Tatashev, 2015)

where $\delta(k, p^r)$ is calculated in accordance with (2).

Proof. If $r_1, r_2 \in R(N)$, then, for $N_1 = \overline{p}^{\overline{r}_1}$, $N_2 = \overline{p}^{\overline{r}_2}$, $N_1 \neq N_2$ we have

$$W_{N_1}^{copr} \cap W_{N_2}^{copr} = \emptyset.$$

Therefore any orbit is contained either in $W_{N_1}^{copr}$ or in $W_{N_2}^{copr}$. From this fact and Lemmas 2 and 3, Theorem 4 follows.

10. Algebra W_N for an even N

In Section 10, we prove theorems about Bernoulli algebras in the case of even values of N.

Consider the case of an even N. Suppose h is a natural number,

$$N = 2^h N_1,$$

where $N_1 = p_1^{s_1} \dots p_l^{s_l}$, $2 < p_1 < \dots < p_l$ are prime numbers.

Lemma 4. *The element i/N ($i = 2^h k$) of the subalgebra W_N belongs to an orbit with period m if and only if the element k/N of the algebra $W_{N/2^h}$ also belongs to an orbit with period m.*

Proof. The number $N/2^h$ is odd. The element k/N of the algebra $W_{N/2^h}$ belongs to an orbit in accordance with Lemma 2. The period of this orbit is equal to $m(N_1)$, where $m(N_1)$ is the minimum l such that in the equation

$$2^l \cdot \frac{k}{N_1} = \frac{k}{N_1} + a \tag{4}$$

the value a is a natural number. We can rewrite (4) as

$$2^l \cdot \frac{2^h k}{2^h N_1} = \frac{2^h k}{2^h N_1} + a.$$

Hence, if we apply the Bernoulli shift operation to the element $\frac{i}{N}$ of the algebra W_N $m(N_1)$ times, then we obtain the same element, and, if we apply the Bernoulli shift operation to the element i/N less than $m(N_1)$ times, we do not obtain the same element. From this, Lemma 7 follows.

Lemma 5. *Suppose the element $\frac{i}{N}$ is such that i is no multiple of 2^h. Then the element i/N is inessential.*

Proof. If we apply the Bernoulli operation to the element i/N 2^h times, then we obtain j/N, where j is a multiple of 2^h. We do not obtain the element i/N again if we apply the Bernoulli operation any times. From this, the lemma follows.

Theorem 5. *Suppose h is a natural number,*

$$N = 2^h N_1,$$

where $N_1 = p_1^{s_1} \dots p_l^{s_l}$, $2 < p_1 < \cdots < p_L$ *is prime numbers. Then algebras* W_N *and* $W_{N/2^h}$ *contain the same number of orbits with each period.*

Proof. Theorem 5 follows from lemmas 4 and 5.

Theorem 6. *Suppose*

$$N = 2^h N_1,$$

where $N_1 = p_1^{s_1} \dots p_l^{s_l}$, $2 < p_1 < \cdots < p_l$, *are prime numbers. Then the algebra* W_N *contains* N_1 *essential elements. The other* $N - N_1$ *elements are nonessential elements.*

Proof. If the number of the Bernoulli shift algebra is odd, then, in accordance with Theorem 4, all elements of this algebra are essential. From this fact and Theorem 5, Theorem 6 follows.

Theorem 7. *Suppose h is a natural number,*

$$N = 2^h N_1,$$

where $N_1 = p_1^{s_1} \dots p_l^{s_l}$, $2 < p_1 < \cdots < p_l$ *are prime numbers. Then the graph of the algebra* W_N *contains, as its subgraphs,* N_1 *trees, and the depth, or, in other words, height, of each of these tree equals h. Each of* N_1 *essential elements of the algebra* W_N *is the root of just one tree, and the degree of the root equals 1. The degree of other vertices, for any tree, is equal to* 2 *save the vertices of the level h. Any tree contains* 2^k *elements. One of these elements is the root of the tree, and there are* 2^{i-1} *elements that are vertices of the level i for any* $i = 1, \dots, h$.

Proof. We shall prove that the roots of trees are essential elements of the set M_0 containing elements

$$\frac{2^h i}{N},\ i = 0, 1, \dots, N_1 - 1,$$

and the vertices of the level 1 are elements of the set M_1

$$\frac{2^{h-1}(2i+1)}{N},\ i = 0, 1, \dots, N_1 - 1.$$

Indeed, suppose $i < \frac{N_1-1}{2}$. If we apply the Bernoulli shift operation to the element $\frac{2^{h-1}(2i+1)}{N}$, we obtain the element $\frac{2^h(2i+1)}{N}$. Suppose $i = \frac{N_1-1}{2}$, the number $N_1 - 1$ is a multiple of 2 as N_1 is even. If we apply the Bernoulli shift to the element $\frac{(2i+1)2^{h-1}}{N}$, we obtain the element $0/N$. Suppose $i > \frac{N_1-1}{2}$. Let us apply the Bernoulli shift to the element $\frac{2^{h-1}(2i+1)}{N}$. We have

$$2 \cdot \frac{2^{h-1}(2i+1)}{N} = 2 \cdot \frac{2\left(\frac{N_1}{2} + i + \frac{1}{2} - \frac{N_1}{2}\right)2^{h-1}}{N} =$$

$$= \frac{2^h \cdot (2i + 1 - N_1)}{N}.$$

Hence each element of the set M_1 corresponds to exactly one element of the set M_0, i.e., each element of the set M_1 corresponds to its successor belonging to the set M_0. Let M_j be the set of elements

$$\frac{2^{h-j}(2i+1)}{N},$$

$j = 2, \dots, h$, $i = 0, 1, \dots, 2^{j-1}N_1 - 1$. It is easy to see that numerators of the elements of the set M_j contain numbers that are multiples of 2^{h-j} but are not multiples of 2^{h-j+1}. We shall prove that M_j is the set of the vertices of the level j, $j = 1, 2, \dots, h$, and, if $j < h$, any vertex of the set M_j is the vertex of degree 2. Assume that $i = i_0$ ($0 \le i_0 \le 2^{j-2}N_1$). In the set M_j, there is a successor of the elements $\frac{2^{h-j}(2i_0+1)}{N}$ and $\frac{2^{h-j}(2i_0+2^{j-1}N_1+1)}{N}$. Namely, this successor is $\frac{2^{h-j+1}(2i_0+1)}{N}$, $j = 1, \dots, h$. There are no any other predecessors of the element $\frac{2^{h-j+1}(2i_0+1)}{N}$ of the set M_{j-1}. From this the theorem follows.

11. Algebras W_N and W_d, Where d is a Divisor of N

In Section 11, we prove theorems which allow to compare algebra W_N with algebra W_d, where d is a divisor of N.

Consider algebras W_N and W_d, where d is a divisor of the number N.

Theorem 8. *Let d be a divisor of the number N. If the element* i/d *of the algebra* W_d *belongs to the orbit with period m, then the element* $\frac{N}{d}\frac{i}{N}$ *of the algebra* W_N *belongs to an orbit with period m. If the elements* i/d *and* j/d *of the algebra* W_d *belong to the same orbit, then elements* $\frac{N}{d}\frac{i}{N}$, $\frac{N}{d}\frac{j}{N}$ *of the algebra* W_N *also belong to the same orbit. If the elements* i/d

and jd of the algebra W_d belong to different orbits, then the elements $\frac{N}{d}\frac{i}{2N}$ and $\frac{N}{d}\frac{j}{2N}$ of the algebra W_N also belong to the different orbits. If the element i/d of the algebra W_d is inessential, then the element $\frac{N}{d}\frac{i}{N}$ of the algebra W_N is also inessential. Elements $\frac{N}{d}\frac{i}{N}$ and $\frac{N}{d}\frac{j}{N}$ of the algebra W_N belong to the same components if and only if the elements i/N and j/N of the algebra W_N also belong to the same connected component.

Proof. The value of the element $\frac{N}{d}\frac{i}{N}$ of the algebra W_N and the value of element i/d of the algebra W_d are the same for any $i = 1, 2, \ldots, d-1$. From this, Theorem 8 follows.

Theorem 9. *Let d be a divisor of the number N. Then the number of orbits with period m, the number of the connected components and the number of elements W_N are not less than in the algebra W_d.*

Theorem 9 follows from Theorem 8.

12. Calculation of Numbers and Periods of Orbits of the Algebra W_{225}

In Section 12, we give an example. We consider algebra W_{225}. We calculate number of orbits of each period.

1) Divisors of the number $225 = (3, 5)^{(2,2)}$, not equal to 1 and 225, are 3, 5, 9, 15, 25, 45, 75.

2) Values of the generalized Euler's function e for divisors, not equal to 1 and 225, of the number 225 are $e(3) = 2$, $e(9) = 6$, $e(15) = 8$, $e(25) = 20$, $e(45) = 12$, $e(75) = 30$, $e(225) = 60$.

3) There exist no divisors, not equal to 1 and 2, of the number $e(3) = 2$, Therefore, $m(3) = e(3) = 2$.

4) There exist a divisor 2 of the number $e(5) = 4$. The number 5 is not a divisor of $2^2 - 1$. Therefore, $m(5) = e(5) = 4$.

5) There exist divisors 2 and 3 of the number $e(9) = 6$. The number 9 not a divisor of $2^2 - 1$ or $2^3 - 1$. $m(9) = e(9) = 6$.

6) There exist divisors 2 and 4 of the number $e(15) = 8$. The number 15 is not a divisor of $2^2 - 1$ but 15 is a divisor of $2^4 - 1$. $m(15) = 4$.

7) There are divisors 2, 4, 5, 10, not equal to 1 and 20, of the number $E(25) = 20$. The number 25 is not a divisor of $2^2 - 1$, $2^4 - 1$, $2^5 - 1$, or $2^{10} - 1$. Therefore, $m(25) = 20$.

8) There exist divisors 2, 3, 4, 6 of the number $e(45) = 12$. The number 45 is not a divisor of $2^2 - 1$, $2^3 - 1$, $2^4 - 1$, or $2^6 - 1$. Therefore, $m(45) = 12$.

9) There exist divisors 2, 4, 5, 10 of the number $e(75) = 20$. The number 75 is not a divisor of $2^2 - 1$, $2^4 - 1$, $2^5 - 1$, $2^8 - 1$, $2^{10} - 1$. Therefore, $m(75) = 20$.

10) Let us calculate the number $m(225)$. There exist divisors 2, 3, 4, 6, 10, 12, 15, 20, 30, 60 of the number $e(225) = 60$. The number 225 is not a divisor of $2^2 - 1$, $2^3 - 1$, $2^4 - 1$, $2^6 - 1$, $2^8 - 1$, $2^{10} - 1$, $2^{12} - 1$, $2^{15} - 1$, $2^{20} - 1$, or $2^{30} - 1$. Therefore, $m(225) = 60$.

11) Let us calculate the number

$$\delta\left(m\left(\overline{p}^{\overline{r}}\right), \overline{p}^{\overline{r}}\right)$$

for divisors of the number 225, not equal to 1, i.e., for any $\overline{r} \in R \cup \overline{s}$:

$$m(3) = 2, \delta(2, 3) = \delta(m(3), 3) = \frac{E(3)}{m(3)} = \frac{2}{2} = 1,$$

$$m(5) = 4, \delta(4, 5) = \delta(m(5), 5) = \frac{E(4)}{m(4)} = \frac{4}{4} = 1,$$

$$m(9) = 6, \delta(6, 9) = \delta(m(9), 9) = \frac{E(9)}{m(9)} = \frac{6}{6} = 1,$$

$$m(15) = 4, \delta(4, 15) = \delta(m(15), 15) = \frac{E(15)}{m(15)} = \frac{8}{4} = 2,$$

$$m(25) = 20, \delta(20, 25) = \delta(m(25), 25) = \frac{E(25)}{m(25)} = \frac{20}{20} = 1,$$

$$\delta(12, 45) = \delta(m(45), 45) = \frac{E(45)}{m(45)} = \frac{24}{12} = 2,$$

$$\delta(20, 75) = \delta(m(75), 75) = \frac{E(75)}{m(75)} = \frac{40}{20} = 2,$$

$$\delta(60, 225) = \delta(m(225), 225) = \frac{E(120)}{m(60)} = 2.$$

12) Let us calculate the value $A(k, 225)$ for any k such that $\delta(k, 225) > 0$:

$$A(2, 225) = \delta(2, 3) = 1,$$

$$A(4, 225) = \delta(4, 5) + \delta(4, 15) = 3,$$

$$A(6, 225) = \delta(6, 9) = 1,$$

$$A(12, 225) = \delta(12, 45) = 2,$$

$$A(20, 225) = \delta(20, 25) + \delta(20, 75) = 3,$$

$$A(60, 225) = \delta(225, 60) = 2.$$

Thus the algebra W_{225} contains an orbit with period 1; an orbit with period 2; 3 orbits with period 4; one orbit with period 6; 2 orbits with period 12; 3 orbits with period 20; 2 orbits with period 60. Since the number 225 is odd, the depth of each orbit equals 0.

13. Conclusion

We can use the concept of Bernoulli algebras in analysis of a dynamical system, (Kozlov, Buslaev, & Tatashev, 2015). The same mathematical subject, which is considered in the paper, can be interpreted as an algebra, a Markov process, or a graph. and related to the binary representation of numbers and a dynamical system, called a bipendulum. We can consider a more general system. There are M particles $P_1, \ldots, P_M$ and $K > 1$ vertices $V_1, \ldots, V_K$. Behavior of the particle P_i is determined by its plan a_i, $i = 1, \ldots, M$. The plan of each particle is the K-ary representation of a number which belongs to the segment $[0, 1]$. The generalized Bernoulli algebra $W_N(K)$ is related to the dynamical system, $N \geq 1$. It is the algebra of proper fractions $\{\frac{0}{N}, \frac{1}{N}, \ldots, \frac{N-1}{N}\}$ with respect to a single unary operation. If this operation is applied to a fraction, then the fraction is multiplied by K and the integer part is excluded. If $K = 2$, then this algebra is a Bernoulli algebra, i.e., $W_N(2) = W_N$.

References

Borovkov, A.A. (1986) *Probability theory (2nd ed.).* Moscow, Nauka. (In Russian.)

Broido, W.P., Ilyina, O.P. (2006). *Computer and systems architecture.* Saint Petersburg, Piter. (In Russian.)

Gantmacher, F.R. (2004). *Theory of matrices (5th ed.).* Moscow, Fizmatlit. (In Russian.)

Harary, F. (1969). *Graph theory.* Massachusetts, Addison-Wesley Publishing Company.

Kozlov, V.V., Buslaev, A.P., & Tatashev, A.G. (2015). On real-valued oscillations of a bipendulum. *Applied Mathematics Letters* (Vol. 46, pp. 44 – 49). http://dx.doi.org/10.1016/j.aml.2015.02.003

Schuster, H.G. (1984). *Deterministic chaos: An introduction.* Weinheim, Physik–Verlag.

Uteshev, A.Ju., Cherkasov, T.M., Shaposhnikov, A.A. (2001). *Digits and codes.* Saint Petersburg, Saint Petersburg University. (In Russian.)

Vinogradov, I.M. (1972). *Foundations of number theory.* Moscow, Nauka. (In Russian.)

Copyrights