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Solving Boundary-Layer Problems by Residual-Power-Series Method

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Abstract

In this paper, the so-called residual-power-series (RPS) method is presented for solving nonlinear boundary-layer equations. The RPS method provides a single unified treatment for the linear and nonlinear terms in the equations. The accuracy and efficiency of the RPS method is demonstrated for both a single and a system of two coupled boundary-layer equations on an unbounded domain.

Keywords: boundary-layer problem, residual-power-series, diagonal Padé approximation

1. Introduction

The residual-power-series (RPS) method, first proposed by (Abu Arqub, 2013), is a powerful method for solving linear and nonlinear problems. Most recently, Abu Arqub et al., (2013) employed the RPS method to solve Lane- Emden type equations. The RPS method is straightforward and simple toapply. The RPS method yields a Taylor expansion of the solution, and as a result, the exact solution is obtained whenever it is a polynomial. Moreover, the solution and all of its derivatives are applicable for each arbitrary point in a given interval. The RPS method has small computational requirements and high precision, and furthermore, it requires less time.

Many problems of interest in fluid mechanics are reduced, by introduction of suitable similarity variables, to nonlinear ordinary differential equations with appropriate boundary conditions (see for example (Vajravelu, 2001), (Kuiken, 1981), (Mishra and Mishra, 2012). The solutions of these nonlinear two-point boundary value problems are normally obtained by using, for example, the traditional finite difference methods. Approximate analytical treatments based on the Adomian decomposition method of the equations given by (Vajravelu, 2001) and (Kuiken,1981) were presented by (Kechil and Hashim, 2007a, 2007b). In this work, we shall extend for the first time the applicability of the RPS to boundary-layer equations on an unbounded domain.

2. Residual-Power-Series (RPS) Method

To describe the basic ideas of the RPS method (Abu Arqub, 2013) and to achieve our goal, we consider the system of initial value problem (IVP)

$$\Psi_i^{(n)}(\eta) = \Gamma_i\left(\eta, \Psi_i(\eta), \Psi_i'(\eta), \dots, \Psi_i^{(n-1)}(\eta)\right), \quad i = 1,2,\dots,r \tag{1}$$

subject to the initial conditions

$$\Psi_i(\eta_0) = \Psi_{i,0}, \Psi_i'(\eta_0) = \Psi_{i,1}, \Psi_i^{(n-1)}(\eta_0) = \Psi_{i,n-1} \tag{2}$$

Where $\Gamma_i: (\eta_0 - \varepsilon, \eta_0 + \varepsilon) \times \mathcal{R}^n \to \mathcal{R}^n$ is a nonlinear analytic function, η denotes the independent variable, $\Psi_i(\eta), \Psi_i'(\eta), \dots, \Psi_i^{(n-1)}(\eta)$ are unkown functions, and η_0, ε are real.

Assume that $y_i(\eta)$ are analytic functios on the given interval.

Therefore, these solutions can be represented as a power series as follows:

$$\Psi_i(\eta) = \sum_{m=0}^{+\infty} c_{i,m}(\eta - \eta_0)^m \tag{3}$$

where the coefficients $c_{i,m}$ are given by

$$c_{i,m} = \frac{\Psi_i^{(m)}(\eta_0)}{m!} = \frac{\Psi_{i,m}}{m!}, \qquad m = 0,1,\dots,n-1 \tag{4}$$

According to equations (2)-(4), the series solution can be written as

$$\Psi_i(\eta) = \Psi_{i,0} + \Psi_{i,1}(\eta-\eta_0) + \frac{\Psi_{i,2}}{2!}(\eta-\eta_0)^2 + \cdots + \frac{\Psi_{i,n-1}}{(n-1)!}(\eta-\eta_0)^{n-1} + \sum_{m=0}^{+\infty} c_{i,m}(\eta-\eta_0)^m \tag{5}$$

In practice, we approximate the solution by the *k*th-truncated series

$$\Psi_i(\eta) = \Psi_{i,0} + \Psi_{i,1}(\eta-\eta_0) + \frac{\Psi_{i,2}}{2!}(\eta-\eta_0)^2 + \cdots + \frac{\Psi_{i,n-1}}{(n-1)!}(\eta-\eta_0)^{n-1} + \sum_{m=0}^{k} c_{i,m}(\eta-\eta_0)^m \tag{6}$$

Now, to determine the rest of the coefficients $c_{i,m}$ for $m = n, n+1, \dots, k$ we define *k*th-residual function as follows

$$R_i(\eta) = \Psi_i^{(n)}(\eta) - \Gamma_i\left(\eta, \Psi_i(\eta), \Psi_i'(\eta), \dots, \Psi_i^{(n-1)}(\eta)\right) \tag{7}$$

It is clear that $R_i(\eta) = 0$ for each η

$$\sum_{m=0}^{k} c_{i,m}(\eta-\eta_0)^m \tag{8}$$

Now, to determine the rest of the coefficients $c_{i,m}$ for $m = n, n+1, \dots, k$ we define *k*th-residual function as follows

$$R_i(\eta) = \Psi_i^{(n)}(\eta_0) - \Gamma_i\left(\eta - \Psi_i(\eta), \Psi_i'(\eta), \dots, \Psi_i^{(n-1)}(\eta)\right) \tag{9}$$

It is clear that $R_i(\eta) = 0$ for each $\in (t_0 - \varepsilon, \eta_0 + \varepsilon)$, this is to confirm that these residual functions are differentiable infinitely many times at $\eta = \eta_0$. Moreover,

$$\left.\frac{d^m}{dt^m} R_i(\eta)\right|_{\eta=\eta_0} = 0 \tag{10}$$

Equation (6) and equation (10) for $m = n, n+1, \dots, k$, generate $k - n + 1$set of linear and nonlinear algebraic equations, respectively. These equations can be easily solved by symbolic computation software such as Maple and Mathematica for the unknown coefficients $c_{i,m}$.

3. Numerical Examples

3.1 Example 1

First we consider the following nonlinear boundary-value problem (Vajravelu, 2001)

$$f''' + ff'' - \frac{2r}{r+1}(f')^2 = 0 \tag{11}$$

$$f(0) = 0, f'(0) = 1, f'(\infty) = 0 \tag{12}$$

where $r$is a real number and the primes denote differentiation with respect to η. Equations (11) and (12) model many viscous flow problems. One such example is the velocity field in the flow and heat transfer phenomenon over a nonlinearly stretching sheet (Vajravelu, 2001). We note that the special case $r = 1$ admits the exact solution $f(\eta) = 1 - exp(-\eta)$ with $f''(0) = -1$.

To construct the solutions of system (11)-(12) by using the RPS, we should first rewrite the boundary conditions (12) in the form of initial conditions as follows

$$f(0) = 0, f'(0) = 1, f''(0) = \alpha \tag{13}$$

where $\alpha = f''(0)$ is to be determined from the boundary conditions at infinity in (12). Clearly, the first three terms of the approximation of $f(\eta)$ are$f_0(\eta) = 0, f_1(\eta) = \eta$ and $f_2(\eta) = \frac{\alpha}{2!}\eta^2$. Then the *k*th-truncated series has the form

$$f(\eta) = \eta + \frac{\alpha}{2!}\eta^2 + \sum_{m=3}^{k} c_m(\eta-\eta_0)^m = \frac{\alpha}{2!}\eta^2 + c_3\eta^3 + c_4\eta^4 + \cdots + c_k\eta^k \tag{14}$$

Table 1. Numerical values of α with the corresponding values of *r* usingdiagonal Padé approximants of f_{32}'

r	[5/5]	[7/7]	[10/10]	α of [3]
1	1.0003410482	–0.9999999051	–1.0000000000	–1.0000
5	1.1982743450	–1.1986417560	–1.1985277159	–1.1945
10	1.2365525777	–1.2334477611	–1.2369318045	–1.2348

To find the coefficients c_m by our RPS algorithm, we construct the residual function as follows.

$$R(\eta) = \sum_{m=3}^{k} m(m-1)(m-2)c_m(\eta-\eta_0)^{m-3} + \left(\eta + \frac{\alpha}{2!}\eta^2 + \sum_{m=3}^{k} c_m(\eta-\eta_0)^m\right) \times (\alpha + \sum_{m=3}^{k} m(m-1)c_m(\eta-\eta_0)^{m-2})^2 - \frac{2r}{r+1}(1+\alpha\eta+\sum_{m=3}^{k} mc_m(\eta-\eta_0)^{m-1})^2 \quad (15)$$

We have obtained the 32-term approximation to $f(\eta)$, but for the lack ofspace, only the first six terms are given below.

$$f_{32}(\eta) = \eta + \frac{\alpha}{2}\eta^2 + \frac{r}{3r+3}\eta^3 + \frac{\alpha(3r-1)}{24(r+1)}\eta^4 + \frac{[-\alpha^2+(3\alpha^2+4)r^2+2(\alpha^2-2)r]}{120(r+1)^2}\eta^5 + \frac{\alpha(19r^2-18r+3)}{720(r+1)^2}\eta^6 + \cdots \quad (16)$$

Now to determine the value of α we impose the condition at infinity in (12). The difficulty at infinity is overcome by employing the diagonal Padé approximants (Boyd, 1997) that approximatef_{32}. Table 1 shows that the values of -1 obtained via diagonal Padé approximants converge to the exact value -1 for the case$r = 1$. For the cases$r = 5$ and$r = 10$, we make a comparison withthe results of Vajravelu, (2001)who solved the problem numerically using theintegration scheme of the fourth-order Runge-Kutta. It is observed that thenumerical results are in well agreement with that of Vajravelu, (2001). Figure 1demonstrates the agreement of the rational function Padé approximant[10/10] of $f$andf' with the exact solution at $r = 1$ and also illustrates thevariations of$f(\eta)$and $f'(\eta)$.

3.2 Example 2

Next we shall apply the RPS for solving a nonlinear system of coupled ordinary equations. Kuiken, (1981) considered the problem of cooling of a low-heat-resistance sheet that moves downwards in a viscous fluid which he modeled by the following nonlinear boundary-value problem,

$$f'''(\eta) + \theta(\eta) - (f'(\eta))^2 = 0 \quad (17)$$

$$\theta''(\eta) - 3\sigma f'(\eta)\theta(\eta) = 0 \quad (18)$$

subject to the boundary conditions

$$f(0) = 0, f'(0) = 1, f'(+\infty) = 0 \quad (19)$$

$$\theta(0) = 1, \theta(+\infty) = 0 \quad (20)$$

where the primes denote differentiation with respect to $\eta$and $\sigma$is a constant.

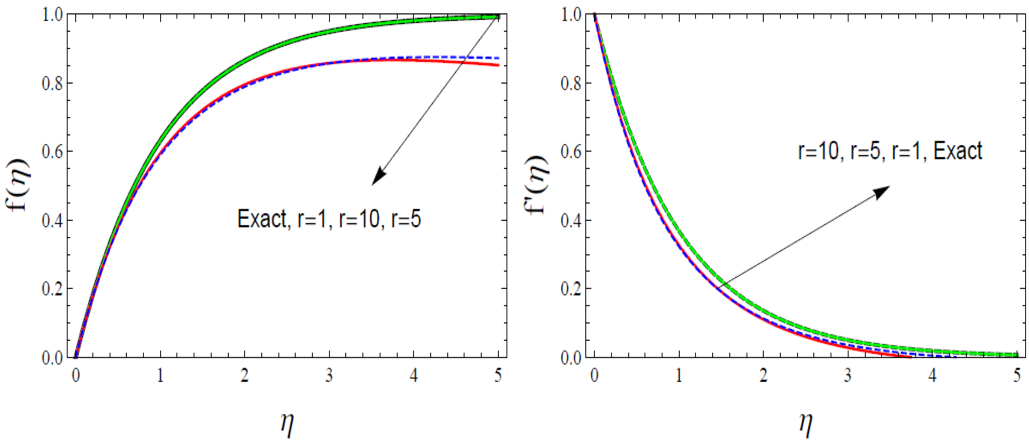

Figure 1. Variations of $f(\eta)$and $f'(\eta)$using $f_{32[10/10]}$and comparisonwith the exact solution for $r = 1$.

To construct the solutions of system (17)-(20) by using the RPS, we takethe following initial conditions

$$f(0) = 0, f'(0) = 1, f''(0) = \alpha \quad (21)$$

$$\theta(0) = 1, \theta'(0) = \delta \quad (22)$$

where $\alpha = f''(0)$ and $\delta = \theta'(0)$ are to be determined from the boundary conditions at infinity. Taking the first three terms of the approximation of $f(\eta)$ as $f_0(\eta) = 0, f_1(\eta) = 0$ and $f_2(\eta) = \frac{\alpha}{2!}\eta^2$, then the *k*th-truncatedseries has the form

$$f(\eta) = \frac{\alpha}{2!}\eta^2 + \sum_{m=3}^{k} c_{1,m}(\eta - \eta_0)^m = \frac{\alpha}{2!}\eta^2 + c_{1,3}\eta^3 + c_{1,4}\eta^4 + \cdots + c_{1,k}\eta^k \tag{23}$$

Next if we select the first two terms of the approximation of $\theta(\eta)$ as $\theta_0(\eta) = 1$ and $\theta_1(\eta) = \delta\eta$, then the *k*th-truncated series has the form

$$\theta(\eta) = 1 + \delta\eta + \sum_{m=2}^{k} c_{2,m}(\eta - \eta_0)^m = 1 + \delta\eta + c_{2,2}\eta^2 + c_{2,3}\eta^3 + \cdots + c_{2,k}\eta^k \tag{24}$$

To find the coefficients $c_{i,m}$by RPS, we construct the residual functions as follows

$$R_1(\eta) = \sum_{m=3}^{k} m(m-1)(m-2)c_{1,m}(\eta - \eta_0)^{m-3} + 1 + \delta\eta + \sum_{m=2}^{k} c_{2,m}(\eta - \eta_0)^m - \left(\alpha\eta + \sum_{m=3}^{k} m c_{1,m}(\eta - \eta_0)^{m-1}\right)^2 \tag{25}$$

$$R_2(\eta) = \sum_{m=2}^{k} m(m-1)c_{2,m}(\eta - \eta_0)^{m-2} - 3\sigma\left(\alpha\eta + \sum_{m=3}^{k} m c_{1,m}(\eta - \eta_0)^{m-1}\right) \times \left(1 + \delta\eta + \sum_{m=2}^{k} c_{2,m}(\eta - \eta_0)^m\right) \tag{26}$$

We have obtained the 23th-order and 22th-order approximations to $f(\eta)$ and $\theta(\eta)$, respectively, but for the lack of space, only the first five terms produced below.

$$f_{23}(\eta) = \frac{\alpha}{2}\eta^2 - \frac{1}{6}\eta^3 - \frac{\beta}{24}\eta^4 + \frac{\alpha^2}{60}\eta^5 + \frac{1}{240}(-\alpha\sigma - 2\sigma)\eta^6 + \frac{(-6\alpha\beta\sigma - 8\alpha\beta + 3\sigma + 6)}{5040}\eta^7 + \frac{(5\alpha^3 + 3\beta\sigma + 5\beta)}{10080}\eta^8 + \frac{(-18\alpha^2\sigma^2 - 21\alpha^2\sigma - 66\alpha^2 + 6\beta^2\sigma + 10\beta^2)}{181440}\eta^9 - \frac{\alpha(15\alpha\beta\sigma^2 + 19\alpha\beta\sigma + 42\alpha\beta - 24\sigma^2 - 31\sigma - 56)}{604800}\eta^{10} + \cdots \tag{27}$$

$$\theta_{22}(\eta) = 1 + \beta\eta + \frac{\alpha}{2}\eta^3 + \frac{\sigma(2\alpha\beta - 1)}{8}\eta^4 - \frac{\beta\sigma}{10}\eta^5 + \frac{\sigma(6\alpha^2\sigma + \alpha^2 - 2\beta^2)}{120}\eta^6 + \frac{\alpha\sigma(15\alpha\beta\sigma + 5\alpha\beta - 24\sigma - 3)}{840}\eta^7 - \frac{\sigma(123\alpha\beta\sigma + 22\alpha\beta - 24\sigma - 3)}{6720}\eta^8 + \frac{\sigma(126\alpha^3\sigma^2 + 126\alpha^3\sigma + 10\alpha^3 - 168\alpha\beta^2\sigma - 28\alpha\beta^2 + 195\beta\sigma + 31\beta)}{60480}\eta^9 + \cdots \tag{28}$$

The undetermined values of α and β are calculated from the boundary onditions at infinity in (19) and (20). The results presented in Tables 2 and 3 are in good agreement with that given by (Kuiken, 1981). Figures 2-3 illustrate the variation of $f(\eta)$,$f(\eta)$ and $\theta(\eta)$ approximated by the diagonal Padéapproximants in the cases$\sigma = 0.1$, $\sigma = 1$ and $\sigma = 10$.

Table 2. Numerical values of α usingdiagonal Padé approximants of f'_{23} and θ_{22}.

σ	[4/4]	[5/5]	[6/6]	α of [4]
0.001	1.1135529418	1.1272760416	1.1252849854	1.1231381347
0.01	1.0631737963	1.0741895683	1.0638385351	1.0633808585
0.1	0.9128082210	0.9238226280	0.9242158493	0.9240830397
1	0.6941230861	0.6998750497	0.6932195158	0.6932116298
10	0.4511240728	0.4502429544	0.4476712316	0.4471165250
100	0.2679197151	0.2681474363	0.2641295627	0.2645235434
1000	0.2204061432	0.1524783266	0.1500456755	0.1512901971
10000	0.0858587180	0.0858519249	0.0844775473	0.0855408524

Table 3. Numerical values of α usingdiagonal Padé approximants of f'_{23} and θ_{22}.

σ	[4/4]	[5/5]	[6/6]	α of [4]
0.001	–0.0371141028	–0.0415417739	–0.0436188230	–0.0468074648
0.01	–0.1274922800	–0.1221616907	–0.1351353865	–0.1357607439
0.1	–0.3621215470	–0.3505589981	–0.3499273453	–0.3500596733
1	–0.7694165843	–0.7614765813	–0.7698955992	–0.7698611967
10	–1.5028543431	–1.5007437650	–1.4985484075	–1.4970992078
100	–2.7627624234	–2.7637067330	–2.7445541894	–2.7468855016
1000	–5.7787858408	–4.9468469883	–4.9104728566	–4.9349476252
10000	–8.8057265644	–8.8032691004	–8.7384279086	–8.8044492660

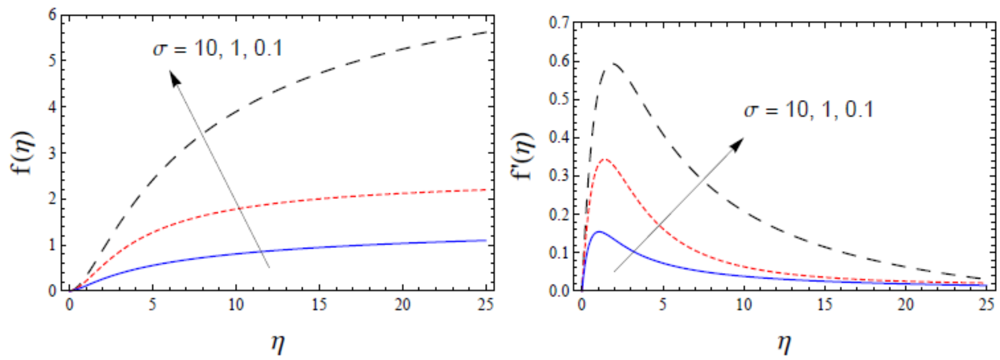

Figure 2. Variations of $f(\eta)$and $f'(\eta)$using $f_{23[6,6]}$for $\sigma = 0.1$, $\alpha = 0.9242158493$and $\beta = -0.3499273453$.$f_{23[5,5]}$for $\sigma = 1$, $\alpha = 0.6941230861$and $\beta = -0.7694165843$.$f_{23[4,4]}$for $\sigma = 10$, $\alpha = 0.4476712316$ and $\beta = -1.4985484075$.

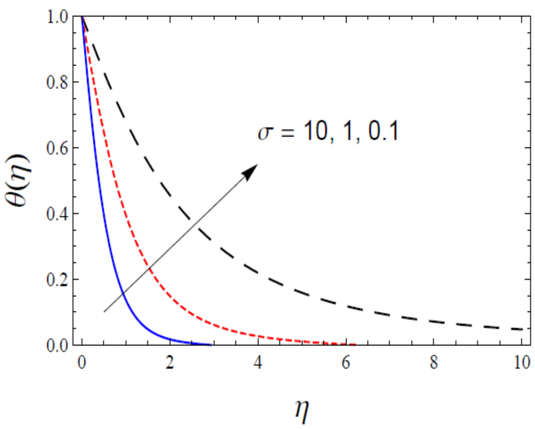

Figure 3. Variations of $\theta(\eta)$ and $\theta_{22[6,6]}$ for $\sigma = 0.1$, $\alpha = 0.9242158493$ and $\beta = -0.3499273453$. $\theta_{22[5,5]}$ for $\sigma = 1$, $\alpha = 0.6932195158$ and $\beta - 0.7698955992$.$\theta_{22[4,4]}$ for $\sigma = 10$, $\alpha = 0.4476712316$ and $\beta = -1.4985484075$.

4. Conclusion

The residual-power-series method was employed to solve nonlinear boundary-value problems. The RPS combined with Padé approximants are also shown to be a promising tool in solving two-point boundary value problems consisting of systems of nonlinear differential equations. The RPS method provides a single unified treatment for the linear and nonlinear termsin the equations. The accuracy and efficiency of the RPS method isdemonstrated for both a single and a system of two coupled boundary-layerequations on an unbounded domain.

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