

Fixed Point Theorems of Multi-Valued Zamfirescu Mapping

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The research is financed by Commission on Higher Education(Sponsoring information)

Abstract

In this paper the concept of Zamfirescu single-valued maps is extend to multi-valued maps and then we study the fixed point properties. We show that the multi-valued Zamfirescu mapping has an iteration which converges to a fixed point and give a rate of convergence. Data dependence problems on multi-valued Zamfirescu mapping are studied.

Keywords: Multi-valued mapping, Fixed point, Weakly Picard operator, Zamfirescu mapping, Data dependence

1. Introduction

The original theorem on fixed point theory is the contraction mapping principle theorem which is stated in Theorem 1.1 by Banach (Banach, S.,1922).

Theorem 1.1 Let (X, d) be a complete metric space and $T : X \rightarrow X$ be an a -contraction mapping, i.e., there exists a constant $a \in (0, 1)$ such that $d(Tx, Ty) \leq ad(x, y)$, for all $x, y \in X$. Then T has a unique fixed point.

In 1972, Zamfirescu (Zamfirescu, T.,1972) extended the condition on T in contraction mapping principle theorem to the class of maps which may not be continuous, called *Zamfirescu mappings* defined in Definition 1.2 and showed that such a mapping has a unique fixed point in Theorem 1.3, respectively.

Definition 1.2 Let (X, d) be a metric space, $T : X \rightarrow X$ is called *Zamfirescu mapping* iff there exist real numbers a, b and c satisfying $0 \leq a < 1$, $0 \leq b < \frac{1}{2}$ and $0 \leq c < \frac{1}{2}$ such that for each $x, y \in X$ at least one of the followings is true:

$$(z_1) \quad d(Tx, Ty) \leq ad(x, y)$$

$$(z_2) \quad d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)]$$

$$(z_3) \quad d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)].$$

Theorem 1.3 Let (X, d) be a complete metric space and T a self map. If T is a Zamfirescu mappings, then T has a unique fixed point.

Zamfirescu' theorem is a generalization of (z_1) Banach's theorem (Banach, S.,1922), (z_2) Kannan' s theorem (Kannan, R.,1968) and (z_3) Chatterjea's theorem (Chatterjea, S.K.,1972).

In 1967, Nadler (Nadler, S.B.,1967) initiated the idea for multi-valued contraction mapping. Later, in 1968,1969 Markin (Markin, J.T.,1968) and Nadler (Nadler, S.B.,1969) independently studied the fixed point of multi-valued contraction mapping.

Let $\mathcal{P}(X)$ be the family of all nonempty subsets of X and let T be a multi-valued mapping, i.e., $T : X \rightarrow \mathcal{P}(X)$. An element $x \in X$ such that $x \in Tx$ is called a fixed point of T . We denote by F_T the set of all fixed points of T , i.e., $F_T = \{x \in X : x \in Tx\}$ and by $G(T)$ the graph of multi-valued operator T , i.e., $G(T) = \{(x, y) : x \in X \text{ and } y \in Tx\}$.

Let (X, d) be a metric space and let $\mathcal{CB}(X)$ denote the family of all nonempty bounded closed subsets of X . For $x \in X$, $A, B \subseteq X$, we consider

$$d(x, A) = \inf\{d(x, a) : a \in A\}, \text{ the distance between } x \text{ and } A,$$

$$d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}, \text{ the distance between } A \text{ and } B,$$

$$\delta(A, B) = \sup\{d(a, b) : a \in A, b \in B\}, \text{ the diameter of } A \text{ and } B,$$

$$h(A, B) = \sup\{d(a, B) : a \in A\},$$

$H(A, B) = \max\{h(A, B), h(B, A)\}$, the Hausdorff-Pompeiu metric on $\mathcal{P}(X)$ induced by d .

The following result, usually referred as Nadler's fixed point theorem, gives a multi-valued version of Theorem 1.1.

Theorem 1.4 (Nadler, S.B., 1969) Let (X, d) be a metric space and $T : X \rightarrow \mathcal{CB}(X)$ a multi-valued a -contraction, i.e., there exists a constant $a \in (0, 1)$ such that $H(Tx, Ty) \leq ad(x, y)$, for all $x, y \in X$. Then T has at least one fixed point.

In this work, we extend the definition of Zamfirescu mapping in case of multi-valued and study the fixed point properties. In section 2, we show that the multi-valued Zamfirescu mapping has an iteration which converges to a fixed point and give a rate of convergence. Data dependence problem on multi-valued Zamfirescu mapping are studied in section 3. Finally, we give an example of such a map on which Nadler's theorem cannot be applied but our criteria can be used.

2. Multi-valued Zamfirescu mapping

We extend the definition of Zamfirescu mapping to case of multi-valued mapping in Definition 2.1.

Definition 2.1 Let (X, d) be a metric space and $T : X \rightarrow \mathcal{CB}(X)$ be a multi-valued mapping. T is said to be a *multi-valued Zamfirescu mapping* iff there exist real numbers a, b and c satisfying $0 \leq a < 1$, $0 \leq b < \frac{1}{2}$ and $0 \leq c < \frac{1}{2}$ such that for each $x, y \in X$ at least one of the followings is true:

- ($\tilde{z}_1$) $H(Tx, Ty) \leq ad(x, y)$
- ($\tilde{z}_2$) $H(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)]$
- ($\tilde{z}_3$) $H(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)]$.

Now, we define three important abstract notions given in (Rus, I.A., Petruşel, A., & Sîntămărian, A., 2003).

Definition 2.2 Let (X, d) be a metric space and $T : X \rightarrow \mathcal{P}(X)$ be a multi-valued operator. T is said to be a *multi-valued weakly Picard (briefly MWP) operator* iff for each $x \in X$ and any $y \in Tx$, there exists a sequence $\{x_n\}_{n=0}^{\infty}$ such that

- (i) $x_0 = x, x_1 = y$;
- (ii) $x_{n+1} \in Tx_n$ for all $n = 0, 1, 2, \dots$ and
- (iii) the sequence $\{x_n\}_{n=0}^{\infty}$ is convergent and its limit is a fixed point of T .

Remark 2.3 A sequence $\{x_n\}_{n=0}^{\infty}$ satisfying conditions (i) and (ii) in Definition 2.2 is also called a sequence of successive approximations of T starting from (x, y) .

Definition 2.4 Let (X, d) be a metric space and $T : X \rightarrow \mathcal{P}(X)$ be a MWP operator. Then we define the *multi-valued operator* $T^{\infty} : G(T) \rightarrow \mathcal{P}(F_T)$, by the formula;

$$T^{\infty}(x, y) := \{z \in F_T \mid \text{there exists a sequence of successive approximations of } T \text{ starting from } (x, y) \text{ which converges to } z\}.$$

Definition 2.5 Let (X, d) be a metric space, $T : X \rightarrow \mathcal{P}(X)$ be a MWP operator and $c > 0$. Then T is a *c-multi-valued weakly Picard (briefly c-MWP) operator* if for every $(x, y) \in G(T)$ there exists a selection $t^{\infty}(x, y)$ in $T^{\infty}(x, y)$ such that $d(x, t^{\infty}(x, y)) \leq cd(x, y)$.

At the beginning, we shall present some examples of *c-MWP* operators given in (Rus, I.A., Petruşel, A., & Sîntămărian, A., 2003).

Example 2.6 Let (X, d) be a metric space and $T : X \rightarrow \mathcal{CB}(X)$ be a multi-valued a -contraction ($0 < a < 1$). Then T is a *c-MWP* operator with $c = (1 - a)^{-1}$.

Example 2.7 Let (X, d) be a metric space and $T : X \rightarrow \mathcal{CB}(X)$ be a multi-valued operator for which there exist positive real numbers α, β and γ such that $\alpha + \beta + \gamma < 1$ and

$$H(Tx, Ty) \leq \alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty),$$

for all $x, y \in X$. Then T is a *c-MWP* operator with $c = (1 - \gamma)[1 - (\alpha + \beta + \gamma)]^{-1}$.

Example 2.8 Let (X, d) be a metric space and $T : X \rightarrow \mathcal{CB}(X)$ be a closed multi-valued operator for which there exist positive real numbers α and β such that $\alpha + \beta < 1$ and

$$H(Tx, Ty) \leq \alpha d(x, y) + \beta d(y, Ty),$$

for all $x \in X$ and $y \in Tx$. Then T is a *c-MWP* operator with $c = (1 - \beta)[1 - (\alpha + \beta)]^{-1}$.

Example 2.9 Let (X, d) be a metric space, $x_0 \in X$ and $r > 0$. Let $T : \tilde{B}(x_0, r) \rightarrow CB(X)$, where $\tilde{B}(x_0, r) = \{x \in X : d(x_0, x) \leq r\}$ be a multi-valued operator for which there exist $\alpha, \beta, \gamma \in \mathbb{R}_+$ and $\alpha + \beta + \gamma < 1$ such that

- (i) $H(Tx, Ty) \leq \alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty), \forall x, y \in \tilde{B}(x_0, r)$
- (ii) $\delta(x_0, Tx_0) < [1 - (\alpha + \beta + \gamma)](1 - \gamma)^{-1}r$.

Then T is a c -MWP operator with $c = (1 - \gamma)[1 - (\alpha + \beta + \gamma)]^{-1}$.

To prove the main results of this paper, we shall need the following lemmas which can be found in (Kunze, H.E., La Torre, D., & Vrscay, E.R., 2007) and (Rus, I.A., 2001).

Lemma 2.10 (Kunze, H.E., La Torre, D., & Vrscay, E.R., 2007) Let (X, d) be a metric space, $x, y \in X$ and A, B and C are subsets of X . The following statements hold.

1. If $A \subseteq B$, then $d(A, C) \geq d(B, C)$ and $h(A, C) \leq h(B, C)$ and $h(C, A) \geq h(C, B)$.
2. $d(x, A) \leq d(x, y) + d(y, A)$.
3. $d(x, A) \leq d(x, y) + d(y, B) + h(B, A)$.
4. $d(x, A) \leq d(x, B) + h(B, A)$.

Remark 2.11 If $A \subseteq B$, then $\delta(A, C)$ and $\delta(B, C)$ cannot be compared.

Lemma 2.12 (Rus, I.A., 2001) Let (X, d) be a metric space, $A, B \in \mathcal{P}(X)$ and $q \in \mathbb{R}, q > 1$ be given. Then for every $a \in A$ there exists $b \in B$ such that $d(a, b) \leq qH(A, B)$.

The next theorem is our main result of this section. It shows that any a multi-valued Zamfirescu mapping is a MWP operator, moreover it is a c -MWP operator.

Theorem 2.13 Let (X, d) be a complete metric space and $T : X \rightarrow CB(X)$ be a multi-valued Zamfirescu mapping. Then

1. T is a MWP operator;
2. for any $x_0 \in X$, there exists an orbit $\{x_n\}_{n=0}^{\infty}$ of T at a point x_0 that converges to a fixed point u of T , for which the following estimates holds:

$$\begin{aligned} d(x_n, u) &\leq \frac{\alpha^n}{1 - \alpha} d(x_0, x_1), & n = 0, 1, 2, \dots \\ d(x_n, u) &\leq \frac{\alpha}{1 - \alpha} d(x_{n-1}, x_n), & n = 1, 2, 3, \dots \end{aligned}$$

for a certain constant $\alpha < 1$.

Proof : Let $x_0 \in X$ and $x_1 \in Tx_0$.

If $H(Tx_0, Tx_1) = 0$, then $Tx_0 = Tx_1$, i.e., $x_1 \in Tx_1$, this means that $F_T \neq \emptyset$.

Let $H(Tx_0, Tx_1) > 0$. Choose $q, 1 < q < \min\{\frac{1}{a}, \frac{1}{2b}, \frac{1}{2c}\}$. By Lemma 2.12 there exists $x_2 \in Tx_1$ such that $d(x_1, x_2) \leq qH(Tx_0, Tx_1)$.

If x_0, x_1 satisfy $(\tilde{z}_1)$, then we have $d(x_1, x_2) \leq qad(x_0, x_1)$,

if x_0, x_1 satisfy $(\tilde{z}_2)$, then we have

$$\begin{aligned} d(x_1, x_2) &\leq qb[d(x_0, Tx_0) + d(x_1, Tx_1)] \\ &\leq qb[d(x_0, x_1) + d(x_1, x_2)] \end{aligned}$$

and hence $d(x_1, x_2) \leq (\frac{qb}{1 - qb})d(x_0, x_1)$,

and if x_0, x_1 satisfy $(\tilde{z}_3)$, then we have

$$\begin{aligned} d(x_1, x_2) &\leq qc[d(x_0, Tx_1) + d(x_1, Tx_0)] \\ &= qcd(x_0, Tx_1) \leq qcd(x_0, x_2) \\ &\leq qc[d(x_0, x_1) + d(x_1, x_2)] \end{aligned}$$

and hence $d(x_1, x_2) \leq (\frac{qc}{1 - qc})d(x_0, x_1)$.

Therefore, for $\alpha = \max\{qa, \frac{qb}{1-qb}, \frac{qc}{1-qc}\}$, we have $0 \leq \alpha < 1$ and thus

$$d(x_1, x_2) \leq \alpha d(x_0, x_1).$$

If $H(Tx_1, Tx_2) = 0$ then $Tx_1 = Tx_2$, i.e., $x_2 \in Tx_1$, this means that $F_T \neq \emptyset$.

Let $H(Tx_1, Tx_2) > 0$. Again by Lemma 2.12, there exists $x_3 \in Tx_2$ such that

$$d(x_2, x_3) \leq \alpha d(x_1, x_2).$$

In this manner we obtain an orbit $\{x_n\}_{n=0}^\infty$ of T at a point x_0 satisfying

$$d(x_n, x_{n+1}) \leq \alpha d(x_{n-1}, x_n), \quad n = 1, 2, 3, \dots \quad (1)$$

and hence by (1) we inductively obtain

$$d(x_n, x_{n+1}) \leq \alpha^n d(x_0, x_1), \quad (2)$$

and, respectively,

$$d(x_{n+k-1}, x_{n+k}) \leq \alpha^k d(x_{n-1}, x_n), \quad k \in \mathbb{N} \cup \{0\}. \quad (3)$$

Thus, for any $n, p \in \mathbb{N}$ by (2), we obtain

$$\begin{aligned} d(x_n, x_{n+p}) &\leq \sum_{k=n}^{n+p-1} d(x_k, x_{k+1}) \\ &\leq \sum_{k=n}^{n+p-1} \alpha^k d(x_0, x_1) \\ &\leq \frac{\alpha^n(1-\alpha^p)}{\alpha^n - \alpha} d(x_0, x_1) \\ &\leq \frac{1-\alpha^p}{1-\alpha} d(x_0, x_1) \end{aligned} \quad (4)$$

and so by (3), we arrive at

$$d(x_n, x_{n+p}) \leq (\alpha + \alpha^2 + \dots + \alpha^p) d(x_{n-1}, x_n) \leq \frac{\alpha}{1-\alpha} d(x_{n-1}, x_n). \quad (5)$$

Since $0 \leq \alpha < 1$, it results in $\alpha^n \rightarrow 0$ (as $n \rightarrow \infty$). Together with (4), this shows that $\{x_n\}_{n=0}^\infty$ is a Cauchy sequence, and so $\{x_n\}_{n=0}^\infty$ converges to some $u \in X$ as (X, d) is complete.

Lemma 2.10 (2) implies that

$$d(u, Tu) \leq d(u, x_{n+1}) + d(x_{n+1}, Tu) \leq d(u, x_{n+1}) + H(Tx_n, Tu).$$

If x_n, u satisfy $(\tilde{z}_1)$, then

$$d(u, Tu) \leq d(u, x_{n+1}) + \alpha d(x_n, u), \quad (6)$$

if x_n, u satisfy $(\tilde{z}_2)$, then

$$\begin{aligned} d(u, Tu) &\leq d(u, x_{n+1}) + b[d(x_n, Tx_n) + d(u, Tu)] \\ &\leq d(u, x_{n+1}) + b[d(x_n, x_{n+1}) + d(u, Tu)], \end{aligned} \quad (7)$$

and if x_n, u satisfy $(\tilde{z}_3)$, then

$$\begin{aligned} d(u, Tu) &\leq d(u, x_{n+1}) + c[d(x_n, Tu) + d(u, Tx_n)] \\ &\leq d(u, x_{n+1}) + c[d(x_n, Tu) + d(u, x_{n+1})]. \end{aligned} \quad (8)$$

Therefore, letting $n \rightarrow \infty$ in (6), (7) and (8), we have $d(u, Tu) = 0$.

Since Tu is closed, this implies $u \in Tu$.

To prove (2), we use (4) and (5) and the continuity of the metric and so by letting $p \rightarrow \infty$, the proof completes. $\square$

Given a point $x \in X$ and a compact set $A \subset X$, we know that there exist $a^* \in A$ such that $d(x, a^*) = d(x, A)$. We call a^* the *projection* of the point x on the set A and denote it as $a^* = \pi_x A$. Obviously, a^* is not unique but we choose one of it.

Let $T : X \rightarrow \mathcal{P}(X)$ be a multi-valued mapping such that Tx is a compact set for all $x \in X$. We now define the following projection associated with a multi-valued T by $Px = \pi_x(Tx)$. For $x_0 \in X$, we define $x_{n+1} = Px_n$, $n = 0, 1, 2, \dots$ and we call the sequence $\{x_n\}_{n=0}^\infty$ in this manner a *Picard projection iteration* sequence of T .

Theorem 2.14 Let (X, d) be a complete metric space and $T : X \rightarrow \mathcal{CB}(X)$ be a multi-valued Zamfirescu mapping such that Tx is compact for all $x \in X$. Then

1. $F_T \neq \emptyset$;
2. for any $x_0 \in X$, the Picard projection iteration $\{x_n\}_{n=0}^\infty$ converges to some $x^* \in F_T$;
3. The following estimates

$$\begin{aligned} d(x_n, x^*) &\leq \frac{\alpha^n}{1-\alpha} d(x_0, x_1), & n = 0, 1, 2, \dots \\ d(x_n, x^*) &\leq \frac{\alpha}{1-\alpha} d(x_{n-1}, x_n), & n = 1, 2, 3, \dots \end{aligned}$$

hold, for a certain constant $\alpha < 1$.

Proof: Let T be a multi-valued Zamfirescu mapping, i.e., for each $x, y \in X$ satisfying at least one condition $(\tilde{z}_1)$, $(\tilde{z}_2)$ or $(\tilde{z}_3)$, and Tx is compact for all $x \in X$.

Let $x_0 \in X$ be arbitrary and let $\{x_n\}_{n=0}^\infty$ be the Picard projection iteration.

For $x_1, x_2 \in X$ we see that;

$$d(x_1, x_2) = d(x_1, Px_1) = d(x_1, Tx_1) \leq h(Tx_0, Tx_1) \leq H(Tx_0, Tx_1)$$

, if x_0, x_1 satisfy $(\tilde{z}_1)$, then $d(x_1, x_2) \leq \alpha d(x_0, x_1)$,

if x_0, x_1 satisfy $(\tilde{z}_2)$, then

$$\begin{aligned} d(x_1, x_2) &\leq b[d(x_0, Tx_0) + d(x_1, Tx_1)] \\ &= b[d(x_0, x_1) + d(x_1, x_2)] \end{aligned}$$

and hence $d(x_1, x_2) \leq \frac{b}{1-b} d(x_0, x_1)$,

if x_0, x_1 satisfy $(\tilde{z}_3)$, then

$$\begin{aligned} d(x_1, x_2) &\leq c[d(x_0, Tx_1) + d(x_1, Tx_0)] \\ &= cd(x_0, Tx_1) \\ &\leq c[d(x_0, x_1) + d(x_1, Tx_1)] \\ &= c[d(x_0, x_1) + d(x_1, x_2)] \end{aligned}$$

and hence $d(x_1, x_2) \leq \frac{c}{1-c} d(x_0, x_1)$.

Therefore, for $\alpha = \{a, \frac{b}{1-b}, \frac{c}{1-c}\}$, we have $0 \leq \alpha < 1$ and thus $d(x_1, x_2) \leq \alpha d(x_0, x_1)$.

Moreover, for each $n \in \mathbb{N}$ we have;

$$d(x_n, x_{n+1}) \leq \alpha d(x_{n-1}, x_n). \quad (9)$$

We can use the same argument as in Theorem 2.13 for the rest of the proof. $\square$

Corollary 2.15 Let (X, d) be a complete metric space. If T is a multi-valued Zamfirescu mapping then T is a *c-MWP* operator with $c = \frac{1}{1-\alpha}$ for a certain constant $\alpha < 1$.

Proof : It follows immediately from Theorem 2.13 (2). $\square$

Theorem 2.16 Let (X, d) be a complete metric space. If T is a multi-valued Zamfirescu mapping, then F_T is complete.

Proof : Let $\{x_n\}_{n=0}^\infty$ be a Cauchy sequence in F_T . Since X is complete, there exist $u \in X$ such that $d(x_n, u) \rightarrow 0$ as $n \rightarrow \infty$. By Lemma 2.10 (3),

$$\begin{aligned} d(u, Tu) &\leq d(u, x_n) + d(x_n, Tx_n) + h(Tx_n, Tu) \\ &\leq d(u, x_n) + d(x_n, Tx_n) + H(Tx_n, Tu) \\ &= d(u, x_n) + H(Tx_n, Tu). \end{aligned}$$

If x_n, u satisfy $(\tilde{z}_1)$, then

$$d(u, Tu) \leq d(u, x_n) + \alpha d(x_n, u), \quad (10)$$

if x_n, u satisfy $(\tilde{z}_2)$, then

$$d(u, Tu) \leq d(u, x_n) + b[d(x_n, Tx_n) + d(u, Tu)] = d(u, x_n) + bd(u, Tu), \quad (11)$$

and if x_n, u satisfy $(\tilde{z}_3)$, then

$$\begin{aligned} d(u, Tu) &\leq d(u, x_n) + c[d(x_n, Tu) + d(u, Tx_n)] \\ &\leq d(u, x_n) + c[d(x_n, Tu) + d(u, x_n) + d(x_n, Tx_n)] \\ &= d(u, x_n) + c[d(x_n, Tu) + d(u, x_n)]. \end{aligned} \quad (12)$$

Hence, letting $n \rightarrow \infty$ in (10), (11) and (12), we get $d(u, Tu) = 0$. Since Tu is closed, $u \in Tu$. Therefore we have the theorem. $\square$

In example 2.17, we give an example of a mapping T such that Nadler's fixed point theorem can not be applied while we can use our Theorem 2.14.

Example 2.17 Let $X = [0, 1]$ and $T : X \rightarrow \mathcal{CB}(X)$ defined by

$$Tx = \begin{cases} [0, \frac{1}{4}] & ; x \in [0, 1) \\ [0, \frac{1}{2}] & ; x = 1. \end{cases}$$

Then T satisfies $(\tilde{z}_3)$ by choosing $c = \frac{1}{3}$. Hence, by Theorem 2.14, T has a fixed point. For $y = 1, x \in [\frac{3}{4}, 1)$, we observe that $|x - y| \leq \frac{1}{4} = H(Tx, Ty)$. These imply that T is not multi-valued contraction, so we cannot apply Nadler's theorem with this example.

3. Data dependence

Let (X, d) be a metric space and $T_1, T_2 : X \rightarrow \mathcal{CB}(X)$ such that F_{T_1} and F_{T_2} are nonempty. Data dependence problem concern on $H(F_{T_1}, F_{T_2})$. In this section we give some upper bound on $H(F_{T_1}, F_{T_2})$.

One of the main results concerning c -MWP operators is the following:

Theorem 3.1 (Rus, I.A., Petruşel, A., & Sîntămărian, A., 2003) Let (X, d) be a metric space and $T_1, T_2 : X \rightarrow \mathcal{P}(X)$ be two multi-valued operators. We suppose that:

- (i) T_i is a c_i -MWP operator, for $i \in \{1, 2\}$, and
- (ii) there exists $\eta > 0$ such that $H(T_1x, T_2x) \leq \eta$, for all $x \in X$.

Then $H(F_{T_1}, F_{T_2}) \leq \eta \max\{c_1, c_2\}$.

We have an immediate Corollaries.

Corollary 3.2 Let (X, d) be a complete metric space and T_1, T_2 two multi-valued Zamfirescu mappings on X and there exists $\eta > 0$ such that $H(T_1x, T_2x) \leq \eta$, for all $x \in X$. Then $H(F_{T_1}, F_{T_2}) \leq \eta \max\{c_1, c_2\}$ for certain constants c_1 and c_2 .

Proof: It follows from the fact T_1 and T_2 are c -MWP operators. $\square$

Let (X, d) be metric space and $T_1, T_2 : X \rightarrow \mathcal{CB}(X)$ multi-valued mappings. Define the distance $d_\infty(T_1, T_2) = \sup_{x \in X} H(T_1x, T_2x)$.

Clearly d_∞ is a metric on a class of multi-valued mappings.

Corollary 3.3 Let (X, d) be a complete metric space and T_1, T_2 two multi-valued Zamfirescu mappings on X . Then $H(F_{T_1}, F_{T_2}) \leq \frac{d_\infty(T_1, T_2)}{1 - \max\{\alpha_1, \alpha_2\}}$ for certain constants $\alpha_1, \alpha_2 < 1$.

Proof: Since $H(T_1x, T_2x) \leq d_\infty(T_1, T_2)$ for all $x \in X$, by Theorem 3.1(ii), we have the corollary. $\square$

Corollary 3.4 follows immediately from Corollary 3.3.

Corollary 3.4 Let $T_n : X \rightarrow \mathcal{CB}(X)$ be a sequence of multi-valued Zamfirescu mapping such that $\sup_n \alpha_n = c < 1$ where α_n is a certain constant for T_n . Suppose that $T_n \rightarrow T$ in the d_∞ metric where T is multi-valued Zamfirescu mapping, then $F_{T_n} \rightarrow F_T$ in the Hausdorff metric.

Acknowledgment

The first author would like to thank the Commission on Higher Education for financial support.

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