# The Mathematical Model and Analytical Solution of Electromagnetic Rail in the Loading Condition 

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#### Abstract

The Making the experiment on electromagnetic launcher, the rail supported by the containment and the insulator is modeled as a cantilever beam of finite length sitting on the elastic foundation. The mathematical model and the dynamic equation of the rail is given in the loading condition, as well as the analytical solution of the equation. The study will paves the way for mathematic model building and solution of rail gun with uneven pressure.


Keywords: Rail gun, Elastic beam, Mathematical model, Detached variable, Analytical solution

## 1. Introduction

Since 1980, particularly over the past decade, with the development of new technologies, new materials, electromagnetic launch technology has become a concern around the world. Electromagnetic launch technology is an emerging propulsion technology, and suitable for big load fired in the short stroke as well as widely used in the military, civilian and industry. ( $\mathrm{Li}, 2001$ ) However, electromagnetic launch technology is still in the experimental stage in the world. There are many technical problems still unresolved to electromagnetic gun from the laboratory onto the battlefield.

Figure 1 shows the principle of magnetic rail. When electric current pass through the rails and armature, it cause the electromagnetic force to promote armature and projectile movement. (Wang, 1995, pp1-124)But the electromagnetic force is very short-lived. In this paper, we establish the mechanical model of the structure response to the rail caused instantly by the electromagnetic force. Meanwhile the containment and insulator as elastic foundation, so the rail can be considered as a cantilever elastic beam to study.

In this paper, the rail is simplified as Timoshenko beams sitting on the elastic foundation. We research the dynamic response of the rail under the magnetic pressure, and obtain the mathematical model and differential equation, boundary conditions and initial conditions of the Timoshenko beam under a loading condition. Using the detached variable, the form solution of equation are obtained, further the complete solution obtained by Lagrange s equation too. It will provide the theoretical basis of mathematics to guide the design and manufacture of the rail.

## 2. Mathematic model

Figure 2 show the simplified force model of the rail sitting on elastic foundation, and a seriate electromagnetic pressure $p(x, t)=q[1-H(x-v t)]$ on the rail in the course of the armature movement. (Jerome T., 2003) Where $H(t)$ is the unit step function, and $q$ is the constant.
Bernoulli-Euler beam theory builts on the plane-section assumption and ignores the effects of shear deformation. In1920s, Timoshenko proposed the amendments theory under the premise of retaining the plane-section assumption, and considered shear deformation and rotational inertia of cross section. (Zhu, 2003, pp1-9)On the two broad-based displacement of the beam theory to the shear deformation, we could assume that the displacement of cross-section can be expressed as two parameters $\omega$ and $\varphi$. (Jerome T., 2005, pp246-250)Considering the issue of plane bending, and ingore the inertial force when the rail is working, the deflection and the corner of the rail satisfy the following equation

$$
\begin{equation*}
\kappa G A\left(\frac{\partial \varphi(x, t)}{\partial x}-\frac{\partial^{2} \omega(x, t)}{\partial^{2} x}\right)+\rho A \frac{\partial^{2} \omega(x, t)}{\partial^{2} t}+k \omega(x, t)=p(x, t) \tag{1}
\end{equation*}
$$

$$
\begin{equation*}
\kappa G A\left(\frac{\partial \omega(x, t)}{\partial x}-\varphi(x, t)\right)+E I \frac{\partial^{2} \varphi(x, t)}{\partial^{2} x}-\rho I \frac{\partial^{2} \varphi(x, t)}{\partial^{2} t}=0 \tag{2}
\end{equation*}
$$

Ignoring the rotary inertia $\rho I \frac{\partial^{2} \varphi(x, t)}{\partial^{2} t}$, these equations can be combined into one higher-order equation on the deflection of the beam $\omega(x, t)$ as follow

$$
\begin{equation*}
E I \frac{\partial^{4} \omega(x, t)}{\partial^{4} x}+\rho A \frac{\partial^{2} \omega(x, t)}{\partial^{2} x}-E I \frac{\rho}{\kappa G} \frac{\partial^{4} \omega(x, t)}{\partial^{2} t \partial^{2} x}+k \omega(x, t)=p(x, t) \tag{3}
\end{equation*}
$$

Here, $E$ is elastic modulus, $G$ is shear modulus, $I$ is the moment of interior of the rail cross sectional, $\kappa$ is shear correction coefficient, $A$ is cross sectional of rail, $k$ is the elastic constant of the elastic foundation, $\rho$ is the density of rail material, and $p(x, t)$ is the intensity of load.

## 3. The analytical solution of equation

### 3.1 Using Detached Variable to resolve the General Solution of Equation

Let(Zhu, 2005)

$$
\begin{equation*}
\omega(x, t)=\theta(x) \phi(t) \tag{4}
\end{equation*}
$$

Substituting (4) into the equation of (3) yields

$$
\begin{equation*}
E I \frac{\omega^{(4)}(x)}{\theta(x)}+m_{0} \frac{\phi^{(2)}(t)}{\phi(t)}-E I \frac{\rho}{\kappa G} \frac{\theta^{(2)}(x)}{\theta(x)}+k=0 \tag{5}
\end{equation*}
$$

From (5), we assume

$$
\begin{gather*}
m_{0} \frac{\phi^{(2)}(t)}{\phi(t)}=-\lambda^{2}(\lambda>0)  \tag{6}\\
E I \frac{\omega^{(4)}(x)}{\theta(x)}-E I \frac{\rho}{\kappa G} \frac{\theta^{(2)}(x)}{\theta(x)}+k-m_{0} \lambda^{2}=0 \tag{7}
\end{gather*}
$$

with $\beta^{4}=\frac{m_{0}}{E I}\left(\lambda^{2}-\frac{k}{m_{0}}\right), \alpha^{2}=\frac{\rho}{2 \kappa G}$
and

$$
\begin{equation*}
\theta^{(4)}(x)-2 \alpha^{2} \theta^{(2)}(x)-\beta^{4} \theta(x)=0 \tag{8}
\end{equation*}
$$

Solution of (6) and (8) can be expressed as:(Zhu, 2005)

$$
\begin{gather*}
\phi(t)=A \cos \lambda t+B \sin \lambda t  \tag{9}\\
\theta(x)=C_{1} \cosh m x+C_{2} \sinh m x+C_{3} \cosh n x+C_{4} \sinh n x \tag{10}
\end{gather*}
$$

where $m=\sqrt{\alpha^{2}-\sqrt{\alpha^{4}+\beta^{4}}}, \quad n=\sqrt{\alpha^{2}+\sqrt{\alpha^{4}+\beta^{4}}}$
According to the actual situation of bound(Chen, 2004), the boundary condition can be represented by
at $\quad x=0\left\{\begin{array}{l}\theta(x)=0 \\ \frac{\partial \theta(x)}{\partial x}=0\end{array}\right.$
and at $\quad x=L\left\{\begin{array}{l}\frac{\partial^{2} \theta(x)}{\partial^{2} x}=0 \\ \frac{\partial^{3} \theta(x)}{\partial^{3} x}=0\end{array}\right.$
The solution of the Equation (10) can be expressed by

$$
\begin{equation*}
\theta_{i}(x)=\cosh m_{i} x-\cosh n_{i} x+\frac{n_{i}^{2} \cosh n_{i} L-m_{i}^{2} \cosh m_{i} L}{m_{i}^{2} \sinh m_{i} L-m_{i} n_{i} \sinh n_{i} L}\left(\sinh m_{i} x-\frac{m_{i}}{n_{i}} \sinh n_{i} x\right) \tag{11}
\end{equation*}
$$

So
$\omega(x, t)=\sum_{i} \theta_{i}(x) \phi_{i}(t)=\sum_{i} \cosh m_{i} x-\cosh n_{i} x+\frac{n_{i}^{2} \cosh n_{i} L-m_{i}^{2} \cosh m_{i} L}{m_{i}^{2} \sinh m_{i} L-m_{i} n_{i} \sinh n_{i} L}\left(\sinh m_{i} x-\frac{m_{i}}{n_{i}} \sinh n_{i} x\right) \times\left(A_{i} \cos \lambda_{i} t+B_{i} \sin \lambda_{i} t\right)$

The series $\omega(x, t)$ is uniform convergence and itemized differentiable(Zhu, 2006, 6), at $t=0$

$$
\begin{gather*}
\omega(x, 0)=\sum_{i} \theta_{i}(x) \phi_{i}(0)=\sum_{i} \theta_{i}(x) A_{i}=\varphi(x)  \tag{13}\\
\frac{\partial \omega(x, 0)}{\partial t}=\sum_{i} \theta_{i}(x) \frac{\partial \phi(0)}{\partial t}=\sum_{i} \theta_{i}(x) B_{i}=\psi(x) \tag{14}
\end{gather*}
$$

The normal modes $\theta_{i}(x)$ are the orthogonal functions, which should satisfy

$$
\int_{0}^{L} \theta_{i} \theta_{j}= \begin{cases}0, & i \neq j  \tag{15}\\ 1, & i=j\end{cases}
$$

We integrate to (13) and (14) respectively, then substitute (15), so we could determine the constants and from

$$
\begin{align*}
A_{i} & =\int_{0}^{L} \theta_{i}(x) \varphi(x) d x  \tag{16}\\
B_{i} & =\frac{1}{\lambda_{i}} \int_{0}^{L} \theta_{i}(x) \psi(x) d x \tag{17}
\end{align*}
$$

Substituting $A_{i}$ and $B_{i}$ into the equation (12), the series solution of equation is obtained. However, $A_{i}$ and $B_{i}$ are integral forms, so the solution are only forms solution of equations.

### 3.2 Analytical Solution

Under the conditions of integrity constraints, the differential equations of beam can be expressed with the generalized coordinates. Usually, that is Lagrange s equation as follows

$$
\begin{equation*}
\frac{d}{d t}\left(\frac{\partial T}{\frac{\partial \phi_{i}}{\partial t}}\right)-\frac{\partial(T-U)}{\partial \phi_{i}}=Q_{i} \tag{18}
\end{equation*}
$$

Where $T$ is the kinetic energy of the system, and can be expressed as

$$
\begin{equation*}
T=\frac{1}{2} \int_{0}^{L} m_{0}\left(\frac{\partial \omega}{\partial t}\right)^{2} d x=\frac{1}{2} \sum_{i} \sum_{j} \frac{\partial \phi_{i}}{\partial t} \frac{\partial \phi_{j}}{\partial t} \int_{0}^{L} m_{0} \theta_{i} \theta_{j} d x \tag{19}
\end{equation*}
$$

Recall the orthogonal function defined in Equation (17), we can rewrite $T$ as

$$
\begin{equation*}
T=\frac{1}{2} \sum_{i} M_{i}\left(\frac{\partial \phi_{i}}{\partial t}\right)^{2} \tag{20}
\end{equation*}
$$

Where

$$
\begin{equation*}
M_{i}=\int_{0}^{L} m_{0} \theta_{i}^{2}(x) d x \tag{21}
\end{equation*}
$$

The total potential of the beam $U$ consists of $U_{b}$, the strain energy of the beam, and $U_{f}$, the strain energy of the elastic foundation, and $U_{g}$, the strain energy of the shear deformation(S. Thmoshenko, 1999),
Where $\quad U_{b}=\frac{1}{2} \int_{0}^{L} E I\left(\frac{\partial^{2} \omega(x, t)}{\partial^{2} t}\right)^{2} d x=\frac{1}{2} \sum_{i} E I \phi_{i}^{2} \int_{0}^{L} \theta^{(4)}(x) \theta(x) d x$

$$
\begin{gather*}
=\frac{E I}{2} \sum_{i} \phi_{i}^{2} \int_{0}^{L}\left(2 \alpha^{2} \theta^{(2)}(x)+\beta^{4} \theta(x)\right) \theta(x) d x=\frac{E I}{2 m_{0}} \sum_{i} \beta_{i}^{4} M_{i} \phi_{i}^{2}+\frac{E I}{2} \sum_{i} \phi_{i}^{2} \int_{0}^{L} 2 \alpha^{2}\left[\theta^{(1)}(x)\right]^{2} d x  \tag{22}\\
U_{f}=\frac{1}{2} \int_{0}^{L} k \omega^{2} d x=\frac{k}{2} \int_{0}^{L}\left(\sum_{i} \theta_{i} \phi_{i}\right)^{2} d x=\frac{k}{2} \sum_{i} \sum_{j} \phi_{i} \phi_{j} \int_{0}^{L} \theta_{i} \theta_{j} d x=\frac{k}{2 m_{0}} M_{i} \sum_{i} \phi_{i}^{2}  \tag{23}\\
U_{g}=\int_{0}^{L} \int_{\frac{-k}{2}}^{\frac{k}{2}} \frac{1}{2 G}\left[\frac{P}{2 I}\left(\frac{h^{2}}{4}-y^{2}\right)\right]^{2} d x d y+\frac{L^{2} h^{2}}{20 G I}(q v t)^{2} \tag{24}
\end{gather*}
$$

So, the total potential energy $U$ is obtained as

$$
\begin{equation*}
U=U_{b}+U_{f}+U_{g}=\sum_{i} M_{i} \phi_{i}^{2} \lambda_{i}^{2}+\frac{E I}{2} \sum_{i} \phi_{i}^{2} \int_{0}^{L} 2 \alpha^{2}\left[\theta^{(1)}(x)\right]^{2} d x+\frac{P^{2} L^{2} h^{2}}{20 G I}=\frac{L^{2} h^{2}}{20 G I}(q v t)^{2} \tag{25}
\end{equation*}
$$

For a given virtual displacement $\delta \phi_{i}$, the virtual work done by the electromagnetic force $p(x, t)=q[1-H(x-v t)]$ can be expressed as:

$$
\begin{equation*}
\partial W=\int_{0}^{L} p(x, t) \delta w_{i} d x=\sum_{i} \partial \phi_{1} Q_{i} \tag{26}
\end{equation*}
$$

Where we define $Q_{i}$ as the generalized force, and

$$
\begin{align*}
Q_{i} & =\int_{0}^{L} p(x, t) \theta_{i}(x) d x=\int_{0}^{L} q[1-H(x-v t)] \theta_{i}(x) d x=\int_{0}^{v t} q \theta_{i}(x) d x \\
& \left.=q\left[\left(\frac{1}{m_{i}} \sinh m_{i} v t-\frac{1}{n_{i}} \sinh n_{i} v t\right)+\frac{n_{i}^{2} \cosh n_{i} L-m_{i}^{2} \cosh m_{i} L}{m_{i}^{2} \sinh m_{i} L-m_{i} n_{i} \sinh n_{i} L} \times\left(\frac{1}{m_{i}} \cosh m_{i} v t-\frac{m_{i}}{n_{i}^{2}} \cosh n_{i} v t\right)-\frac{1}{m_{i}}+\frac{m_{i}}{n_{i}^{2}}\right)\right] \tag{27}
\end{align*}
$$

Substituting $T, U, Q_{i}$ into the Equation (18), the ordinary differential with second order is obtained in terms of $\phi_{i}(t)$ as follows

$$
\begin{equation*}
\frac{\partial^{2} \phi_{i}}{\partial^{2} t}+\frac{M_{i} \lambda_{i}^{2}+\frac{1}{m_{0}} \alpha^{2} E I}{M_{i}} \phi_{i}=\frac{Q_{i}(t)}{M_{i}} \tag{28}
\end{equation*}
$$

Let

$$
\begin{equation*}
\eta_{i}^{2}=\lambda_{i}^{2}+\frac{\alpha^{2} E I}{m_{0} M_{i}} \tag{29}
\end{equation*}
$$

So the equation (30) can be expressed as

$$
\begin{equation*}
\frac{\partial^{2} \phi_{i}}{\partial^{2} t}+\eta_{i}^{2} \phi_{i}=\frac{Q_{i}(t)}{M_{i}} \tag{30}
\end{equation*}
$$

Assuming that the initial conditions are as at $t=0\left\{\begin{array}{l}\phi(0)=0 \\ \frac{\partial \phi(0)}{\partial t}\end{array}\right.$
The solution of equation (28) is
$\left.\phi_{i}(t)=\frac{q}{M_{i} \eta_{i}} \int_{0}^{L}\left(\frac{1}{m_{i}} \sinh m_{i} v \sigma-\frac{1}{n_{i}} \sinh n_{i} v \sigma\right)+\frac{n_{i}^{2} \cosh n_{i} L-m_{i}^{2} \cosh m_{i} L}{m_{i}^{2} \sinh m_{i} L-m_{i} n_{i} \sinh n_{i} L} \times\left(\frac{1}{m_{i}} \cosh m_{i} v \sigma-\frac{m_{i}}{n_{i}^{2}} \cosh n_{i} v \sigma\right)-\frac{1}{m_{i}}+\frac{m_{i}}{n_{i}^{2}}\right) \times\left(\sin \eta_{i}(t-\sigma)\right) d \sigma$

$$
\begin{equation*}
=\frac{q}{M_{i} \eta_{i}}\left(I_{1}+I_{2}+I_{3}+I_{4}+I_{5}+I_{6}\right) \tag{31}
\end{equation*}
$$

Where

$$
\begin{gather*}
I_{1}=\int_{0}^{L} \frac{1}{m_{i}} \sinh m_{i} v \sigma \sin \eta_{i}(t-\sigma) d \sigma=\frac{1}{m_{i}}\left(m_{i}^{2} v^{2}+\eta_{i}^{2}\right) \times\left(\eta_{i} \sinh m_{i} v t-m_{i} v \sin \eta_{i} t\right)  \tag{32}\\
I_{2}=-\int_{0}^{L} \frac{1}{n_{i}} \sinh n_{i} v \sigma \sin \eta_{i}(t-\sigma) d \sigma=-\frac{1}{n_{i}}\left(n_{i}^{2} v^{2}+\eta_{i}^{2}\right) \times\left(\eta_{i} \sinh n_{i} v t-n_{i} v \sin \eta_{i} t\right)  \tag{33}\\
I_{3}=\frac{n_{i}^{2} \cosh n_{i} L-m_{i}^{2} \cosh m_{i} L}{m_{i}^{2} \sinh m_{i} L-m_{i} n_{i} \sinh n_{i} L} \int_{0}^{L} \frac{1}{m_{i}} \cosh m_{i} v \sigma \sin \eta_{i}(t-\sigma) d \sigma \\
=\frac{n_{i}^{2} \cosh n_{i} L-m_{i}^{2} \cosh m_{i} L}{m_{i}^{2} \sinh m_{i} L-m_{i} n_{i} \sinh n_{i} L} \frac{\eta_{i}}{m_{i}\left(m_{i}^{2} v^{2}+\eta_{i}^{2}\right)} \times\left(\cosh m_{i} v t-\cos \eta_{i} t\right)  \tag{34}\\
I_{4}=-\frac{n_{i}^{2} \cosh n_{i} L-m_{i}^{2} \cosh m_{i} L}{m_{i}^{2} \sinh m_{i} L-m_{i} n_{i} \sinh n_{i} L} \int_{0}^{L} \frac{m_{i}}{n_{i}^{2}} \cosh n_{i} v \sigma \sin \eta_{i}(t-\sigma) d \sigma \\
=-\frac{n_{i}^{2} \cosh n_{i} L-m_{i}^{2} \cosh m_{i} L}{m_{i}^{2} \sinh m_{i} L-m_{i} n_{i} \sinh n_{i} L} \frac{\eta_{i} m_{i}}{n_{i}^{2}\left(m_{i}^{2} v^{2}+\eta_{i}^{2}\right)} \times\left(\cosh n_{i} v t-\cos \eta_{i} t\right) \tag{35}
\end{gather*}
$$

$$
\begin{gather*}
I_{5}=-\frac{n_{i}^{2} \cosh n_{i} L-m_{i}^{2} \cosh m_{i} L}{m_{i}^{2} \sinh m_{i} L-m_{i} n_{i} \sinh n_{i} L} \int_{0}^{L} \frac{1}{m_{i}} \sin \eta_{i}(t-\sigma) d \sigma=-\frac{n_{i}^{2} \cosh n_{i} L-m_{i}^{2} \cosh m_{i} L}{m_{i}^{2} \sinh m_{i} L-m_{i} n_{i} \sinh n_{i} L} \frac{1}{m_{i} \eta_{i}} \times\left(1-\cos \eta_{i} t\right)  \tag{36}\\
I_{6}=\frac{n_{i}^{2} \cosh n_{i} L-m_{i}^{2} \cosh m_{i} L}{m_{i}^{2} \sinh m_{i} L-m_{i} n_{i} \sinh n_{i} L} \int_{0}^{L} \frac{m_{i}}{n_{i}^{2}} \sin \eta_{i}(t-\sigma) d \sigma=\frac{n_{i}^{2} \cosh n_{i} L-m_{i}^{2} \cosh m_{i} L}{m_{i}^{2} \sinh m_{i} L-m_{i} n_{i} \sinh n_{i} L} \frac{m_{i}}{n_{i}^{2} \eta_{i}} \times\left(1-\cos \eta_{i} t\right) \tag{37}
\end{gather*}
$$

The analytical solution can be obtained by substituting (32)-(37) into (31), then substituting (31) into (12). So we obtain the solution of the equation (3).

## 4. Numerical example

There are material parameters of rail[10] with $E=120 \mathrm{GPa}, I=2.5 \times 10^{-9} \mathrm{~m}^{4}, \rho=8320 \mathrm{~kg} / \mathrm{m}^{3}, G=47 \mathrm{GPa}, \mathrm{A}=$ $3 \times 10^{-4} \mathrm{~m}^{2}, K=2.5 \times 10^{7} \mathrm{~N} / \mathrm{m}^{2}, L=1 \mathrm{~m}$. Since the beginning speed of the projectile is $1000 \mathrm{~m} / \mathrm{s}$, movement-time of the projectile is 0.01 seconds in the chamber. The curve change of the rail deflection are given only in 0.4 ms as follow[11][12] Considering the influence of the speed [13] of load, the shear modified coefficient and the time on the deflection, Figure 3 and Figure 4 show that the deflection change dramatically in 0 . 2 milliseconds, follow slowly. Figure4 illustrates the shear modified coefficient make a full impact on the deflections, so we can not overlook the shear deformation of the rail.

## 5. Conclusion

(1) The rail is a cantilever beam sitting on the elastic fountion. So we establish the mathematical model, and consider the shear deformation of the beam.
(2) Using the detached variable and Lagrange equation of generalized coordinate, the analytical solution of governing equation is deduced.
(3) Making use of the Matlab software, we analysis the impact of the load speed and shear modified coefficient on dynamic response of the rail which is is more significant. The shear modified coefficient is the greater, deformation of the beam is the more obvious. The effect of the shear to rail can't be ignored.

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Chunyan Zhou, female, born in 1982 in Zhang Jiakou, graduatestudent of college of science in Yanshan University, now. Major research are the electromagnetic force device to build the mathematical model and computer simulation.

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