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Oscillation Results for Second Order Neutral Equations with Distributed Deviating Arguments

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Abstract

The oscillation of second order neutral equations with distributed deviating arguments is studied. By using a class of parameter functions $\Phi(t, s, l)$ and the generalized Riccati technique, some new oscillation criteria for the equations are obtained. The obtained results are different from most known ones and can be applied to many cases which are not covered by existing results. Two examples are also included to show the significance of our results.

Keywords: Neutral equation, Oscillation, Distributed deviating arguments

1. Introduction

We are concerned here with the oscillatory behavior of the second order neutral equations with distributed deviating arguments of the form

$$\left\{r(t)\Psi(x(t))\left[x(t) + \sum_{i=1}^{h} c_i(t)x(\tau_i(t))\right]'\right\}' + \int_a^b F(t, \xi, x[g(t, \xi)])d\sigma(\xi) = 0, \quad t \geq t_0. \tag{1}$$

We assume throughout this paper that the following conditions hold.

(A1) $r(t) \in C(I, R_+), \int_{t_0}^{\infty} ds/r(s) = \infty, I = [t_0, \infty), R_+ = (0, \infty)$;

(A2) $\Psi(x) \in C(R, R), 0 < \Psi(x) \leq L^{-1}$ for $x \neq 0$, L is a constant;

(A3) $c_i(t) \in C(I, R_0), i = 1, 2, \cdots, h$, and $\sum_{i=1}^{h} c_i(t) \leq 1, R_0 = [0, \infty)$;

(A4) $\tau_i(t) \in C(I, R), \tau_i(t) \leq t$ for $t \in I$ and $\lim_{t\to\infty} \tau_i(t) = \infty, i = 1, 2, \cdots, h$;

(A5) $g(t, \xi) \in C(I \times [a, b], R), g(t, \xi) \leq t$ for $\xi \in [a, b], g(t, \xi)$ is nondecreasing with respect to t and ξ, respectively, $g'(t, a) > 0$ for $t \in I$, and $\liminf_{t\to\infty, \xi\in[a,b]}\{g(t, \xi)\} = \infty$;

(A6) $F(t, \xi, x) \in C(I \times [a, b] \times R, R), \sigma(\xi) \in C([a, b], R)$ is nondecreasing, and the integral of Eq. (1) is a Stieltjes one.

We restrict our attention to solutions $x(t)$ of Eq. (1) which exist on some half-line $[t, \infty)$ and nontrivial for all large t. It is tacitly assumed that such solutions exist. As is customary, a solution $x(t)$ of Eq. (1) is called oscillatory if it has arbitrarily large zeros. Otherwise, it is said to be nonoscillatory. Eq. (1) is called oscillatory if all its solutions are oscillatory.

The oscillation problem for various particular cases of Eq. (1) such as

$$\{r(t)\Psi(x(t))[x(t) + c(t)x(\tau(t))]'\}' + q(t)f(x(\sigma(t))) = 0$$

and

$$\{r(t)[x(t) + c(t)x(t - \tau)]'\}' + \int_a^b q(t, \xi)x[g(t, \xi)]d\sigma(\xi) = 0,$$

has been studied by many authors, e.g., see(Li and Liu, 1995, pp.45-53, Ruan, 1993, pp.485-496, Şahiner,2004, pp.697-706, Wang and Li,2003, pp.407-418, Wang, 2004, pp.1935-1946, Xu and Weng, 2007, pp.460-477).

As we can see, an important tool in the study of oscillatory behavior of solutions for the equations above is the averaging technique, which involves a function class X which is defined by Philos(1989, pp.482-492) and used extensively. Say a function $H = H(t, s)$ belongs to the function class X, If $H \in C(D, R_0)$, where $D = \{(t, s) : t \geq s \geq t_0\}$, which satisfies $H(t, t) = 0, H(t, s) > 0$ for $t > s$, and has partial derivative $\partial H/\partial s$ and $\partial H/\partial t$ on D such that

$$\frac{\partial H}{\partial t} = h_1(t, s)\sqrt{H(t, s)} \qquad \text{and} \qquad \frac{\partial H}{\partial s} = -h_2(t, s)\sqrt{H(t, s)}, \tag{2}$$

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where $h_1, h_2 \in L^2_{loc}(D, R)$.

Following the ideas of Sun(2004, pp.341-351), Sun and Meng(2007, pp.1310-1316), Yang(2007, pp.900-907) and Dubé and Mingarelli(2005, pp.208-220), in this paper, we define another function class Y. We say that a function $\Phi = \Phi(t, s, l)$ belongs to the function class Y, denoted by $\Phi \in Y$, if $\Phi \in C(E, R)$, where $E = \{(t, s, l) : t \geq s \geq l \geq t_0\}$, which satisfies $\Phi(t, t, l) = \Phi(t, l, l) = 0$ and has the partial derivative $\Phi_s = \partial\Phi/\partial s$ on E such that $\Phi_s \in L^2_{loc}(E, R)$.

It is interesting to note that $H_1(t, s)H_2(s, l) \in Y$ for any $H_1, H_2 \in X$.

In Sections 2 and 3 of this paper, we will establish some new oscillation results for Eq. (1) by using the auxiliary function $\Phi \in Y$. Our results are different from most known ones in the sense that they are given in the form that $\limsup_{t\to\infty}[\cdot]$ is greater than a constant, rather than in the form $\limsup_{t\to\infty}[\cdot] = \infty$ as usual. Thus, our results can be applied to many cases, which are not covered by existing ones. Finally in Section 4, two examples that show the importance of our results are included.

2. Oscillation criteria of Kamenev type

Theorem 2.1 Suppose that there exist functions $q(t, \xi) \in C(I \times [a, b], R_0)$, which is not eventually zero on any ray $[t_\mu, \infty) \times [a, b]$ for $t_\mu \geq t_0$, and $f(x) \in C(R, R)$ such that

$$F(t, \xi, x)\mathrm{sgn}x \geq q(t, \xi) f(x)\mathrm{sgn}x \tag{3}$$

and

$$-f(-x) \geq f(x) \geq \lambda x > 0, \quad x > 0, \ \lambda \text{ is a constant.} \tag{4}$$

If there exist functions $\Phi \in Y$ and $\rho(t) \in C^1(I, R_+)$ such that for each $l \geq t_0$,

$$\limsup_{t\to\infty} \int_l^t \left\{\Phi^2(t, s, l)Q_1(s) - \frac{\rho(s)r[g(s, a)]}{Lg'(s, a)}\Theta^2(t, s, l)\right\}ds > 0, \tag{5}$$

where

$$Q_1(s) = \lambda\rho(s) \int_a^b q(s, \xi)\left\{1 - \sum_{i=1}^{h} c_i[g(s, \xi)]\right\}d\sigma(\xi), \tag{6}$$

$$\Theta(t, s, l) = \Phi_s(t, s, l) + \frac{\rho'(s)}{2\rho(s)}\Phi(t, s, l), \tag{7}$$

then Eq. (1) is oscillatory.

Proof Assume that $x(t)$ is an eventually positive solution of Eq. (1). From (A4) and (A5), there exists a $t_1 \geq t_0$ such that $x(t) > 0, x(\tau_i(t)) > 0$ and $x[g(t, \xi)] > 0$ for $t \geq t_1, \xi \in [a, b], i = 1, 2, \cdots, h$.

Letting

$$y(t) = x(t) + \sum_{i=1}^{h} c_i(t)x(\tau_i(t)), \tag{8}$$

from (A3) and (8), we have $y(t) \geq x(t) > 0$, and from (1), (3) and (4), we have $(r(t)\Psi(x(t))y'(t))' \leq 0$, for $t \geq t_1$.

Next, we show that $y'(t) \geq 0$ for $t \geq t_1$. In fact, if there exists a $t_2 \geq t_1$ with $y'(t_2) < 0$, then by $(r(t)\Psi(x(t))y'(t))' \leq 0$, we have

$$r(t)\Psi(x(t))y'(t) \leq r(t_2)\Psi(x(t_2))y'(t_2) \triangleq \mu < 0, \quad t \geq t_2.$$

Dividing both sides by $r(t)\Psi(x(t)) > 0$, from (A2), we obtain

$$y'(t) \leq \frac{\mu}{\Psi(x(t))}\frac{1}{r(t)} \leq \frac{\mu L}{r(t)}. \tag{9}$$

Integrating (9) from t_2 to t leads to

$$y(t) \leq y(t_2) + \mu L \int_{t_2}^{t} \frac{1}{r(s)}ds, \quad t \geq t_2,$$

therefore, from (A1), we conclude that $\lim_{t\to\infty} y(t) = -\infty$, this contradicts $y(t) > 0, t \geq t_1$.

From (1), (3), (4) and (8), we get

$$\begin{aligned} 0 &= (r(t)\Psi(x(t))y'(t))' + \int_a^b F(t,\xi,x[g(t,\xi)])d\sigma(\xi) \\ &\geq (r(t)\Psi(x(t))y'(t))' + \int_a^b q(t,\xi)f(x[g(t,\xi)])d\sigma(\xi) \\ &\geq (r(t)\Psi(x(t))y'(t))' + \lambda\int_a^b q(t,\xi)\Big\{y[g(t,\xi)] - \sum_{i=1}^h c_i[g(t,\xi)]x(\tau_i[g(t,\xi)])\Big\}d\sigma(\xi). \end{aligned}$$

Using $y'(t) \geq 0$, and $y(t) \geq x(t), t \geq t_1$, we have $y[g(t,\xi)] \geq y(\tau_i[g(t,\xi)]) \geq x(\tau_i[g(t,\xi)])$ for $i = 1, 2, \cdots, h$. Thus,

$$(r(t)\Psi(x(t))y'(t))' + \lambda\int_a^b q(t,\xi)\Big\{1 - \sum_{i=1}^h c_i[g(t,\xi)]\Big\}y[g(t,\xi)]d\sigma(\xi) \leq 0, \quad t \geq t_1.$$

Further, observing that $g(t,\xi)$ is nondecreasing with respect to ξ, we have $y[g(t,a)] \leq y[g(t,\xi)]$ for $t \geq t_1$ and $\xi \in [a,b]$. Thus,

$$(r(t)\Psi(x(t))y'(t))' + \lambda y[g(t,a)]\int_a^b q(t,\xi)\Big\{1 - \sum_{i=1}^h c_i[g(t,\xi)]\Big\}d\sigma(\xi) \leq 0, \quad t \geq t_1.$$

Define

$$z(t) = \rho(t)\frac{r(t)\Psi(x(t))y'(t)}{y[g(t,a)]}, \quad t \geq t_1.$$

Noting that $g(t,a) \leq t$ and $(r(t)\Psi(x(t))y'(t))' \leq 0$ for $t \geq t_1$, we have $r(t)\Psi(x(t))y'(t) \leq r[g(t,a)]\Psi(x[g(t,a)])y'[g(t,a)]$ for $t \geq t_1$. Therefore,

$$\begin{aligned} z'(t) &= \frac{\rho'(t)}{\rho(t)}z(t) + \rho(t)\frac{(r(t)\Psi(x(t))y'(t))'}{y[g(t,a)]} - \rho(t)\frac{r(t)\Psi(x(t))y'(t)y'[g(t,a)]g'(t,a)}{y^2[g(t,a)]} \\ &\leq \frac{\rho'(t)}{\rho(t)}z(t) - Q_1(t) - \frac{g'(t,a)}{\rho(t)r[g(t,a)]\Psi(x[g(t,a)])}z^2(t) \\ &\leq \frac{\rho'(t)}{\rho(t)}z(t) - Q_1(t) - \frac{Lg'(t,a)}{\rho(t)r[g(t,a)]}z^2(t), \end{aligned} \tag{10}$$

where $Q_1(t)$ is defined as in (6). Multiplying (10), with t replaced by s, by $\Phi^2(t,s,l)(t \geq l \geq t_1)$ and integrating from l to t, we obtain

$$\begin{aligned} \int_l^t \Phi^2(t,s,l)Q_1(s)ds &\leq -\int_l^t \Phi^2(t,s,l)z'(s)ds + \int_l^t \Phi^2(t,s,l)\frac{\rho'(s)}{\rho(s)}z(s)ds - \int_l^t \Phi^2(t,s,l)\frac{Lg'(s,a)}{\rho(s)r[g(s,a)]}z^2(s)ds \\ &= \int_l^t 2\Phi(t,s,l)\Theta(t,s,l)z(s)ds - \int_l^t \Phi^2(t,s,l)\frac{Lg'(s,a)}{\rho(s)r[g(s,a)]}z^2(s)ds \\ &= -\int_l^t \left\{\sqrt{\frac{Lg'(s,a)}{\rho(s)r[g(s,a)]}}\Phi(t,s,l)z(s) - \sqrt{\frac{\rho(s)r[g(s,a)]}{Lg'(s,a)}}\Theta(t,s,l)\right\}^2 ds \\ &\quad + \int_l^t \frac{\rho(s)r[g(s,a)]}{Lg'(s,a)}\Theta^2(t,s,l)ds \\ &\leq \int_l^t \frac{\rho(s)r[g(s,a)]}{Lg'(s,a)}\Theta^2(t,s,l)ds, \end{aligned}$$

i.e.,

$$\int_l^t \Phi^2(t,s,l)Q_1(s)ds \leq \int_l^t \frac{\rho(s)r[g(s,a)]}{Lg'(s,a)}\Theta^2(t,s,l)ds, \tag{11}$$

where $\Theta(t,s,l)$ is defined as in (7). This implies that

$$\limsup_{t\to\infty}\int_l^t \Big\{\Phi^2(t,s,l)Q_1(s) - \frac{\rho(s)r[g(s,a)]}{Lg'(s,a)}\Theta^2(t,s,l)\Big\}ds \leq 0,$$

which contradicts the assumption (5).

If $x(t)$ is an eventually negative solution of Eq. (1), let $w(t) = -x(t)$, then Eq. (1) will transfer the following equation

$$\Big\{r(t)\Psi(-w(t))\Big[w(t) + \sum_{i=1}^h c_i(t)w(\tau_i(t))\Big]'\Big\}' + \int_a^b F^*(t,\xi,w[g(t,\xi)])d\sigma(\xi) = 0, \quad t \geq t_0, \tag{12}$$

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in which $F^*(t,\xi,w[g(t,\xi)]) \equiv -F(t,\xi,-w[g(t,\xi)])$. It is easy to see that $w(t)$ is an eventually positive solution of Eq. (12). From (3) and (4), we can obtain

$$F^*(t,\xi,w[g(t,\xi)]) \equiv -F(t,\xi,-w[g(t,\xi)]) \geq q(t,\xi)\{-f(-w[g(t,\xi)])\} \geq q(t,\xi)f(w[g(t,\xi)]).$$

Then, Eq. (12) satisfies the conditions of Theorem 2.1. Defining $y(t) = w(t)+\sum_{i=1}^{h} c_i(t)w(\tau_i(t))$, $z(t) = \rho(t)\frac{r(t)\Psi(-w(t))y'(t)}{y[g(t,a)]}$,and using the above-mentioned method, we can also get a contradiction. This completes the proof of Theorem 2.1.

Under the appropriate choices of the functions $\Phi(t,s,l)$, we can derive many new oscillation criteria for Eq. (1) from Theorem 2.1. For instance, let $\Phi(t,s,l) = \sqrt{H(t,s)H(s,l)}$, where $H \in X$. By Theorem 2.1, we have the following oscillation result.

Theorem 2.2 Suppose that (3) and (4) hold. If there exist functions $H \in X$ and $\rho(t) \in C^1(I,R_+)$ such that for each $l \geq t_0$,

$$\limsup_{t\to\infty} \int_l^t H(t,s)H(s,l)\left\{Q_1(s) - \frac{\rho(s)r[g(s,a)]}{4Lg'(s,a)}\left[\frac{\rho'(s)}{\rho(s)} + \frac{h_1(s,l)}{\sqrt{H(s,l)}} - \frac{h_2(t,s)}{\sqrt{H(t,s)}}\right]^2\right\}ds > 0,$$

where $Q_1(t)$ is defined as in (6), and $h_1(s,l), h_2(t,s)$ are defined as in (2), then Eq. (1) is oscillatory.

If we choose $\Phi(t,s,l) = \sqrt{\phi(s)(t-s)^m(s-l)^n}$, where $\phi(t) \in C^1(I,R_+)$ and $m,n > 1$ are constants, then we have the following oscillation theorem by Theorem 2.1.

Theorem 2.3 Suppose that (3) and (4) hold. If there exist functions $\rho(t), \phi(t) \in C^1(I,R_+)$ and constants $m,n > 1$ such that for each $l \geq t_0$,

$$\limsup_{t\to\infty} \int_l^t \phi(s)(t-s)^m(s-l)^n\left\{Q_1(s) - \frac{\rho(s)r[g(s,a)]}{4Lg'(s,a)}\left[\frac{\phi'(s)}{\phi(s)} + \frac{\rho'(s)}{\rho(s)} + \frac{nt-(m+n)s+ml}{(t-s)(s-l)}\right]^2\right\}ds > 0,$$

where $Q_1(t)$ is defined as in (6), then Eq. (1) is oscillatory.

Define

$$R(t) = \int_\tau^t \frac{1}{r[g(s,a)]}ds, \quad t \geq \tau \geq t_0,$$

and let

$$\Phi(t,s,l) = \sqrt{\phi(s)[R(t)-R(s)]^m[R(s)-R(l)]^n},$$

where $\phi(t) \in C^1(I,R_+)$, and $m,n > 1$ are constants. According to the simple computation, we get the following oscillation criterion by Theorem 2.1.

Theorem 2.4 Suppose that (3) and (4) hold. If there exist functions $\rho(t), \phi(t) \in C^1(I,R_+)$ and constants $m,n > 1$ such that for each $l \geq t_0$,

$$\limsup_{t\to\infty} \int_l^t \phi(s)[R(t)-R(s)]^m[R(s)-R(l)]^n\left\{Q_1(s)-\frac{\rho(s)r[g(s,a)]}{4Lg'(s,a)}\left[\frac{\phi'(s)}{\phi(s)}+\frac{\rho'(s)}{\rho(s)}+\frac{nR(t)-(m+n)R(s)+mR(l)}{r[g(s,a)][R(t)-R(s)][R(s)-R(l)]}\right]^2\right\}ds > 0,$$

where $Q_1(t)$ is defined as in (6), then Eq. (1) is oscillatory.

Taking $\rho(t) \equiv 1$ and $\phi(t) \equiv 1$, by Theorem 2.4, we have the following interesting theorem.

Theorem 2.5 Suppose that (3) and (4) hold, $\lim_{t\to\infty} R(t) = \infty$ and $g'(t,a) \geq k > 0$ for $t \in I$, where k is a constant. If there exist constants $m,n > 1$ such that for each $l \geq t_0$,

$$\limsup_{t\to\infty} \frac{1}{R^{m+n-1}(t)} \int_l^t [R(t)-R(s)]^m[R(s)-R(l)]^n kLQ_2(s)ds > mn(m+n-2)\frac{\Gamma(m-1)\Gamma(n-1)}{4\Gamma(m+n)}, \tag{13}$$

where

$$Q_2(s) = \lambda \int_a^b q(s,\xi)\left\{1 - \sum_{i=1}^{h} c_i[g(s,\xi)]\right\}d\sigma(\xi), \tag{14}$$

then Eq. (1) is oscillatory.

Proof Assume that Eq. (1) has a nonoscillatory solution $x(t) > 0$. By using the same arguments as in the proof of Theorem 2.1, we conclude that (11) with $\rho(t) \equiv 1$ is satisfied, i.e.,

$$\int_l^t \Phi^2(t,s,l)Q_2(s)ds \leq \int_l^t \frac{r[g(s,a)]}{Lg'(s,a)}\Theta^2(t,s,l)ds.$$

Letting $\Phi(t,s,l) = \sqrt{[R(t)-R(s)]^m[R(s)-R(l)]^n}$, we have that for $t \geq l \geq t_1$,

$$\int_l^t [R(t)-R(s)]^m[R(s)-R(l)]^n Q_2(s)ds \leq \int_l^t [R(t)-R(s)]^{m-2}[R(s)-R(l)]^{n-2}\frac{[nR(t)-(m+n)R(s)+mR(l)]^2}{4Lg'(s,a)r[g(s,a)]}ds.$$

Since $g'(t,a) \geq k > 0$ for $t \in I$, we have

$$\begin{aligned}&\int_l^t [R(t)-R(s)]^m[R(s)-R(l)]^n kLQ_2(s)ds\\ &\leq \frac{1}{4}\int_l^t [R(t)-R(s)]^{m-2}[R(s)-R(l)]^{n-2}\{n[R(t)-R(s)]-m[R(s)-R(l)]\}^2 dR(s).\end{aligned} \tag{15}$$

By setting $u = R(s)-R(l)$ and $v = R(t)-R(l)$, we get

$$\begin{aligned}&\int_l^t [R(t)-R(s)]^{m-2}[R(s)-R(l)]^{n-2}\{n[R(t)-R(s)]-m[R(s)-R(l)]\}^2 dR(s)\\ &= \int_0^{R(t)-R(l)} u^{n-2}[R(t)-R(l)-u]^{m-2}\{n[R(t)-R(l)-u]-mu\}^2\,du\\ &= \int_0^v u^{n-2}(v-u)^{m-2}[n(v-u)-mu]^2\,du\\ &= n^2\int_0^v u^{n-2}(v-u)^m du - 2mn\int_0^v u^{n-1}(v-u)^{m-1}du + m^2\int_0^v u^n(v-u)^{m-2}du.\end{aligned} \tag{16}$$

Letting $x = \frac{u}{v}$, and using the following Euler's Beta function,

$$\int_0^1 x^{\beta-1}(1-x)^{\gamma-1}dx = \frac{\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\beta+\gamma)}, \quad Re(\beta,\gamma) > 0,$$

we obtain that

$$\begin{aligned}&\int_0^v u^{n-2}(v-u)^{m-2}[n(v-u)-mu]^2\,du\\ &= n^2v^{m+n-1}\frac{\Gamma(n-1)\Gamma(m+1)}{\Gamma(m+n)} - 2mnv^{m+n-1}\frac{\Gamma(n)\Gamma(m)}{\Gamma(m+n)} + m^2v^{m+n-1}\frac{\Gamma(n+1)\Gamma(m-1)}{\Gamma(m+n)}\\ &= v^{m+n-1}\frac{\Gamma(n-1)\Gamma(m-1)}{\Gamma(m+n)}[n^2m(m-1)-2mn(n-1)(m-1)+m^2n(n-1)]\\ &= mn(m+n-2)\frac{\Gamma(m-1)\Gamma(n-1)}{\Gamma(m+n)}v^{m+n-1}.\end{aligned} \tag{17}$$

Substituting back in for $v = R(t)-R(l)$, (16) and (17) give

$$\begin{aligned}&\int_l^t [R(t)-R(s)]^{m-2}[R(s)-R(l)]^{n-2}\{n[R(t)-R(s)]-m[R(s)-R(l)]\}^2\,dR(s)\\ &= mn(m+n-2)\frac{\Gamma(m-1)\Gamma(n-1)}{\Gamma(m+n)}[R(t)-R(l)]^{m+n-1}.\end{aligned} \tag{18}$$

From (15) and (18), we can easily obtain

$$\limsup_{t\to\infty}\frac{1}{R^{m+n-1}(t)}\int_l^t [R(t)-R(s)]^m[R(s)-R(l)]^n kLQ_2(s)ds \leq mn(m+n-2)\frac{\Gamma(m-1)\Gamma(n-1)}{4\Gamma(m+n)},$$

which contradicts the assumption (13) . This completes the proof of Theorem 2.5.

Based on Theorem 2.5 we obtain the following corollary.

Corollary 2.1 Suppose that (3) and (4) hold, $\lim_{t\to\infty} R(t) = \infty$ and $g'(t,a) \geq k > 0$ for $t \in I$, where k is a constant. If there exists a constant $\alpha > \frac{1}{2}$ such that for each $l \geq t_0$ either

$$\text{(i)}\quad \limsup_{t\to\infty}\frac{1}{R^{2\alpha+1}(t)}\int_l^t [R(t)-R(s)]^{2\alpha}[R(s)-R(l)]^2 kLQ_2(s)ds > \frac{\alpha}{(2\alpha-1)(2\alpha+1)}$$

or

$$\text{(ii)}\quad \limsup_{t\to\infty}\frac{1}{R^{2\alpha+1}(t)}\int_l^t [R(t)-R(s)]^2[R(s)-R(l)]^{2\alpha} kLQ_2(s)ds > \frac{\alpha}{(2\alpha-1)(2\alpha+1)}, \tag{19}$$

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where $Q_2(t)$ is defined as in (14), then Eq. (1) is oscillatory.

Proof (i) In (13), replaced m, n by 2α and 2, respectively, we obtain

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{R^{2\alpha+1}(t)}\int_l^t [R(t)-R(s)]^{2\alpha}[R(s)-R(l)]^2 kLQ_2(s)ds\\ &> (2\alpha)2(2\alpha+2-2)\frac{\Gamma(2\alpha-1)\Gamma(2-1)}{4\Gamma(2\alpha+2)}\\ &= \frac{2\alpha^2\Gamma(2\alpha-1)}{(2\alpha+1)(2\alpha)(2\alpha-1)\Gamma(2\alpha-1)} = \frac{\alpha}{(2\alpha-1)(2\alpha+1)}.\end{aligned}$$

(ii) In (13), replaced m, n by 2 and 2α, respectively, the remainder of the proof is similar to that of (i) and hence omitted.

3. Interval oscillation criteria

In this section, we will establish several new interval oscillation criteria for Eq.(1), that is, criteria given by the behavior of Eq.(1) only on a sequence of subintervals of $[t_0, \infty)$ rather than on the whole half-line.

Theorem 3.1 Suppose that (3) and (4) hold. If for each $l \geq t_0$, there exist functions $\Phi \in Y, \rho(t) \in C^1(I, R_+)$ and two constants $d > c \geq l$ such that

$$\int_c^d \left\{\Phi^2(d,s,c)Q_1(s) - \frac{\rho(s)r[g(s,a)]}{Lg'(s,a)}\Theta^2(d,s,c)\right\}ds > 0, \tag{20}$$

where $Q_1(t)$ and $\Theta(d, s, c)$ are respectively defined as in (6) and (7), then Eq. (1) is oscillatory.

Proof As in the proof of Theorem 2.1, with t and l replaced by d and c, respectively. We can easily see that every solution of Eq. (1) has at least one zero in (c, d), i.e., every solution of Eq. (1) has arbitrarily large zero on $[t_0, \infty)$. This completes the proof of Theorem 3.1.

As consequences of Theorem 3.1 we get the following interval oscillation criteria for Eq. (1).

Corollary 3.1 Suppose that (3) and (4) hold. If for each $l \geq t_0$, there exist functions $H \in X, \rho(t) \in C^1(I, R_+)$ and two constants $d > c \geq l$ such that

$$\int_c^d H(d,s)H(s,c)\left\{Q_1(s) - \frac{\rho(s)r[g(s,a)]}{4Lg'(s,a)}\left[\frac{\rho'(s)}{\rho(s)} + \frac{h_1(s,c)}{\sqrt{H(s,c)}} - \frac{h_2(d,s)}{\sqrt{H(d,s)}}\right]^2\right\}ds > 0,$$

where $Q_1(t)$ is defined as in (6), and $h_1(s, c), h_2(d, s)$ are defined as in (2), then Eq. (1) is oscillatory.

Corollary 3.2 Suppose that (3) and (4) hold. If for each $l \geq t_0$, there exist functions $\rho(t), \phi(t) \in C^1(I, R_+)$, two constants $m, n > 1$, and two constants $d > c \geq l$ such that

$$\int_c^d \phi(s)(d-s)^m(s-c)^n\left\{Q_1(s) - \frac{\rho(s)r[g(s,a)]}{4Lg'(s,a)}\left[\frac{\phi'(s)}{\phi(s)} + \frac{\rho'(s)}{\rho(s)} + \frac{nd-(m+n)s+mc}{(d-s)(s-c)}\right]^2\right\}ds > 0,$$

where $Q_1(t)$ is defined as in (6), then Eq. (1) is oscillatory.

Corollary 3.3 Suppose that (3) and (4) hold. If for each $l \geq t_0$, there exist functions $\rho(t), \phi(t) \in C^1(I, R_+)$, two constants $m, n > 1$, and two constants $d > c \geq l$ such that

$$\int_c^d \phi(s)[R(d)-R(s)]^m[R(s)-R(c)]^n\left\{Q_1(s) - \frac{\rho(s)r[g(s,a)]}{4Lg'(s,a)}\left[\frac{\phi'(s)}{\phi(s)} + \frac{\rho'(s)}{\rho(s)} + \frac{nR(d)-(m+n)R(s)+mR(c)}{r[g(s,a)][R(d)-R(s)][R(s)-R(c)]}\right]^2\right\}ds > 0,$$

where $Q_1(t)$ is defined as in (6), then Eq. (1) is oscillatory.

Remark 3.1 Theorems 2.1-2.5 and 3.1, Corollaries 2.1 and 3.1-3.3 are new because we introduce a new class of kernel functions $\Phi(t, s, l)$ which is basically a product $H(t, s)H(s, l)$ for a kernel $H(t, s)$ of Philos' type.

Remark 3.2 Since the integral of Eq. (1) is a Stieltjes integral, the criteria in this paper are adapted to the following equation:

$$\left\{r(t)\Psi(x(t))\left[x(t) + \sum_{i=1}^h c_i(t)x(\tau_i(t))\right]'\right\}' + \sum_{j=1}^k f_j(t, x[g_j(t)]) = 0, \quad t \geq t_0.$$

4. Examples

In this section, we will present two examples to illustrate our results. To the best of our knowledge, no previous criteria for oscillation can be applied to these examples.

We first give an example to illustrate Corollary 2.1.

Example 4.1 Consider the equation

$$\left\{\frac{1}{t(1+e^{-|x(t)|})}\left[x(t)+\frac{1}{t+1}x(\delta_1 t)+\frac{1}{t+2}x(\delta_2 t)\right]'\right\}' + \int_{-1}^{0}\frac{\theta(t+\xi+1)(t+\xi+2)}{t^3[(t+\xi)(t+\xi+1)-1]\sin(\xi+2)}x(t+\xi)e^{x^2(t+\xi)}d\xi = 0, \quad t \geq 2, \tag{21}$$

where $0 < \delta_1, \delta_2 \leq 1$ and $\theta > 1$ are constants, $a = -1,\ b = 0,\ r(t) = \frac{1}{t},\ \Psi(x) = \frac{1}{1+e^{-|x|}},\ c_1(t) = \frac{1}{t+1},\ c_2(t) = \frac{1}{t+2},\ g(t,\xi) = t+\xi,\ F(t,\xi,x) = \frac{\theta(t+\xi+1)(t+\xi+2)}{t^3[(t+\xi)(t+\xi+1)-1]\sin(\xi+2)}xe^{x^2}$.

Noting that

$$\frac{\theta(t+\xi+1)(t+\xi+2)}{t^3[(t+\xi)(t+\xi+1)-1]\sin(\xi+2)}xe^{x^2}\text{sgn}x \geq \frac{\theta(t+\xi+1)(t+\xi+2)}{t^3[(t+\xi)(t+\xi+1)-1]}xe^{x^2}\text{sgn}x, \quad t \geq 2, \ -1 \leq \xi \leq 0,$$

choosing $q(t,\xi) = \frac{\theta(t+\xi+1)(t+\xi+2)}{t^3[(t+\xi)(t+\xi+1)-1]}, f(x) = xe^{x^2}$, then the conditions in Corollary 2.1 hold for $L = k = \lambda = 1$, and $R(t) = \int_\tau^t (s-1)ds = \frac{1}{2}(t-1)^2 - \frac{1}{2}(\tau-1)^2,\ t \geq \tau \geq 2, \lim_{t\to\infty} R(t) = \infty$.

$$Q_2(t) = \int_{-1}^{0}\frac{\theta(t+\xi+1)(t+\xi+2)}{t^3[(t+\xi)(t+\xi+1)-1]}\left(1-\frac{1}{t+\xi+1}-\frac{1}{t+\xi+2}\right)d\xi = \frac{\theta}{t^3}.$$

For any $l \geq 2$,

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{R^{2\alpha+1}(t)}\int_l^t [R(t)-R(s)]^2[R(s)-R(l)]^{2\alpha}kLQ_2(s)ds \\
&= \theta\lim_{t\to\infty}\frac{\int_l^t [R(s)-R(l)]^{2\alpha}\frac{1}{s^3}ds}{R^{2\alpha-1}(t)} - 2\theta\lim_{t\to\infty}\frac{\int_l^t R(s)[R(s)-R(l)]^{2\alpha}\frac{1}{s^3}ds}{R^{2\alpha}(t)} + \theta\lim_{t\to\infty}\frac{\int_l^t R^2(s)[R(s)-R(l)]^{2\alpha}\frac{1}{s^3}ds}{R^{2\alpha+1}(t)} \\
&= \theta\lim_{t\to\infty}\frac{[R(t)-R(l)]^{2\alpha}}{(2\alpha-1)R^{2\alpha-2}(t)t^3(t-1)} - \theta\lim_{t\to\infty}\frac{[R(t)-R(l)]^{2\alpha}}{\alpha R^{2\alpha-2}(t)t^3(t-1)} + \theta\lim_{t\to\infty}\frac{[R(t)-R(l)]^{2\alpha}}{(2\alpha+1)R^{2\alpha-2}(t)t^3(t-1)} \\
&= \frac{\theta}{4}\frac{1}{\alpha(2\alpha-1)(2\alpha+1)}.
\end{aligned}$$

For any $\theta > 1$, there exists $\alpha > \frac{1}{2}$, such that $\frac{\theta}{4} > \alpha^2$, i.e.,

$$\frac{\theta}{4}\frac{1}{\alpha(2\alpha-1)(2\alpha+1)} > \frac{\alpha}{(2\alpha-1)(2\alpha+1)}.$$

Thus, (19) holds for $\theta > 1$. By Corollary 2.1 we see that Eq.(21) is oscillatory for $\theta > 1$.

The second example illustrates Theorem 3.1.

Example 4.2 Consider the equation

$$\left\{\frac{1}{e^{2t}(1+x^2(t))}\left[x(t)+(1-e^{-t})x(t-\tau_1)+e^{-t-1}x(t-\tau_2)\right]'\right\}' + \int_1^2\frac{\eta te^{\frac{1}{2}\xi t-t+1}}{e-1}x(\frac{1}{2}\xi t)\left[1+x^{\frac{2}{3}}(\frac{1}{2}\xi t)\right]\arctan\xi d\xi = 0, \quad t \geq 1, \tag{22}$$

where $\tau_1, \tau_2 \geq 0$ and $\eta > \frac{10}{\pi}$ are constants, $a = 1,\ b = 2,\ r(t) = \frac{1}{e^{2t}},\ \Psi(x) = \frac{1}{1+x^2},\ c_1(t) = 1-e^{-t},\ c_2(t) = e^{-t-1},\ g(t,\xi) = \frac{1}{2}\xi t,\ F(t,\xi,x) = \frac{\eta te^{\frac{1}{2}\xi t-t+1}}{e-1}x(1+x^{\frac{2}{3}})\arctan\xi$.

Choosing $q(t,\xi) = \frac{\pi}{4}\frac{\eta e^{\frac{1}{2}\xi t-t+1}}{e-1},\ f(x) = x(1+x^{\frac{2}{3}})$ and $\rho(t) = e^t$, then the conditions in Theorem 3.1 hold for $L = \lambda = 1$.

$$Q_1(t) = e^t\int_1^2\frac{\pi}{4}\frac{\eta e^{\frac{1}{2}\xi t-t+1}}{e-1}\left[1-(1-e^{-\frac{1}{2}\xi t}+e^{-\frac{1}{2}\xi t-1})\right]d\xi = \frac{\pi}{4}\eta, \quad \Theta(t,s,l) = \Phi_s(t,s,l)+\frac{1}{2}\Phi(t,s,l).$$

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For any $l \geq 1$, there exists $K \in \{0, 1, 2, \cdots\}$ such that $2K\pi \geq l$. Let $c = 2K\pi$, $d = 2K\pi + \pi$, and $\Phi(d, s, c) = \sqrt{\sin(d-s)\sin(s-c)} = \sin s$ for $c \leq s \leq d$, then we have

$$\begin{aligned}
&\int_c^d \left\{\Phi^2(d, s, c)Q_1(s) - \frac{\rho(s)r[g(s, a)]}{Lg'(s, a)}\Theta^2(d, s, c)\right\}ds \\
&= \int_{2K\pi}^{2K\pi+\pi} \left[\frac{\pi}{4}\eta\sin^2 s - 2(\cos s + \frac{1}{2}\sin s)^2\right]ds \\
&= \int_0^{\pi} \left[\left(\frac{\pi}{8}\eta - \frac{5}{4}\right) - \left(\frac{\pi}{8}\eta + \frac{3}{4}\right)\cos 2s - \sin 2s\right]ds \\
&= \left(\frac{\pi}{8}\eta - \frac{5}{4}\right)\pi > 0,
\end{aligned}$$

for $\eta > \frac{10}{\pi}$. This means that (20) holds. Therefore, by Theorem 3.1, Eq.(22) is oscillatory for $\eta > \frac{10}{\pi}$.

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