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Non-wandering Operator in Bargmann Space

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Abstract

In this paper the Bargmann space is denoted by F. This space's roots can be found in mathematical problem of relativistic physics or in quantum optics. In physics the Bargmann space contains the canonical coherent states, so it is the main tool for studying the bosonic coherent state theory of radiation field and for other application .This paper deals with the unilateral backward shift operator T on a Bargmann space F. We provide a sufficient condition for an unbounded operator to be non-wandering operator, and then apply the condition to give a necessary and sufficient condition in order that T be a non-wandering operator.

Keywords: Non-wandering operator, Unbounded operator, Bargmann space, Hypercyclic operator

1. Introduction

It is well known that linear operators in finite-dimensional linear spaces can't be chaotic but the nonlinear operator may be. Only in infinite-dimensional linear spaces can linear operators have chaotic properties. This has attracted widely attention (Godefroy G, 1991; Lixin Tian, 2005; Jiangbo Zhou, 2001; Shaoguang Shi, 2006; Lixin Tian, 2006). Lixin Tian and other researchers introduced non-wandering operators in infinite-dimensional Banach space, which are the generalization of Axiom A dynamic system but different from it. They are new linear chaotic operators and relative to hypercyclic operators, but different from them (Lixin Tian, 2005). In recent years, Jiangbo Zhou discussed the hereditayily hypercyclic decomposition of non-wandering operators in infinite dimensional Frechet space (Jiangbo Zhou, 2001); Shaoguang Shi obtained non-wandering operator sequences on Banach space (Shaoguang Shi, 2006) ; Lihong Ren studied n-multiple non-wandering operator (Lixin Tian, 2006); Minggang Wang studied the pseudo orbit tracing property of invertible non-wandering operator (Lixin Tian, 2007) and Non-wandering Property of Differentiation Operator(Minggang Wang, 2008).

In this paper the Bargmann space is denoted by F . This space have been studied by many authors(I.E. Segal, 1963; J.R. Klauder, 1968; J.R. Klauder, 1985; H.Emamirad, 1997). Bargmann space's roots can be found in mathematical problem of relativistic physics (I.E. Segal, 1963) or in quantum optics (J.R. Klauder, 1968). In physics the Bargmann space contains the canonical coherent states, so it is the main tool for studying the bosonic coherent state theory of radiation field (J.R. Klauder, 1985) and for other application (H.Emamirad, 1997).

In finite-dimensional separable Banach space, for the bounded linear operators, Lixin Tian and other researchers have given the definition of non-wandering operator (Lixin Tian, 2005). However, this definition is restricted for the bounded linear operators. In this paper, we consider the non-wandering property of the unbounded operators. Let Tbe an unbounded operator on a separable infinite dimensional Banach space X. It may happen that a vector $x$is in the domain of T, but Txfails to be in the domain of T. For this reason, in order to consider the non-wandering property of the unbounded operator, we should firstly suppose that if x in the domain of T then for every integer $n \geq 1$the vector $T^n x$is in the domain of T.

On the basis of the above research, in this paper, we first provide a sufficient condition for an unbounded operator to be non-wandering operator(see Theorem 1), and then apply the condition to give a necessary and sufficient condition in order that T be a non-wandering operator. (see Theorem 2)

2. Basic notation and definitions

Definition 2.1(Lixin Tian, 2005) Let$(X, \|\cdot\|)$ be an infinite dimensional separable *Banach*space. Suppose $T \in L(X)$

(1) Assume that there exists a closed subspace $E \subset X$, which has hyperbolic structure:$E = E^u \oplus E^s$,$TE^u = E^u, TE^s = E^s$,where $E^u, E^s$are closed subspaces. In addition, there exists constants $\tau\,(0 < \tau < 1)$ and $C > 0$, such that for any $\xi \in E^u, k \in N, \left\|T^k \xi\right\| \geq C\tau^{-k} \|\xi\|$, and for any $\eta \in E^s, k \in N, \left\|T^k \eta\right\| \leq C\tau^k \|\eta\|$;

(2) Assume also that $Per(T)$ is dense in$E$. Then $T$is said to be a non-wandering operator relative to$E$.

definition 2.2 Suppose $T \in L(X)$ and $\{e_i\}_1^\infty$ is a basis in$X$, then T is called a unilateral backward shift operator relative to $\{e_i\}_1^\infty$ if $Te_n = e_{n-1}\ (n > 1)$ and $Te_1 = 0$.

3. Main results

Theorem 1 Let$(X, \|\cdot\|)$ be an infinite dimensional separable *Banach*space.$T$is an unbounded operator, if for $\forall n \geq 1$, $T^n$is the closed operator and Tsatisfy (1) there exists a closed subspace $E \subset X$, which has hyperbolic structure; (2) $Per(T)$ is dense in$E$. Then $T$is a non-wandering operator relative to$E$.

Proof By the Closed Graph Theorem, we can easily obtain this result.

Remark In fact, from Theorem 1, if an unbounded operator T has non-wandering property, then T need to satisfy: (1) $T^n$is the closed operator, $\forall n \geq 1$; (2) there exists a closed subspace $E \subset X$, which has hyperbolic structure relative to T; (3) $\overline{Per(T)} = E$

In the following, we will apply Theorem 1 to the unilateral backward shift operator T on a Bargmann space F.

3.1 Non-wandering operator in Bargmann space

Let $\{w_n\}_{n \in N}$ be an arbitrary weight sequence, we define the iterated unbounded back-ward shift T^nin Bargmann space by

$$T^n\left(\sum_{k\geq 0} C_k \frac{x^k}{\sqrt{k!}}\right) = \sum_{k\geq 0}\left(\prod_{j=k}^{n+k-1} w_j\right) C_{n+k}\frac{x^k}{\sqrt{k!}}$$

with its domain in F, and we define

$$D(T^n) = \left\{ f(x) = \sum_{k\geq 0} C_k \frac{x^k}{\sqrt{k!}} \left| \sum_{k\geq 0} |C_k|^2 < \infty; \sum_{k\geq 0} \left|\prod_{j=k}^{m+k-1} w_j\right|^2 |C_{k+m}|^2 < \infty \right.\right\}$$

for all $m \in N, 1 \leq m \leq n$.

Theorem 2 A linear unbounded backward shift operator $T{:}F \to F$ is non-wandering operator if the positive series

$$\sum_{n=0}^{\infty}\prod_{j=0}^{n-1}\frac{1}{|w_j|^2}$$

converges.

Proof From Theorem 1, we need three steps to proof the theorem:

Firstly, we proof that for $\forall n \in N$, $T^n$is closed.

By the definition, we choose $\{f_j\} \in D(T^n)$,since F is a Hilber space, then $\{f_j\} \to f_0$ in F, so $f_j(x) = \sum\limits_{k\geq 0} C_{k,j}\frac{x^k}{\sqrt{k!}} \quad \to$ $f_0(x) = \sum\limits_{k\geq 0} C_k^0 \frac{x^k}{\sqrt{k!}} \quad j \to \infty$

$$\Rightarrow \left(\prod_{j=k}^{n+k-1} w_j\right) C_{n+k,j} \to \left(\prod_{j=k}^{n+k-1} w_j\right) C_{n+k}^0,$$

Let $T^n f_j \to g_0$, then $g_0(x) = \sum\limits_{k\geq 0} \xi_k^0 \frac{x^k}{\sqrt{k!}}$

We can conclude that

$$\xi_k^0 = \left(\prod_{j=k}^{n+k-1} w_j\right) C_{n+k}^0$$

This proves that $f_0 \in D(T^n)$ and $T^n f_0 = g_0$.

(2)Since the positive series $\sum\limits_{n=0}^{\infty}\prod\limits_{j=0}^{n-1}\frac{1}{|w_j|^2}$ converges, then $\prod\limits_{j=0}^{n-1}\frac{1}{|w_j|^2} \to 0$, that is, $\prod\limits_{j=0}^{n} w_j \to \infty \quad n \to \infty$

We choose $w_j$is an increasing sequence.

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For $\forall \lambda \in C$, let$f_\lambda(x) = \sum_{k\geq 0}\left(\prod_{j=0}^{k-1} \frac{\lambda}{w_j}\right)\frac{x^k}{\sqrt{k!}}$, then for $\forall 0 < \upsilon < 1$ and $n \in N$large enough, we have$|\lambda| \leq \upsilon |w_n|$, so

$$\sum_{k\geq 0}\prod_{j=0}^{k-1}\left|\frac{\lambda}{w_j}\right|^2 = \sum_{k=0}^{n}\prod_{j=0}^{k-1}\left|\frac{\lambda}{w_j}\right|^2 + \sum_{k=n+1}^{\infty}\prod_{j=0}^{k-1}\left|\frac{\lambda}{w_j}\right|^2$$

$$\leq \sum_{k=0}^{n}\prod_{j=0}^{k-1}\left|\frac{\lambda}{w_j}\right|^2 + \left(\frac{1}{1-\upsilon}\right)\prod_{j=0}^{n}\left|\frac{\lambda}{w_j}\right|^2 < \infty$$

Thus we get $f_\lambda(x) \in F$ and

$$Tf_\lambda(x) = \sum_{k\geq 0} w_k\left(\prod_{j=0}^{k} \frac{\lambda}{w_j}\right)\frac{x^k}{\sqrt{k!}} = \lambda\sum_{k\geq 0}\left(\prod_{j=0}^{k-1} \frac{\lambda}{w_j}\right)\frac{x^k}{\sqrt{k!}} = \lambda f_\lambda(x)$$

Therefore, $f_\lambda(x)$is the eigenvector corresponding to the eigenvalue λ, furthermore, by the arbitrarily of λ, we can construct the sets:

$$V_1 = \{\lambda : \quad |\lambda| > 1 \quad \lambda \in \delta_P(T)\}$$

$V_2 = \{\lambda : \quad 0 < |\lambda| < 1 \quad \lambda \in \delta_P(T)\}$, where $\delta_p(T)$ is the spectrum of T.

Let $E^u = \overline{span\ \{f_\lambda; \quad \lambda \in V_1\}}$ for $\forall \xi \in E^u$, then $\xi = \sum_{i=1}^{\infty}\alpha_i f_{\lambda_i} = \sum_{i=1}^{\infty}\alpha_i\sum_{k\geq 0}\left(\prod_{j=0}^{k-1}\frac{\lambda_i}{w_j}\right)\frac{x^k}{\sqrt{k!}}$ and for $\forall k \in N$, we have

$$\left\|T^k(\xi)\right\| = \left\|T^k\left(\sum_{i=1}^{\infty}\alpha_i f_{\lambda_i}\right)\right\| = \left\|\sum_{i=1}^{\infty}\alpha_i\lambda_i^k\sum_{k\geq 0}\left(\prod_{j=0}^{k-1}\frac{\lambda_i}{w_j}\right)\frac{x^k}{\sqrt{k!}}\right\|$$

$$\geq \mu^k\left|\sum_{i=1}^{\infty}\sum_{k\geq 0}\left(\prod_{j=0}^{k-1}\frac{\lambda_i}{w_j}\right)\frac{x^k}{\sqrt{k!}}\right| = \mu^k|\xi| \tag{1}$$

where $\mu = \min\ \{|\lambda_i| \quad \lambda_i \in V_1\} > 1$. Let $\tau = \frac{1}{\mu}$, then we can easily get $0 < \tau < 1$. So by (1) we have $\left\|T^k(\xi)\right\| = \left\|T^k\left(\sum_{i=1}^{\infty}\alpha_i f_{\lambda_i}\right)\right\| = \left\|\sum_{i=1}^{\infty}\alpha_i\lambda_i^k f_{\lambda_i}\right\| \geq \tau^{-k}|\xi|$.

Next, we will prove E^u is the invariant subspace of T.

Since for $\forall \xi \in E^u$, then $\xi = \sum_{i=1}^{\infty}\alpha_i f_{\lambda_i} = T\sum_{i=1}^{\infty}\frac{\alpha_i}{\lambda_i}f_{\lambda_i} = TE^u \Rightarrow \xi \in TE^u$, so $E^u \subset TE^u$. In the other hand, for $\forall \eta \in TE^u$, then there exists $\varphi \in E^u \Rightarrow \varphi = \sum_{i=1}^{\infty}\beta_i f_{\lambda_i}$, such that, $\eta = T\varphi = \sum_{i=1}^{\infty}\lambda_i\beta_i f_{\lambda_i} \in E^u$. So we can get $TE^u \subset E^u$, therefore, $TE^u = E^u$.

Similarly, let $E^s = \overline{span\ \{f_\lambda; \quad \lambda \in V_2\}}$, then $TE^s = E^s$, for $\forall \eta \in E^s$,we have $\eta = \sum_{i=1}^{\infty}\beta_i f_{\lambda_i} = \sum_{i=1}^{\infty}\beta_i\sum_{k\geq 0}\left(\prod_{j=0}^{k-1}\frac{\lambda}{w_j}\right)\frac{x^k}{\sqrt{k!}}$, so for $\forall k \in N$,$\left\|T^k(\eta)\right\| = \left\|T^k\left(\sum_{i=1}^{\infty}\beta_i f_{\lambda_i}\right)\right\| = \left\|\sum_{i=1}^{\infty}\beta_i\lambda_i^k f_{\lambda_i}\right\| \leq \tau^k|\eta|$where $0 < \tau = \max\ \{|\lambda_i| \quad \lambda_i \in V_2\} < 1$and $TE^s = E^s$

Let $E = E^u \oplus E^s$, then we can easily get E has hyperbolic structure.

(3) From the definition, we have $T^n\left(\sum_{k\geq 0}C_k\frac{x^k}{\sqrt{k!}}\right) = \sum_{k\geq 0}\left(\prod_{j=k}^{n+k-1}w_j\right)C_{n+k}\frac{x^k}{\sqrt{k!}}$, if T has the $N-$periodic point, then we have $\left(\prod_{j=k}^{n+k-1}w_j\right)C_{n+k} = C_k, \forall k \geq 0$, so for $\forall l = 0, 1\cdots n-1$, $k \geq 1$, we have $C_{kn+l} = \left(\prod_{j=l}^{kn+l-1}\frac{1}{w_j}\right)C_l$, thus for $\forall \upsilon \geq 0 \quad n \geq \upsilon$,$n \in N$, we can construct

$$g_{v,n}(x) = \frac{x^v}{\sqrt{v!}} + \sum_{k=1}^{\infty}\left(\prod_{j=v}^{kn+v-1}\frac{1}{w_j}\right)\frac{x^{kn+v}}{\sqrt{(kn+v)!}} \quad .$$

Since the series $\sum_{n=0}^{\infty}\prod_{j=0}^{n-1}\frac{1}{|w_j|^2}$ converges, then we have $\sum_{k=1}^{\infty}\prod_{j=v}^{kn+v-1}\frac{1}{w_j} < \infty$, so, we get $g_{v,n} \in F$ and

$$T^N g_{v,n}(x) = T^N\left(\frac{x^v}{\sqrt{v!}} + \sum_{k=1}^{\infty}\left(\prod_{j=v}^{kn+v-1}\frac{1}{w_j}\right)\frac{x^{kn+v}}{\sqrt{(kn+v)!}}\right) = \frac{x^v}{\sqrt{v!}} + \sum_{k=1}^{\infty}\left(\prod_{j=v}^{kn+v-1}\frac{1}{w_j}\right)\frac{x^{kn+v}}{\sqrt{(kn+v)!}}.$$

Thus $g_{v,n}(x)$ is the $N-$periodic point of T.

Let $E_0 = span\{g_{v,n}(x)\}$, in the following we will prove that E_0 is dense on F.

Since for $\forall f(x) \in F$, let $f(x) = \sum\limits_{v=0}^{m} C_v \frac{x^v}{\sqrt{v!}}$, by the definition of Bargmann space, we have

$$|C_v \prod_{j=0}^{v-1} w_j| < \infty.$$

Suppose $|C_v \prod_{j=0}^{v-1} w_j| < 1$, then there exists $g(x) \in E_0$ and

$$g(x) = \sum_{v=0}^{m} C_v g_{v},_n(x)$$

so that

$$\|g - f\| = \|\sum_{v=0}^{m} C_v(g_{v},_n(x) - \frac{x^v}{\sqrt{v!}})\| = \|\sum_{v=0}^{m}(C_v \prod_{j=0}^{v-1} w_j)\sum_{k=1}^{\infty}(\prod_{j=0}^{kn+v-1}\frac{1}{w_j}\frac{x^k v + n}{\sqrt{(kn+v)!}}\|$$

$$\leq \sum_{v=0}^{m}\|\sum_{k=1}^{\infty}(\prod_{j=0}^{kn+v-1}\frac{1}{w_j}\frac{x^k v + n}{\sqrt{(kn+v)!}}\| \tag{2}$$

Furthermore, from the series $\sum\limits_{n=0}^{\infty}\prod\limits_{j=0}^{n-1}\frac{1}{|w_j|^2}$ converges, the there exists $n \geq m$ such that for $\forall \varepsilon > 0$, we have

$$\sum_{k \geq n+1}(\prod_{j=0}^{k}\frac{1}{w_j}\varepsilon_k\frac{x^k}{\sqrt{k!}}\| < \frac{\varepsilon}{m+1}$$

where ε_k taking values 0 or 1 , so we can get when $n \geq m$, then (2) $< \varepsilon$.

Therefore, E_0 is dense on F.

Corollary 1 The operator of differentiation $D: \ f \ \rightarrow \ f'$ defined on

$$\delta = \left\{f \in F \left| f' \in F\right.\right\}$$

is the non-wandering operator on F.

Proof Since $\frac{x^k}{\sqrt{k!}}$ is an orthonormal basis in F, then we have

$$D\left(\frac{x^k}{\sqrt{k!}}\right) = k \cdot \frac{x^{k-1}}{\sqrt{k!}} = \sqrt{k}\frac{x^{k-1}}{\sqrt{(k-1)!}} = w_{k-1}\frac{x^{k-1}}{\sqrt{(k-1)!}}$$

where$w_k = \sqrt{k+1}$,

So, we can get the operator of differentiation $D: \ f \ \rightarrow \ f'$ is the weighted backward shift operator, therefore,

$$D\left(\sum_{k \geq 0} C_k \frac{x^k}{\sqrt{k!}}\right) = \sum_{k \geq 0} w_k C_{k+1}\frac{x^k}{\sqrt{k!}} \quad , \quad w_k = \sqrt{k+1}$$

so the series

$$\sum_{n=0}^{\infty}\prod_{j=0}^{n-1}\frac{1}{|w_j|^2} = \sum_{n=0}^{\infty}\prod_{k=0}^{n-1}\frac{1}{k+1}$$

converges in F, thus by the Theorem 2, we have D is the non-wandering operator on F.

4. Conclusion

In this paper, we first extent the non-wandering operator theory to the unbounded operator. We provide a sufficient condition for an unbounded operator to be non-wandering operator and get differentiation operator on the Bargmann space be a non-wandering operator. Therefore, the non-wandering operator theory has been further improved, but also enriched the research of the chaotic operator and the Hypercyclic operator.

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