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Products of Admissible Monomials in the Polynomial Algebra as a Module over the Steenrod Algebra

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Abstract

Let $\mathbf{P}(n) = \mathbb{F}_2[x_1, \ldots, x_n]$ be the polynomial algebra in n variables x_i, of degree one, over the field $\mathbb{F}_2$ of two elements. The mod-2 Steenrod algebra $\mathcal{A}$ acts on $\mathbf{P}(n)$ according to well known rules. A major problem in algebraic topology is that of determining $\mathcal{A}^+\mathbf{P}(n)$, the image of the action of the positively graded part of $\mathcal{A}$. We are interested in the related problem of determining a basis for the quotient vector space $\mathbf{Q}(n) = \mathbf{P}(n)/\mathcal{A}^+\mathbf{P}(\mathbf{n})$. Both $\mathbf{P}(n) = \bigoplus_{d\geq 0}\mathbf{P}^d(n)$ and $\mathbf{Q}(n)$ are graded, where $\mathbf{P}^d(n)$ denotes the set of homogeneous polynomials of degree d. $\mathbf{Q}(n)$ has been explicitly calculated for $n = 1, 2, 3, 4$ but problems remain for $n \geq 5$. In this note we show that if $u = x_1^{m_1}\cdots x_k^{m_k} \in \mathbf{P}^d(k)$ and $v = x_1^{e_1}\cdots x_r^{e_r} \in \mathbf{P}^{d'}(r)$ are an admissible monomials, (that is, u and v meet a criterion to be in a certain basis for $\mathbf{Q}(k)$ and $\mathbf{Q}(r)$ respectively), then for each permutation $\sigma \in S_{k+r}$ for which $\sigma(i) < \sigma(j)$, $i < j \leq k$ and $\sigma(s) < \sigma(t)$, $k < s < t \leq k+r$, the monomial $x_{\sigma(1)}^{m_1}\cdots x_{\sigma(k)}^{m_k}x_{\sigma(k+1)}^{e_1}\cdots x_{\sigma(k+r)}^{e_r} \in \mathbf{P}^{d+d'}(k+r)$ is admissible. As an application we consider a few cases when $n = 5$.

Keywords: steenrod squares, polynomial algebra, hit problem.

1. 1. Introduction

For $n \geq 1$ let $\mathbf{P}(n)$ be the mod-2 cohomology group of the n-fold product of $\mathbb{R}P^\infty$ with itself. Then $\mathbf{P}(n)$ is the polynomial algebra

$$\mathbf{P}(n) = \mathbb{F}_2[x_1, \ldots, x_n]$$

in n variables x_i, each of degree 1, over the field $\mathbb{F}_2$ of two elements. The mod-2 Steenrod algebra $\mathcal{A}$ is the graded associative algebra generated over $\mathbb{F}_2$ by symbols Sq^i for $i \geq 0$, called Steenrod squares subject to the Adem relations (Adem, 1957) and $Sq^0 = 1$. Let $\mathbf{P}^d(n)$ denote the homogeneous polynomials of degree d. The action of the Steenrod squares $Sq^i : \mathbf{P}^d(n) \to \mathbf{P}^{d+i}(n)$ is determined by the formula:

$$Sq^i(u) = \begin{cases} u, & i = 0 \\ u^2, & \deg(u) = i \\ 0, & \deg(u) < i, \end{cases}$$

and the Cartan formula

$$Sq^i(uv) = \sum_{r=0}^{i} Sq^r(u)Sq^{i-r}(v).$$

A polynomial $u \in \mathbf{P}^d(n)$ is said to be hit if it is in the image of the action of $\mathcal{A}$ on $\mathbf{P}(n)$, that is, if

$$u = \sum_{i>0} Sq^i(u_i),$$

for some $u_i \in \mathbf{P}(n)$ of degree $d - i$. Let $\mathcal{A}^+\mathbf{P}(n)$ denote the subspace of all hit polynomials. The problem of determining $\mathcal{A}^+\mathbf{P}(n)$ is called the hit problem and has been studied by several authors, (Singer, 1991) and (Wood, 1989). We are interested in the related problem of determining a basis for the quotient vector space

$$\mathbf{Q}(n) = \mathbf{P}(n)/\mathcal{A}^+\mathbf{P}(n)$$

which has also been studied by several authors, (Kameko, 1990, 2003), (Peterson, 1987) and (Sum, 2007). Some of the motivation for studying these problems is mentioned in (Nam, 2004). It stems from the Peterson conjecture proved in (Wood, 1989) and various other sources (Peterson,1989) and (Singer, 1989).

The following result is useful for determining $\mathcal{A}$-generators for $\mathbf{P}(n)$. Let $\alpha(m)$ denote the number of digits 1 in the binary expansion of m.

In (Wood, 1989)[Theorem 1], R.M.W. Wood proved that:

Theorem 1 (Wood, 1989). *Let $u \in \mathbf{P}(n)$ be a monomial of degree d. If $\alpha(n+d) > n$, then u is hit.*

Thus $\mathbf{Q}^d(n)$ is zero unless $\alpha(n+d) \leq n$ or, equivalently, unless d can be written in the form, $d = \sum_{i=1}^{n}(2^{\lambda_i} - 1)$ where $\lambda_i \geq 0$. Thus $\mathbf{Q}^d(n) \neq 0$ only if $\mathbf{P}^d(n)$ contains monomials $v = x_1^{2^{\lambda_1}-1} \cdots x_n^{2^{\lambda_n}-1}$ called spikes.

$\mathbf{Q}(n)$ has been explicitly calculated by Peterson in (Peterson,1987) for $n = 1, 2$, by Kameko in his thesis (Kameko, 1990) for $n = 3$ and independently by Kameko in (Kameko, 2003) and Sum in (Sum, 2007) for $n = 4$. In this work we shall, unless otherwise stated, be concerned with a basis for $\mathbf{Q}(n)$ consisting of 'admissible monomials', as defined below. Thus when we write $u \in \mathbf{Q}^d(n)$ we mean that u is an admissible monomial of degree d.

We define what it means for a monomial $b = x_1^{e_1} \cdots x_n^{e_n} \in \mathbf{P}(n)$ to be admissible. Write $e_i = \sum_{j\geq 0} \alpha_j(e_i)2^j$ for the binary expansion of each exponent e_i. The expansions are then assembled into a matrix $\beta(b) = (\alpha_j(e_i))$ of digits 0 or 1 with $\alpha_j(e_i)$ in the (i, j)-th position of the matrix. We then associate with b, two sequences,

$$w(b) = (w_0(b), w_1(b), \ldots, w_j(b), \ldots),$$
$$e(b) = (e_1, e_2, \ldots, e_n),$$

where $w_j(b) = \sum_{i=1}^{n} \alpha_j(e_i)$ for each $j \geq 0$. $w(b)$ is called the **weight vector** of the monomial b and $e(b)$ is called the **exponent vector** of the monomial b.

Given two sequences $p = (u_0, u_1, \ldots, u_l, 0, \ldots)$, $q = (v_0, v_1, \ldots, v_l, 0, \ldots)$, we say $p < q$ if there is a positive integer k such that $u_i = v_i$ for all $i < k$ and $u_k < v_k$. We are now in a position to define an order relation on monomials.

Definition 1. Let a, b be monomials in $\mathbf{P}(n)$. We say that $a < b$ if one of the following holds:

1. $w(a) < w(b)$,
2. $w(a) = w(b)$ and $e(a) < e(b)$.

Note that the order relation on the set of sequences is the lexicographical one.

Following Kameko, (Kameko, 1990) we define:

Definition 2. A monomial $b \in \mathbf{P}(n)$ is said to be **inadmissible** if there exist monomials $b_1, b_2, \ldots, b_r \in \mathbf{P}(n)$ with $b_j < b$ for each j, $1 \leq j \leq r$, such that

$$b \equiv \left(\sum_{j=1}^{r} b_j\right) \mod \mathcal{A}^+\mathbf{P}(n).$$

b is said to be **admissible** if it is not inadmissible.

Clearly the set of all admissible monomials in $\mathbf{P}(n)$ form a basis for $\mathbf{Q}(n)$.

Our main result is:

Theorem 2. *If $u = x_1^{m_1} \cdots x_k^{m_k} \in \mathbf{P}^d(k)$ and $v = x_1^{e_1} \cdots x_r^{e_r} \in \mathbf{P}^{d'}(r)$ are admissible monomials, then for each permutation $\sigma \in S_{k+r}$ for which $\sigma(i) < \sigma(j)$, $i < j \leq k$ and $\sigma(s) < \sigma(t)$, $k < s < t \leq k+r$, the monomial $x_{\sigma(1)}^{m_1} \cdots x_{\sigma(k)}^{m_k} x_{\sigma(k+1)}^{e_1} \cdots x_{\sigma(k+r)}^{e_r} \in \mathbf{P}^{d+d'}(k+r)$ is admissible.*

Theorem 2 is a generalization of the following result of the author and Uys proved in (Mothebe and Uys, 2015).

Let $u = x_1^{m_1} \cdots x_{n-1}^{m_{n-1}} \in \mathbf{P}(n-1)$ be a monomial of degree d'. Given any pair of integers (j, λ), $1 \leq j \leq n$, $\lambda \geq 0$, let $h_j^{\lambda}(u)$ denote the monomial $x_1^{m_1} \cdots x_{j-1}^{m_{j-1}} x_j^{2^{\lambda}-1} x_{j+1}^{m_j} \cdots x_n^{m_{n-1}} \in \mathbf{P}^{d'+(2^{\lambda}-1)}(n)$.

Theorem 3. *Let $u \in \mathbf{P}(n-1)$ be a monomial of degree d', where $\alpha(d'+n-1) \leq n-1$. If u is admissible, then for each pair of integers (j, λ), $1 \leq j \leq n$, $\lambda \geq 0$, $h_j^{\lambda}(u)$ is admissible.*

As our main application of Theorem 2 we consider a few cases when $n = 5$. The relevant result in this case is Theorem 4 stated below. To explain the table that appears in the theorem we note that given any explicit admissible monomial basis for $\mathbf{Q}(s)$, $1 \leq s \leq n-1$, one may compute $\mathcal{GLB}(n, d)$, the dimension of the subspace of $\mathbf{Q}^d(n)$ generated by all degree d monomials of the form $x_{\sigma(1)}^{m_1} \cdots x_{\sigma(k)}^{m_k} x_{\sigma(k+1)}^{e_1} \cdots x_{\sigma(k+r)}^{e_r}$ for all triples (k, r, σ) where $k + r = n$ and $\sigma \in S_{k+r}$ satisfies the hypothesis of the theorem. In general $\mathcal{GLB}(n, d) \leq \dim(\mathbf{Q}^d(n))$ but there are cases where equality holds.

In (Sum, 2007) Sum gives an explicit admissible monomial basis for $\mathbf{Q}(4)$ and in addition recalls the results of Kameko (Kameko, 1990) for $\mathbf{Q}(3)$. In this paper we make use of these results to compute $\mathcal{GLB}(5,d)$, $1 \le d \le 30$, and compare these values with $\dim\left(\mathbf{Q}^d(5)\right)$ in the given range. The results are given in Table A in Theorem 4. The table is incomplete as $\dim\left(\mathbf{Q}^d(5)\right)$ has not yet, in general, been calculated for $d \ge 13$. As can be seen from Table A there are cases where $\mathcal{GLB}(5,d) = \dim(\mathbf{Q}^d(5))$. This is demonstrated with the aid of known results for $\dim(\mathbf{Q}^d(5))$ (cited in Table A). The results also show an improvement from the results obtained in (Mothebe and Uys, 2015) by application of Theorem 3.

Theorem 4. *Table A gives lower bounds, $\mathcal{GLB}(5,d)$, for the dimension of* $\mathbf{Q}^d(5)$, $1 \le d \le 30$.

Table A

d	$\dim(\mathbf{Q}^d(5))$	Ref	$\mathcal{GLB}(5,d)$	d	$\dim(\mathbf{Q}^d(5))$	Ref	$\mathcal{GLB}(5,d)$
1	5		5	16			418
2	10		10	17			543
3	25		25	18			680
4	45	(Sum and Phuc, 2013)	45	19	912	(Tin, 2014)	780
5	46	(Mothebe and Uys, 2015)	46	20			591
6	74	(Mothebe and Uys, 2015)	74	21			780
7	110	(Mothebe and Uys, 2015)	110	22			819
8	174	(Tin, 2014)	174	23			993
9	191	(Mothebe and Uys, 2015)	191	24			925
10	280	(Mothebe and Uys, 2015)	280	25			1073
11	315	(Mothebe 2009)	315	26	1024	(Walker and Wood, 2007)	1003
12	190	(Sum and Phuc, 2013)	190	27	315		315
13	-		250	28	480	(Sum and Phuc, 2013)	480
14	-		302	29			491
15	432	(Sum, 2014)	404	30			785

While this approach remains to be explored in general these test results suffice for our purpose in this paper and we hope to make a more general account in subsequent work. We are thus only required to prove Theorem 2. This is the subject of the next section which is also our concluding section.

2. Proof of Theorem 2

In this section we prove Theorem 2. It shall suffice to show that if $u = x_1^{m_1} \cdots x_k^{m_k} \in \mathbf{P}^d(k)$ and $v = x_1^{e_1} \cdots x_r^{e_r} \in \mathbf{P}^{d'}(r)$ are admissible monomials, then $x_1^{m_1} \cdots x_k^{m_k} x_{k+1}^{e_1} \cdots x_{k+r}^{e_r} \in \mathbf{P}^{d+d'}(k+r)$ is admissible. We first note that for any given monomial $u = x_1^{m_1} \cdots x_k^{m_k} \in \mathbf{P}^d(k)$ we have a mapping

$$h_u \ : \ \mathbf{P}^{d'}(r) \to \mathbf{P}^{d+d'}(k+r)$$

given on monomials by $h_u(x_1^{e_1} \cdots x_r^{e_r}) = x_1^{m_1} \cdots x_k^{m_k} x_{k+1}^{e_1} \cdots x_{k+r}^{e_r}$. Unless otherwise stated, we shall use product notation uv for $h_u(v)$. In this way we see that each monomial $u \in \mathbf{P}^d(k)$ determines a subspace (namely $h_u(\mathbf{P}^{d'}(r))$) of $\mathbf{P}^{d+d'}(k+r)$ isomorphic to $\mathbf{P}^{d'}(r)$. Similarly each monomial $v = x_1^{e_1} \cdots x_r^{e_r} \in \mathbf{P}^{d'}(r)$ determines a mapping

$$g_v \ : \ \mathbf{P}^d(k) \to \mathbf{P}^{d+d'}(k+r)$$

given on monomials by $g_v(x_1^{m_1} \cdots x_k^{m_k}) = x_1^{m_1} \cdots x_k^{m_k} x_{k+1}^{e_1} \cdots x_{k+r}^{e_r}$. Let $\pi_u : \mathbf{P}^{d+d'}(k+r) \to h_u(\mathbf{P}^{d'}(r))$ denote the projection of $\mathbf{P}^{d+d'}(k+r)$ onto the summand $h_u(\mathbf{P}^{d'}(r))$ of $\mathbf{P}^{d+d'}(k+r)$. Our aim is to show that h_u induces an isomorphism from $\mathcal{A}^+\mathbf{P}(r) \cap \mathbf{P}^{d'}(r)$ to $\pi_u(\mathcal{A}^+\mathbf{P}(k+r) \cap \mathbf{P}^{d+d'}(k+r))$, given by $Sq^b(z) \mapsto uSq^b(z)$. In other words we claim that $\pi_u(\mathcal{A}^+\mathbf{P}(k+r) \cap \mathbf{P}^{d+d'}(k+r))$ is generated by polynomials of the form $uSq^b(z)$ where

$$Sq^b(z) \in \mathcal{A}^+\mathbf{P}(r) \cap \mathbf{P}^{d'}(r)$$

Under this assumption, suppose that

$$p = u \sum_j u_j$$

is a polynomial generated by elements $uSq^b(z) \in \pi_u(\mathcal{A}^+\mathbf{P}(k+r) \cap \mathbf{P}^{d+d'}(k+r))$. Since h_u is order preserving it follows that if uu_{j_l} is a term of highest order in p, then u_{j_l} is an inadmissible monomial in $\mathbf{P}^{d'}(r)$. A parallel argument holds for the projection $\pi_v : \mathbf{P}^{d+d'}(k+r) \to g_v(\mathbf{P}^d(k))$ associated with the mapping g_v. Clearly this shall suffice for a proof of Theorem 2.

Suppose therefore that a monomial $u \in \mathbf{P}^d(k)$ is given. The important thing to note is that for any integer $s > 0$, $Sq^s(y) \in \mathcal{A}^+\mathbf{P}(k+r) \cap \mathbf{P}^{d+d'}(k+r)$ is a hit polynomial that has a monomial of the form $h_u(v)$ if and only if there exists integers $a \geq 0$ and $b \geq 0$ and monomials w, z with $a+b=s$ and $wz=y$ such that $Sq^a(w) \in \mathcal{A}^+\mathbf{P}(k) \cap \mathbf{P}^d(k)$ is a hit polynomial which has u as a term and $Sq^b(z) \in \mathcal{A}^+\mathbf{P}(r) \cap \mathbf{P}^{d'}(r)$ is a hit polynomial which has v as a term. This is an immediate consequence of the Cartan formula for the action of the Steenrod algebra on polynomials. We then have, by the Cartan formula, $Sq^s(y) = Sq^{a+b}(wz)$ is equal to the sum

$$Sq^a(w)Sq^b(z) + \sum_{t \neq b} Sq^{a+b-t}(w)Sq^t(z). \tag{1}$$

Now $\pi_u(\sum_{t\neq b} Sq^{a+b-t}(w)Sq^t(z)) = 0$ since, clearly, the polynomial

$$\sum_{t\neq b} Sq^{a+b-t}(w)Sq^t(z)$$

has no terms of the form $h_u(c)$. Now suppose that $b > 0$ and that modulo hit monomials $Sq^a(w) = \sum_{j=1}^{l} u_j$. Then $Sq^a(w)Sq^b(z) = (\sum_{j=1}^{l} u_j)Sq^b(z) = \sum_{j=1}^{l} u_j Sq^b(z)$. Monomials of the form uv then occur as terms of the part of $Sq^a(w)Sq^b(z)$ of the form $uSq^b(z)$. Note that if $u_j \neq u$, then $\pi_u(u_j Sq^b(z)) = 0$. On the other hand $uSq^b(z) = Sq^b(uz) + \sum_{t>0} Sq^t(u)Sq^{b-t}(z)$. But

$$\pi_u(\sum_{t>0} Sq^t(u)Sq^{b-t}(z)) = 0.$$

Thus

$$\pi_u(Sq^s(y)) = \pi_u(Sq^b(uz)) = uSq^b(z) = h_u(Sq^b(z))$$

This establishes the isomorphism.

A similar analysis may be drawn to show that $\pi_v(\mathcal{A}^+\mathbf{P}(k+r) \cap \mathbf{P}^{d+d'}(k+r))$ is generated by polynomials of the form $(Sq^a(w))v$ where $Sq^a(w) \in \mathcal{A}^+\mathbf{P}(k) \cap \mathbf{P}^d(k)$. Now suppose that a polynomial $q = (\sum_i v_i)v$ is generated by elements $(Sq^a(w))v \in \pi_v(\mathcal{A}^+\mathbf{P}(k+r) \cap \mathbf{P}^{d+d'}(k+r))$. Since g_v is order preserving it follows that if $v_{i_l}v$ is a term of highest order in q, then v_{i_l} is an inadmissible monomial in $\mathbf{P}^d(k)$.

It follows from our argument above that if u and v are admissible monomials then uv is admissible. Clearly the statement of the theorem remains true if we take any product $\sigma(uv) = x_{\sigma(1)}^{m_1} \cdots x_{\sigma(k)}^{m_k} x_{\sigma(k+1)}^{e_1} \cdots x_{\sigma(k+r)}^{e_r}$ resulting from any permutation $\sigma \in S_{k+r}$ that satisfies the hypothesis of the theorem. Note that if uv is a monomial of highest order in the polynomial p generated by expressions of the form $uSq^b(z)$ then $\sigma(uv) = x_{\sigma(1)}^{m_1} \cdots x_{\sigma(k)}^{m_k} x_{\sigma(k+1)}^{e_1} \cdots x_{\sigma(k+r)}^{e_r}$ is the monomial of highest order in the polynomial $\sigma(p)$ generated by expressions of the form $x_{\sigma(1)}^{m_1} \cdots x_{\sigma(k)}^{m_k} Sq^b(x_{\sigma(k+1)}^{s_1} \cdots x_{\sigma(k+r)}^{s_r})$ where $x_{\sigma(1)}^{m_1} \cdots x_{\sigma(k)}^{m_k} x_{\sigma(k+1)}^{s_1} \cdots x_{\sigma(k+r)}^{s_r} = \sigma(uz)$. This completes the proof of the theorem.

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