Journal of Mathematics Research; Vol. 8, No. 2; April 2016
ISSN 1916-9795 E-ISSN 1916-9809
Published by Canadian Center of Science and Education

An Extension of the Euler Phi-function to Sets of Integers Relatively Prime to 30

Mbakiso Fix Mothebe[1] & Ben T. Modise[1]

[1] Department of Mathematics, University of Botswana, Pvt Bag 00704, Gaborone, Botswana

Correspondence: Mbakiso Fix Mothebe, Department of Mathematics, University of Botswana, Pvt Bag 00704, Gaborone, Botswana. E-mail: mothebemf@mopipi.ub.bw

Received: January 27, 2016 Accepted: February 14, 2016 Online Published: March 10, 2016

doi:10.5539/jmr.v8n2p45 URL: http://dx.doi.org/10.5539/jmr.v8n2p45

Abstract

Let $n \geq 1$ be an integer and let $S = \{1, 7, 11, 13, 17, 19, 23, 29\}$, the set of integers which are both less than and relatively prime to 30. Define $\phi_3(n)$ to be the number of integers $x,\ 0 \leq x \leq n-1$, for which $\gcd(30n, 30x+i) = 1$ for all $i \in S$. In this note we show that ϕ_3 is multiplicative, that is, if $\gcd(m, n) = 1$, then $\phi_3(mn) = \phi_3(m)\phi_3(n)$. We make a conjecture about primes generated by S.

Keywords: Euler phi-function, multiplicative function

1. Introduction

Let $n \geq 1$ be an integer. In (Mothebe & Samuel, 2015) we define $\phi_2(n)$ to be the number of integers $x,\ 1 \leq x \leq n$, for which both $6x-1$ and $6x+1$ are relatively prime to $6n$. We proved that the function is multiplicative and thereby obtained a formula for its evaluation.

Let $S = \{1, 7, 11, 13, 17, 19, 23, 29\}$, the set of integers which are both less than and relatively prime to 30. Define $\phi_3(n)$ to be the number of integers $x,\ 0 \leq x \leq n-1$, for which $\gcd(30n, 30x+i) = 1$ for all $i \in S$. In this note we draw analogy with our study of ϕ_2 and show that ϕ_3 is multiplicative. In the same vain we obtain a formula for evaluating ϕ_3. Our study motivates a conjecture to the effect that there are infinitely many integers x for which there is a set of the form,

$$\{30x+1, 30x+7, 30x+11, 30x+13, 30x+17, 30x+19, 30x+23, 30x+29\},$$

that contains seven primes. We illustrate with some computations. The conjecture in turn implies cases of Alphonse de Polignac's conjecture that for every number k, there are infinitely many prime pairs p and p' such that $p' - p = 2k$.

Let $2, 3, 5, p_t, \ldots, p_k$ be first k consecutive primes in ascending order. One may generalise the above definition and define $\phi_k(n)$ to be the number of integers $x,\ 0 \leq x \leq n-1$, for which $\gcd((\prod_{t=1}^{k} p_t)n, (\prod_{t=1}^{k} p_t)x + i) = 1$ for all integers i which are less than $2.3.5 \ldots p_k$ and relatively prime to each of the primes $2, 3, 5, p_t, \ldots p_k$. These functions may also be shown to be multiplicative.

If p is a prime then $\phi_3(p)$ is easy to evaluate. For example $\phi_3(7) = 0$ since for all x, the set $\{30x + i \mid i \in S\}$ contains an integer divisible by 7. On the other hand if $p \neq 7$, then $\phi_3(p) \neq 0$. It is easy to check that $\phi_3(p) = p$ if $p = 2, 3$ or 5. Further $\phi_3(11) = 11 - 6$ and $\phi_3(p) = p - 8$ if $p \geq 13$. We note also that $\phi_3(1) = 1$.

We now proceed to show that we can evaluate $\phi_3(n)$ from the prime factorization of n. Our arguments are based on those used by Burton in (Burton, 2002), to show that the Euler phi-function is multiplicative. We first note:

Theorem 1. *Let k and s be nonnegative numbers and let $p \geq 13$ be a prime number. Then:*

(i) $\phi_3(q^k) = q^k$ *if* $q = 2, 3$ *or* 5.

(ii) $\phi_3(7^s) = 0$.

(iii) $\phi_3(11^k) = 11^k - 6.11^{k-1} = 11^k\left(1 - \frac{6}{11}\right)$.

(iv) $\phi_3(p^k) = p^k - 8p^{k-1} = p^k\left(1 - \frac{8}{p}\right)$.

Proof. We shall only consider the cases (iii) and (iv) as (i) and (ii) are easy to verify.

(iii) and (iv). Clearly, for each $i \in S$, $\gcd(30x+i, 30p) = 1$ if and only if p does not divide $30x+i$. Further for each

$i \in S$, there exists one integer x between 0 and $p-1$ that satisfies the congruence relation $30x + i \equiv 0 \pmod{p}$. We note however that if $p = 11$, then in S, we have $23 \equiv 1 \pmod{11}$ and $29 \equiv 7 \pmod{11}$. Hence for all x for which $30x + 1 \equiv 0 \pmod{11}$ we also have $30x + 23 \equiv 0 \pmod{11}$ and for all x for which $30x + 7 \equiv 0 \pmod{11}$ we also have $30x + 29 \equiv 0 \pmod{11}$. No such case arises when $p \geq 13$.

Returning to our discussion, it follows that for each $i \in S$ there are p^{k-1} integers between 1 and p^k that satisfy $30x + i \equiv 0 \pmod{p}$. Thus for each $i \in S$, the set

$$\{30x + i \mid 1 \leq x \leq p^k\}$$

contains exactly $p^k - p^{k-1}$ integers x for which $\gcd(30p^k, 30x + i) = 1$. Since these integers x are distinct for distinct elements $i \in S$ it follows that if $p \geq 13$, we must have $\phi_3(p^k) = p^k - 8p^{k-1}$. However if $p = 11$ we must have $\phi_3(11^k) = 11^k - 6.11^{k-1}$.

For example $\phi_3(6.11^2) = 11^2 - 6.11 = 55$ and $\phi_2(13^2) = 13^2 - 8.13 = 65$.

We recall that:

Definition 1. A number theoretic function f is said to be **multiplicative** if $f(mn) = f(m)f(n)$ whenever $\gcd(m, n) = 1$.

From the proof of Theorem 1 it is clear that for all non-negative integers $k,\ s,\ t,\ r$:

$$\phi_3(2^k 3^s 5^t 7^r) = \phi_3(2^k)\phi_3(3^s)\phi_3(5^k)\phi_3(7^r).$$

We now show that the function ϕ_3 is multiplicative. This will enable us to obtain a formula for $\phi_3(n)$ based on a factorization of n as a product of primes.

We require the following results.

Lemma 1. *Given integers m, n, and $i \in S$,* $\gcd(30mn, 30x + i) = 1$ *if and only if* $\gcd(30n, 30x + i) = 1$ *and* $\gcd(30m, 30x + i) = 1$.

This is an immediate consequence of the following standard result.

Lemma 2. *Given integers m, n, k,* $\gcd(k, mn) = 1$ *if and only if* $\gcd(k, m) = 1$ *and* $\gcd(k, n) = 1$.

We note also the following standard result.

Lemma 3. *If $a = bq + r$, then* $\gcd(a, b) = \gcd(b, r)$.

Theorem 2. *The function ϕ_3 is multiplicative, that is, if* $\gcd(m, n) = 1$*, then* $\phi_3(mn) = \phi_3(m)\phi_3(n)$.

Proof. The result holds if either m or n equals 1. Further $\phi_3(mn) = \phi_3(m)\phi_3(n) = 0$ if m or n is equal to 7. We shall therefore assume neither m nor n equals 1 or 7. For each integer x denote the set $\{30x + i \mid i \in S\}$ by $\{30x + i\}$. Arrange the sets $\{30x + i\}$, $1 \leq x \leq mn$, in an $n \times m$ array as follows:

$$\begin{bmatrix} S & \dots & \{30(m-1) + i\} \\ \{30m + i\} & \dots & \{30(2m-1) + i\} \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ \{30((n-1)m) + i\} & \dots & \{30(nm-1) + i\} \end{bmatrix}$$

We know that $\phi_3(mn)$ is equal to the number of sets $\{30x + i\}$ in this matrix for which all elements of $\{30x + i\}$ are relatively prime to $30mn$. By virtue of Lemma 1 this is the same as the number of sets $\{30x + i\}$ in the same matrix for which all elements of $\{30x + i\}$ are relatively prime to each of $30m$ and $30n$. We first note, by virtue of Lemma 3, that for all $i \in S$ and all $x,\ 0 \leq x \leq m-1$, and $q,\ 0 \leq q \leq n-1$:

$$\gcd(30(qm + x) + i, 30m) = \gcd(30x + i, 30m).$$

Therefore each of the sets $\{30(qm + x) + i\}$ in the x^{th} column contains elements which are all relatively prime to $30m$ if and only if the set $\{30x + i\}$ contains elements all of which are relatively prime to $30m$. Therefore only $\phi_3(30m)$ columns contain sets $\{30x + i\}$ for which every element is relatively prime to $30m$ and every other set in the column will constitute of integers all of which are relatively prime to $30m$.

The problem now is to show that in each of these $\phi_3(30m)$ columns there are exactly $\phi_3(30n)$ sets $\{30x + i\}$ all of whose elements are relatively prime to $30n$, for then altogether there would be $\phi_3(30m)\phi_3(30n)$ sets in the table for which every element is relatively prime to both $30m$ and $30n$.

The sets that are in the x^{th} column, $0 \le x \le m-1$, (where it is assumed $\gcd(30x+i, 30m) = 1$ for all i) are:

$$\{30x + i\}, \{30(m + x) + i\}, \ldots, \{30((n-1)m + x) + i\}.$$

There are n sets in this sequence and for no two distinct sets

$$\{30(qm + x) + i\} \ \ \{30(jm + x) + i\}$$

in the sequence can we have

$$30(qm + x) + i \equiv 30(jm + x) + i \pmod{n}$$

for any $i \in S$, since otherwise we would arrive at a contradiction $q \equiv j \pmod{n}$.

Thus the terms of the sequence,

$$x,\ m + x, 2m + x, \ldots, (n-1)m + x$$

are congruent modulo n to $0, 1, 2, \ldots, n-1$ in some order.

Now suppose t is congruent modulo n to $qm+x$. Then the elements of the set $\{30(qm+x)+i\}$ are all relatively prime to $30n$ if and only if the elements of the set $\{30t+i\}$ are all relatively prime to $30n$. The implication is that the x^{th} column contains as many sets, all of whose elements are relatively prime to $30n$, as does the collection $\{S, \{30+i\}, \{2.30+i\}\ldots, \{30(n-1)+i\}\}$, namely $\phi_3(30n)$ sets. Thus the number of sets $\{30x + i\}$ in the matrix each of whose elements is relatively prime to $30m$ and $30n$ is $\phi_3(30m)\phi_3(30n)$. This completes the proof of the theorem.

As a consequence of Theorems 1 and 2 we have:

Theorem 3. *If the integer* $n > 1$ *has the prime factorization*

$$n = 2^{k_1}3^{k_2}5^{k_3}11^{k_4}p_5^{k_5}\ldots p_r^{k_r}$$

with $p_s \ne 7$ *for any* $s \ge 5$, *then*

$$\phi_3(n) \quad = 2^{k_1}3^{k_2}5^{k_3}(11^{k_4} - 6.11^{k_4-1})(p_5^{k_5} - 8p_5^{k_5-1})\ldots(p_r^{k_r} - 8p_r^{k_r-1}).$$

For example $\phi_3(143) = \phi_3(11)\phi_3(13) = 25$.

For each integer $x \ge 0$ the set $\{30x + i\}$ has an element which is divisible by 7. This is the case since for each element j of the integers modulo 7 there is an element $i \in S$ such that $i \equiv j \pmod 7$. For some integers x the sets $\{30x + i\}$ contain seven primes. This in is the case when, for instance, $x = 0, 1, 2, 49, 62, 79, 89, 188$. We shall call such sets **seven-prime sets**. Given an integer n a sieve method (a modified version of the sieve of Eratosthenes) may be employed to find the number of integers $x \le n-1$ for which $\{30x + i\}$ is a seven-prime set. This motivates the following conjecture which is in the same mould as the Twin Prime Conjecture.

Conjecture 1. *There are infinitely many seven-prime sets.*

Definition 2. For $n > 0$, let $\pi_7(n)$ denote the number of integers x, $0 \le x \le n-1$, for which $\{30x + i\}$, is a seven-prime set.

The following table gives the values of $\pi_7(10^k)$ for $k \le 10^8$.

n	$\pi_7(n)$
10	3
10^2	7
10^3	8
10^4	10
10^5	20
10^6	64
10^7	227
10^8	962

In 1894, Alphonse de Polignac made a conjecture that for every number k, there are infinitely many prime pairs p and p' such that $p' - p = 2k$. The case $k = 1$ is the well-known Twin Prime Conjecture. Since the elements of S differ by $2, 4, 6, 8, 10, 12, 16, 18, 22, 28$ we see that Conjecture 1 implies cases of Alphonse de Polignac's conjecture.

References

Burton, D. M. (2002). *Elementary Number Theory.* New York, NY: McGraw-Hill International Edition.

Mothebe M. F., & Samuel S. (2015) *Properties of the Euler phi-function on pairs of positive integers* $(6x - 1, 6x + 1)$. Preprint: University of Botswana.

Copyrights