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Alternating Group A_5 Actions on Homotopy $S^2 \times S^2$

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Abstract

Let X be a smooth, closed 4-manifold which is homotopy equivalent to $S^2 \times S^2$. By the Seiberg-Witten theory, we take $\mathrm{Ind}_{A_5} D_X$ as a virtual A_5-representation and give its concrete representation. We also study $\mathrm{Ind}_{A_5} D_X$ when X is homotopy equivalent to $\sharp_n S^2 \times S^2$. Besides we give an example of our main theorem.

Keywords: homotopy $S^2 \times S^2$, alternating group actions, Seiberg-Witten equations, Dirac operator

1. Introduction

Suppose X is a smooth, closed, connected spin 4-manifold. Let b_i be the i-th Betti number and b_+ be the rank of the maximal positive definite subspace of $H^2(X;\mathbb{R})$. $\sigma(X)$ denotes the signature of X. By Freedman & Quinn 1990 and Bryan 1998, the intersection form of X with non-positive signature should be

$$-2kE_8 \oplus mH, \qquad k \geq 0,$$

where E_8 is the 8 dimension bilinear intersection form and H is the hyperbolic form. Obviously, $m = b_2^+(X)$ and $k = -\sigma(X)/16$.

Suppose X admits a finite G-action which preserves the spin structure. We also suppose there is a Riemannian matric on X so that the G-action is isometric. Under these assumption, the G-action can always be lifted to a $\tilde{G}$-action on the spinor bundles, where $\tilde{G}$ is in the following extension

$$1 \to \mathbb{Z}_2 \to \tilde{G} \to G \to 1.$$

If $\tilde{G}$ contains a subgroup isomorphic to G, then the G-action is called even type. Otherwise, the G-action is called odd type. When G is the alternating group A_5, $\tilde{G}$ is a group isomorphic to $\mathbb{Z}_2 \times A_5$. Since A_5 is a subgroup of $\mathbb{Z}_2 \times A_5$, the spin A_5 action on a spin 4-manifold must be of even type.

By Bryan 1998, for a spin even type G-action on a spin manifold X, the Dirac operator D_X is G-equivariant and $\mathrm{Ind}_G D_X = \ker D_X - \mathrm{coker} D_X \in R(G)$. Suppose $\mathrm{Ind}_{A_5} D_X = a_0\rho_0 + b_0\rho_1 + c_0\rho_2 + d_0\rho_3 + e_0\rho_4$, where $\rho_0, \rho_1, \rho_2, \rho_3$ and ρ_4 are irreducible representations of A_5 of degree 1, 3, 3, 4 and 5 (for detail see section 2), a_0, b_0, c_0, d_0 and e_0 are all integers.

The finite spin group actions on spin 4-manifold are widely studied. Such as Bryan 1998, Fang 2001, Furuta 2001, Liu 2005, Liu 2006 and Liu & Li 2008. In this paper, we mainly study the spin alternating group A_5 action on spin 4-manifolds X which are homotopy equivalent to $S^2 \times S^2$. Let $-X$ denote X with the reversed orientation. Then $-X$ is also homotopy equivalent to $S^2 \times S^2$ and satisfies $\mathrm{Ind}_{A_5} D_X = -\mathrm{Ind}_{A_5} D_{-X}$. Using this property, representation theory, Seiberg-Witten theory and the character formula for K-theory degree, we obtain the following main result.

Theorem 1 *Let X be a closed smooth 4-manifold which is homotopy equivalent to $S^2 \times S^2$. If X admits a smooth spin alternating group A_5 action such that $b_2^+(X/A_5) = b_2^+(X)$, then* $\mathrm{Ind}_{A_5} D_X = a_0(\rho_0 - 2\rho_1 + \rho_4) + c_0(\rho_2 - \rho_1)$, *where a, b are integers.*

Corrolary 2 *Let X be a closed smooth 4-manifold which is homotopy equivalent to $\sharp_n S^2 \times S^2$. If X admits a smooth spin alternating group A_5 action such that $b_2^+(X/A_5) = b_2^+(X)$, then* $\mathrm{Ind}_{A_5} D_X = a_0\rho_0 + b_0\rho_1 + c_0\rho_2 + d_0\rho_3 + e_0\rho_4$ *satisfies* $|b_0 + c_0 + d_0 + 2e_0| \leq \frac{n-1}{2}$.

Theorem 3. *Let X be a closed smooth 4-manifold which is homotopy equivalent to $\sharp_n S^2 \times S^2$. Suppose X admit a smooth spin alternating group A_5 action and $b_2^+(X/A_5) = 0$, $b_2^+(X/<s>) = 0$ and $b_2^+(X/<t>) \neq 0$. Then as an element of*

$R(A_5)$, $Ind_{A_5}D$ is of the form

$$a_0\rho_0 + b_0(\rho_1 + \rho_2) + (a_0 + b_0)\rho_3 - (a_0 + 2b_0)\rho_4,$$

and $n \equiv 0 \bmod 4$.

The rest of this paper consists of three parts. The first one is the introduction about this study. The second one gives the proofs of Theorem 1, Corollary 2 and Theorem 3. The last part contains an example about the main theorem.

2. Preliminaries

In this section, we review some basics about the Seiberg-Witten equations and symmetries on it, conjugacy classes of alternating group A_5, the index of $\mathcal{D}$ and the K-theory degree. Notice that this section largely depends on Bryan 1998. Besides, readers can also refer to Fang 2001, Furuta 2001 and Liu 2006.

2.1 Seiberg-Witten equations and its symmetry

Let $U^{\pm}$ be the positive and negative complex spinor bundles and $U = U^+ \oplus U^-$. Denote by $D : \Gamma(U^+) \to \Gamma(U^-)$ the Dirac operator and $\rho : \Lambda^*_{\mathbb{C}} \to \mathrm{End}_{\mathbb{C}}(U)$ the Clifford multiplication. Then the Seiberg-Witten equations are as follows

$$D\phi + \rho(a)\phi = 0, \qquad \rho(d^+a) - \phi\otimes\phi^* + \frac{1}{2}|\phi|^2\mathrm{id} = 0, \qquad d^*a = 0,$$

where $(a, \phi) \in \Omega^1(X, \sqrt{-1}\mathbb{R}) \times \Gamma(U^+)$. Let V be the L^4_2-completion of $\Gamma(\sqrt{-1}\Lambda^1 \oplus U^+)$ and W' be the L^3_2-completion of $\Gamma(U^- \oplus \sqrt{-1}\mathrm{su}(U^+) \oplus \sqrt{-1}\Lambda^0)$. We could look the Seiberg-Witten equations as the zero set of a map

$$\mathcal{D} + \mathcal{Q} : \ V \to W',$$

where $\mathcal{D}(a, \phi) = (D\phi, \rho(d^+a), d^*a))$, $\mathcal{Q}(a, \phi) = (\rho(a)\phi, \phi\otimes\phi^* - \frac{1}{2}|\phi|^2\mathrm{id}, 0)$.

In fact, the image of $\mathcal{D}+\mathcal{Q}$ is L^2-orthogonal to the constant functions in $\sqrt{-1}\Omega^0 \subset W'$. We denote W to be the orthogonal complement of the constant functions in W' and consider $\mathcal{D} + \mathcal{Q} : V \to W$.

Next we consider the symmetries on the Seiberg-Witten equations. Denote by SU(2). the group of unit quaternions and S^1 the set of elements in the form $e^{\sqrt{-1}\theta}$. Suppose Pin(2) is the normalizer of S^1 in SU(2). Then the elements of Pin(2) should be in the form $e^{\sqrt{-1}\theta}$ or $e^{\sqrt{-1}\theta}J$. The action of Pin(2) on $\Gamma(U^{\pm})$ is the multiplication on the left. The action of $\mathbb{Z}/2$ on $\Gamma(\Lambda^*_{\mathbb{C}})$ is multiplication by ± 1. By this way, we obtain the action of Pin(2) on V, W. Furthermore, the operator $\mathcal{D}$ and $\mathcal{Q}$ are all Pin(2) equivariant.

Assume X is a closed smooth spin 4-manifold and G is a compact Lie group action on X which is isometric and preserves the spin structure. If the action is of even type, then both $\mathcal{D}$ and $\mathcal{Q}$ are $\tilde{G} = \mathrm{Pin}(2) \times G$ equivariant maps (Bryan 1998).

2.2 The Alternating Group A_5

In this paper, we consider the action of the alternating group A_5 on homotopy $S^2 \times S^2$. The alternating group A_5 is the minimal nonabelian finite simple group which consists of even permutations of a set $\{a, b, c, d, e\}$ with 5 elements. It consists of 60 elements which can be divided into the following 5 conjugacy classes:

(1) the identity element 1;

(2) 15 elements of order 2 which is conjugate with $x = (ab)(cd)$;

(3) 20 elements of order 3 which is conjugate with $t = (abc)$;

(4) 12 elements of order 5 which is conjugate with $s = (abcde)$;

(5) 12 elements of order 5 which is conjugate with $s^2 = (abced)$.

Besides, we have the following character table for A_5, where $\omega = e^{2\pi i/5}$. For detail computation, we can refer to Serre 1997.

Table 1. Table title (the character table for A_5)

1	t	x	s	s^2	
χ_0	1	1	1	1	1
χ_1	3	0	-1	$1+\omega+\omega^4$	$1+\omega^2+\omega^3$
χ_2	3	0	-1	$1+\omega^2+\omega^3$	$1+\omega+\omega^4$
χ_3	4	1	0	-1	-1
χ_4	5	-1	1	0	0

2.3 The Index of $\mathcal{D}$ and the Character Formula for the K-theory Degree

Denote by $V_\lambda \subset V$ (resp. $W_\lambda \subset W$) the subspace spanned by the eigenspaces of $\mathcal{D}^*\mathcal{D}$ (resp. $\mathcal{D}\mathcal{D}^*$) with eigenvalues less than or equal to $\lambda \in \mathbb{R}$. Denote $V_{\lambda,\mathbb{C}} = V_\lambda \otimes \mathbb{C}$, $W_{\lambda,\mathbb{C}} = W_\lambda \otimes \mathbb{C}$. Then

$$\mathrm{Ind}\mathcal{D} = [\ker\mathcal{D}] - [\mathrm{Coker}\mathcal{D}] = [V_{\lambda,\mathbb{C}}] - [W_{\lambda,\mathbb{C}}].$$

Let $r : R(\widetilde{G}) \to R(\mathrm{Pin}(2))$ denotes the restriction map. Suppose $\widetilde{1}$ be the non-trivial one dimensional representation in $R(\mathrm{Pin}(2))$, which is obtained by pulling back the non-trivial $\mathbb{Z}/2$ representation by the map $\mathrm{Pin}(2) \to \mathbb{Z}/2$. Denote h_i the 2 dimensional irreducible representation in $R(\mathrm{Pin}(2))$, which is the restriction of the standard representation of SU(2) to $\mathrm{Pin}(2) \subset \mathrm{SU}(2)$ and write $h_1 = h$. Then Furuta determines $\mathrm{Ind}\mathcal{D}$ as a Pin(2) representation, and shows

$$r(\mathrm{Ind}\mathcal{D}) = 2kh - m\tilde{1}.$$

Thus $\mathrm{Ind}\mathcal{D} = sh - t\tilde{1}$, where s and t are polynomials such that $s(1) = 2k$ and $t(1) = m$.

For a spin A_5 action, $\tilde{G} = \mathrm{Pin}(2) \times A_5$. We have

$$s(\rho_0, \rho_1, \rho_2, \rho_3, \rho_4) = a_0\rho_0 + b_0\rho_1 + c_0\rho_2 + d_0\rho_3 + e_0\rho_4,$$

$$t(\rho_0, \rho_1, \rho_2, \rho_3, \rho_4) = a_1\rho_0 + b_1\rho_1 + c_1\rho_2 + d_1\rho_3 + e_1\rho_4,$$

such that $a_0 + 3b_0 + 3c_0 + 4d_0 + 5e_0 = 2k$ and $a_1 + 3b_1 + 3c_1 + 4d_1 + 5e_1 = m = b_2^+(X)$.

Suppose $< g >$ is the cyclic subgroup of A_5 generated by $g \in A_5$. Then by using dimensions of invariant subspaces of $< g >$ and multiplicities of eigenvalue 1 of ρ_i, $(0 \le i \le 4)$ for respective conjugacy classes, we get

$$\dim (H^+(X)^{A_5}) = a_1 = b_2^+(X/A_5),$$

$$\dim (H^+(X)^{<(abc)>}) = a_1 + b_1 + c_1 + 2d_1 + e_1 = b_2^+(X/ < (abc) >),$$
$$\dim (H^+(X)^{<(ab)(cd)>}) = a_1 + b_1 + c_1 + 2d_1 + 3e_1 = b_2^+(X/ < (ab)(cd) >),$$
$$\dim (H^+(X)^{<(abcde)>}) = a_1 + b_1 + c_1 + e_1 = b_2^+(X/ < (abcde) >),$$
$$\dim (H^+(X)^{<(abced)>}) = a_1 + b_1 + c_1 + e_1 = b_2^+(X/ < (abced) >).$$

Moreover, for the Dirac operator of $\mathrm{Ind}_{A_5} D$, we get

$$\dim (\mathrm{Ind}_{A_5} D)^{A_5} = a_0,$$

$$\dim (\mathrm{Ind}_{A_5} D)^{<(abc)>} = a_0 + b_0 + c_0 + 2d_0 + e_0,$$
$$\dim (\mathrm{Ind}_{A_5} D)^{<(ab)(cd)>} = a_0 + b_0 + c_0 + 2d_0 + 3e_0,$$
$$\dim (\mathrm{Ind}_{A_5} D)^{<(abcde)>} = a_0 + b_0 + c_0 + e_0,$$
$$\dim (\mathrm{Ind}_{A_5} D)^{<(abced)>} = a_0 + b_0 + c_0 + e_0.$$

Suppose V and W are two complex G-representations of compact Lie group G. BV and BW are balls in V and W. We construct a G-map $f : BV \to BW$ which preserves the boundaries of BV and BW. Denote by V_g and W_g the subspaces of V and W fixed under the action of $g \in G$ and by $V_g^\perp$ and $W_g^\perp$ the corresponding orthogonal complements. Define $f^g : V_g \to W_g$ to be the restriction of f. Suppose $\lambda_{-1}\beta = \Sigma(-1)^i\lambda^i\beta$. Then we have the following character formula for the degree α_f.

Theorem 4.(Tom Dieck 1979) *Let $f : BV \to BW$ be a G-map preserving boundaries and let $\alpha_f \in R(G)$ be the K-theory degree. Then*

$$\mathrm{tr}_g(\alpha_f) = d(f^g)\mathrm{tr}_g(\lambda_{-1}(W_g^\perp - V_g^\perp)),$$

where tr_g is the trace of the action of an element $g \in G$, $d(f^g)$ is the topological degree of f^g.

Obviously, if $\dim V_g \neq \dim W_g$, then $d(f^g) = 0$. Note that $\lambda_{-1}(\Sigma_i k_i\rho_i) = \prod_i(\lambda_{-1}\rho_i)^{k_i}$. When ρ_i is a 1-dim representation, $\lambda_{-1}\rho_i = (1 - \rho_i)$. When ρ_i is a 2-dim representation h, we have $\lambda_{-1}\rho_i = (2 - h)$. Suppose $\phi \in S^1 \subset \mathrm{Pin}(2)$ is the element generating a dense subgroup of S^1, $J \in \mathrm{Pin}(2)$ is an element in the set of quaternion. The action of ϕ on the 2-dim representation h is nontrivial and on the 1-dim representation $\tilde{1}$ is trivial. J acts on h with two invariant subspaces. The action of J on them is multiplying $\pm\sqrt{-1}$. In the following, to be simple we denote α_f by α, denote V_g and W_g by V and W.

3. Results

Theorem 1 *Let X be a closed smooth 4-manifold which is homotopy equivalent to $S^2 \times S^2$. If X admits a smooth spin alternating group A_5 action with $b_2^+(X/A_5) = b_2^+(X)$, then $\mathrm{Ind}_{A_5} D_X = a_0(\rho_0 - 2\rho_1 + \rho_4) + c_0(\rho_2 - \rho_1)$, where a, b are integers.*

Proof. Obviously, $b_2^+(X/A_5) = b_2^+(X) = 1$, $k = -\sigma(X)/16 = 0$ and $m = b_2^+(X) = 1$. For

$$s(\rho_0,\rho_1,\rho_2,\rho_3,\rho_4) = a_0\rho_0 + b_0\rho_1 + c_0\rho_2 + d_0\rho_3 + e_0\rho_4,$$

and

$$t(\rho_0,\rho_1,\rho_2,\rho_3,\rho_4) = a_1\rho_0 + b_1\rho_1 + c_1\rho_2 + d_1\rho_3 + e_1\rho_4,$$

we have

$$a_0 + 3b_0 + 3c_0 + 4d_0 + 5e_0 = 0,$$
$$a_1 = 1,$$
$$b_1 = c_1 = d_1 = e_1 = 0.$$

Note that $\alpha \in R(\mathrm{Pin}(2) \times A_5)$, then it must in the form

$$\alpha = \alpha_0 + \tilde{\alpha}_0\tilde{1} + \sum_{i=1}^{\infty} \alpha_i h_i,$$

where $\alpha_i = l_i\rho_0 + m_i\rho_1 + n_i\rho_2 + q_i\rho_3 + r_i\rho_4,\ i \geq 0$ and $\tilde{\alpha}_0 = \tilde{l}_0\rho_0 + \tilde{m}_0\rho_1 + \tilde{n}_0\rho_2 + \tilde{q}_0\rho_3 + \tilde{r}_0\rho_4$.

By the action of ϕ,

$$\dim(V(\rho_0,\rho_1,\rho_2,\rho_3,\rho_4))_\phi - \dim(W(\rho_0,\rho_1,\rho_2,\rho_3,\rho_4))_\phi = -(a_1 + 3b_1 + 3c_1 + 4d_1 + 5e_1) = -1.$$

Then from T. tom Dieck's character formula, we get $\mathrm{tr}_\phi\alpha = 0$.

Notice that ϕt acts non-trivially on $V(\rho_0,\rho_1,\rho_2,\rho_3,\rho_4)h$. t acts trivially on ρ_0. The actions of t on ρ_1, ρ_2, ρ_4 all have a 1-dim invariant subspace, while the action of t on ρ_3 has a 2-dim invariant subspace. The above actions give rise to

$$\dim(V(\rho_0,\rho_1,\rho_2,\rho_3,\rho_4))_{\phi t} - \dim(W(\rho_0,\rho_1,\rho_2,\rho_3,\rho_4))_{\phi t} = -(a_1 + b_1 + c_1 + 2d_1 + e_1) = -1.$$

Hence $\mathrm{tr}_{\phi t}\alpha = 0$.

The action of ϕx on $V(\rho_0,\rho_1,\rho_2,\rho_3,\rho_4)h$ is non-trivial while it is trivial on $\tilde{1}$. x acts on ρ_1 and ρ_2 both with a 1-dim invariant subspace while it has a 2-dim invariant subspace on ρ_3 and a 3-dim invariant subspace on ρ_4 respectively.

$$\dim(V(\rho_0,\rho_1,\rho_2,\rho_3,\rho_4))_{\phi x} - \dim(W(\rho_0,\rho_1,\rho_2,\rho_3,\rho_4))_{\phi x} = -(a_1 + b_1 + c_1 + 2d_1 + 3e_1) = -1.$$

Therefore $\mathrm{tr}_{\phi x}\alpha = 0$.

The action of ϕs on $V(\rho_0,\rho_1,\rho_2,\rho_3,\rho_4)h$ is nontrivial. s acts on ρ_0 trivially and with a 1-dim invariant subspace on ρ_1, ρ_2 and ρ_4 respectively. Thus we have

$$\dim(V(\rho_0,\rho_1,\rho_2,\rho_3,\rho_4))_{\phi s} - \dim(W(\rho_0,\rho_1,\rho_2,\rho_3,\rho_4))_{\phi s} = -(a_1 + b_1 + c_1 + e_1) = -1.$$

For the same reason, we have

$$\dim(V(\rho_0,\rho_1,\rho_2,\rho_3,\rho_4))_{\phi s^2} - \dim(W(\rho_0,\rho_1,\rho_2,\rho_3,\rho_4))_{\phi s^2} = -(a_1 + b_1 + c_1 + e_1) = -1.$$

Thus $\mathrm{tr}_{\phi s}\alpha = \mathrm{tr}_{\phi s^2}\alpha = 0$.

In summary, if $b_2^+(X/A_5) = b_2^+(X) = 1$ then we have $\mathrm{tr}_\phi\alpha = \mathrm{tr}_{\phi t}\alpha = \mathrm{tr}_{\phi x}\alpha = \mathrm{tr}_{\phi s}\alpha = \mathrm{tr}_{\phi s^2}\alpha = 0$ which implies that

$$\begin{aligned} 0 &= \mathrm{tr}_\phi\alpha = tr_\phi(\alpha_0 + \tilde{\alpha}_0\tilde{1} + \sum_{i=1}^{\infty}\alpha_i h_i) \\ &= \mathrm{tr}_\phi\alpha_0 + \mathrm{tr}_\phi\tilde{\alpha}_0 + \sum_{i=1}^{\infty}\mathrm{tr}_\phi\alpha_i(\phi^i + \phi^{-i}) \\ &= (l_0 + 3m_0 + 3n_0 + 4q_0 + 5r_0) + (\tilde{l}_0 + 3\tilde{m}_0 + 3\tilde{n}_0 + 4\tilde{q}_0 + 5\tilde{r}_0) + \sum_{i=1}^{\infty}\mathrm{tr}_\phi\alpha_i(\phi^i + \phi^{-i}), \end{aligned}$$

$$\begin{aligned} 0 &= \mathrm{tr}_{\phi t}\alpha = \mathrm{tr}_t(\alpha_0 + \tilde{\alpha}_0\tilde{1} + \sum_{i=1}^{\infty}\alpha_i(\phi^i + \phi^{-i})) \\ &= (l_0 + q_0 - r_0) + (\tilde{l}_0 + \tilde{q}_0 - \tilde{r}_0) + \sum_{i=1}^{\infty}\mathrm{tr}_t\alpha_i(\phi^i + \phi^{-i}), \end{aligned}$$

$$\begin{aligned} 0 &= \mathrm{tr}_{\phi x}\alpha = \mathrm{tr}_x(\alpha_0 + \tilde{\alpha}_0\tilde{1} + \sum_{i=1}^{\infty}\alpha_i(\phi^i + \phi^{-i})) \\ &= (l_0 - m_0 - n_0 + r_0) + (\tilde{l}_0 - \tilde{m}_0 - \tilde{n}_0 + \tilde{r}_0) + \sum_{i=1}^{\infty}\mathrm{tr}_x\alpha_i(\phi^i + \phi^{-i}), \end{aligned}$$

$$\begin{aligned} 0 &= \mathrm{tr}_{\phi s}\alpha = \mathrm{tr}_s(\alpha_0 + \tilde{\alpha}_0\tilde{1} + \sum_{i=1}^{\infty}\alpha_i(\phi^i + \phi^{-i})) \\ &= [l_0 + (1 + \omega + \omega^4)m_0 + (1 + \omega^2 + \omega^3)n_0 - q_0] + \\ &\quad [\tilde{l}_0 + (1 + \omega + \omega^4)\tilde{m}_0 + (1 + \omega^2 + \omega^3)\tilde{n}_0 - \tilde{q}_0] + \sum_{i=1}^{\infty}\mathrm{tr}_s\alpha_i(\phi^i + \phi^{-i}), \end{aligned}$$

$$\begin{aligned} 0 &= \mathrm{tr}_{\phi s^2}\alpha = \mathrm{tr}_s^2(\alpha_0 + \tilde{\alpha}_0\tilde{1} + \sum_{i=1}^{\infty}\alpha_i(\phi^i + \phi^{-i})) \\ &= [l_0 + (1 + \omega^2 + \omega^3)m_0 + (1 + \omega + \omega^4)n_0 - q_0] + \\ &\quad [\tilde{l}_0 + (1 + \omega^2 + \omega^3)\tilde{m}_0 + (1 + \omega + \omega^4)\tilde{n}_0 - \tilde{q}_0] + \sum_{i=1}^{\infty}\mathrm{tr}_{s^2}\alpha_i(\phi^i + \phi^{-i}). \end{aligned}$$

From these equations we can conclude $\alpha_0 = -\tilde{\alpha}_0$ and $\alpha_i = 0, i > 0$, that is $\alpha = \alpha_0(1 - \tilde{1})$.

Since J acts non-trivially on both h and $\tilde{1}$, and $\dim V_J = \dim W_J = 0$, we have $d(f^J) = 1$. Besides, $\mathrm{tr}_J h = 0$ and $\mathrm{tr}_J \tilde{1} = -1$. Then we have $\mathrm{tr}_J(\alpha) = \mathrm{tr}_J((1 - \tilde{1})^m(2 - h)^{-2k}) = 2^{m-2k}$.

Since the action of Jt is non-trivial on Vh and $W\tilde{1}$, we have $d(f^{Jt}) = 1$. Then

$$\begin{aligned} &\mathrm{tr}_{Jt}(\alpha) \\ =\ &\mathrm{tr}_{Jt}[\lambda_{-1}(a_1)\tilde{1} - \lambda_{-1}(a_0 + b_0\rho_1 + c_0\rho_2 + d_0\rho_3 + e_0\rho_4)h] \\ =\ &\mathrm{tr}_{Jt}[(1 - \tilde{1})^{a_1}(1 - h)^{-a_0}(1 - \rho_1 h)^{-b_0}(1 - \rho_2 h)^{-c_0}(1 - \rho_3 h)^{-d_0}(1 - \rho_4 h)^{-e_0}] \\ =\ &\frac{2^{a_1}}{2^{a_0}[2(1 + \varepsilon^2)(1 + \varepsilon)]^{b_0}[2(1 + \varepsilon)(1 + \varepsilon^2)]^{c_0}[2^2(1 + \varepsilon^2)(1 + \varepsilon)]^{d_0}[2(1 + \varepsilon^2)^2(1 + \varepsilon)^2]^{e_0}} \\ =\ &2^{a_1 - (a_0 + b_0 + c_0 + 2d_0 + e_0)}. \end{aligned}$$

Here the 3-dim representation ρ_1 can be decomposed into three complex lines, the actions of t on them are multiplying 1, ε and ε^2, where $\varepsilon = e^{2\pi i/3}$. Similarly, the action of t on the three subspaces of representation ρ_2 is 1, ε^2 and ε. For the 4-dimensional representation ρ_3, the action of t is 1, 1, ε, ε^2. For the 5-dimensional representation ρ_4, the action of t is 1, ε, ε, ε^2, ε^2.

Since Jx acts non-trivially on both $V(\rho_0, \rho_1, \rho_2, \rho_3, \rho_4)h$ and $W(\rho_0, \rho_1, \rho_2, \rho_3, \rho_4)\tilde{1}$, we have

$$\dim(V(\rho_0, \rho_1, \rho_2, \rho_3, \rho_4))_{Jx} - \dim(W(\rho_0, \rho_1, \rho_2, \rho_3, \rho_4))_{Jx} = 0.$$

Consequently, $d(f^{Jx}) = 1$. Then

$$\begin{aligned} \mathrm{tr}_{Jx}(\alpha) &= \mathrm{tr}_{Jx}[\lambda_{-1}(a_1)\tilde{1} - \lambda_{-1}(a_0\rho_0 + b_0\rho_1 + c_0\rho_2 + d_0\rho_3 + e_0\rho_4)h] \\ &= \mathrm{tr}_{Jx}[(1 - \tilde{1})^{a_1}(1 - \rho_0 h)^{-a_0}(1 - \rho_1 h)^{-b_0}(1 - \rho_2 h)^{-c_0}(1 - \rho_3 h)^{-d_0}(1 - \rho_4 h)^{-e_0}] \\ &= 2^{a_1 - (a_0 + 3b_0 + 3c_0 + 4d_0 + 5e_0)}. \end{aligned}$$

Since Js acts non-trivially on both $V(\rho_0, \rho_1, \rho_2, \rho_3, \rho_4)h$ and $W(\rho_0, \rho_1, \rho_2, \rho_3, \rho_4)\tilde{1}$, we have

$$\dim(V(\rho_0, \rho_1, \rho_2, \rho_3, \rho_4))_{Js} - \dim(W(\rho_0, \rho_1, \rho_2, \rho_3, \rho_4))_{Js} = 0.$$

thereby, $d(f^{Js}) = 1$. From tom Dieck formula, we have

$$\begin{aligned} \mathrm{tr}_{Js}(\alpha) &= \mathrm{tr}_{Js}[\lambda_{-1}(a_1)\tilde{1} - \lambda_{-1}(a_0\rho_0 + b_0\rho_1 + c_0\rho_2 + d_0\rho_3 + e_0\rho_4)h] \\ &= 2^{a_1}2^{-a_0}[2(1+\omega^2)(1+\omega^3)]^{-b_0}[2(1+\omega^4)(1+\omega)]^{-c_0} \\ &\quad [(1+\omega^2)(1+\omega^4)(1+\omega)(1+\omega^3)]^{-d_0}[2(1+\omega^2)(1+\omega^4)(1+\omega)(1+\omega^3)]^{-e_0} \\ &= 2^{a_1-(a_0+b_0+c_0+e_0)}[(1+\omega^2)(1+\omega^3)]^{b_0-c_0}. \end{aligned}$$

For the same reasons, we have

$$\mathrm{tr}_{Js^2}(\alpha) = 2^{a_1-(a_0+b_0+c_0+e_0)}[(1+\omega^2)(1+\omega^3)]^{c_0-b_0}.$$

By calculating directly, we have

$$\mathrm{tr}_J\alpha_0 = l_0 + 3m_0 + 3n_0 + 4q_0 + 5r_0 = 2^{m-2k-1} = 1, \tag{1}$$

$$\mathrm{tr}_t\alpha_0 = l_0 + q_0 - r_0 = 2^{a_1-(a_0+b_0+c_0+2d_0+e_0)-1} = 2^{2(b_0+c_0+d_0+2e_0)}, \tag{2}$$

$$\mathrm{tr}_x\alpha_0 = l_0 - m_0 - n_0 + r_0 = 2^{a_1-(a_0+3b_0+3c_0+4d_0+5e_0)-1} = 2^{m-2k-1} = 1, \tag{3}$$

$$\begin{aligned} \mathrm{tr}_s\alpha_0 &= l_0 + (1+\omega+\omega^4)m_0 + (1+\omega^2+\omega^3)n_0 - q_0 \\ &= 2^{a_1-(a_0+b_0+c_0+e_0)-1}[(1+\omega^2)(1+\omega^3)]^{b_0-c_0}, \end{aligned} \tag{4}$$

$$\begin{aligned} \mathrm{tr}_{s^2}\alpha_0 &= l_0 + (1+\omega^2+\omega^3)m_0 + (1+\omega+\omega^4)n_0 - q_0 \\ &= 2^{a_1-(a_0+b_0+c_0+e_0)-1}[(1+\omega^2)(1+\omega^3)]^{c_0-b_0}. \end{aligned} \tag{5}$$

Notice that we have the following relations.

$$\begin{aligned} \mathrm{tr}_{Jx}\alpha &= \mathrm{tr}_x(2\alpha_0) = 2\mathrm{tr}_x\alpha_0, \\ \mathrm{tr}_{Jt}\alpha &= \mathrm{tr}_t(2\alpha_0) = 2\mathrm{tr}_t\alpha_0, \\ \mathrm{tr}_{Js}\alpha &= \mathrm{tr}_s(2\alpha_0) = 2\mathrm{tr}_s\alpha_0, \\ \mathrm{tr}_{Js^2}\alpha &= \mathrm{tr}_{s^2}(2\alpha_0) = 2\mathrm{tr}_{s^2}\alpha_0. \end{aligned}$$

From (1) and (3) we get

$$l_0 + q_0 + 2r_0 = 1,$$

which together with (2) shows us

$$r_0 = \frac{1}{3}[1 - 2^{2(b_0+c_0+d_0+2e_0)}].$$

Since $r_0 \in \mathbb{Z}$, so $b_0 + c_0 + d_0 + 2e_0 \geq 0$.

Now we consider $-X$, the reverse-oriented homotopy $S^2 \times S^2$. If we denote by $\mathrm{Ind}_{A_5}D_{-X} = a'_0\rho_0 + b'_0\rho_1 + c'_0\rho_2 + d'_0\rho_3 + e'_0\rho_4$, from the above discussion we know that $b'_0 + c'_0 + d'_0 + 2e'_0 \geq 0$. On the other hand, we have $\mathrm{Ind}_{A_5}D_X = -\mathrm{Ind}_{A_5}D_{-X}$, so $a'_0 = -a_0, b'_0 = -b_0, c'_0 = -c_0, d'_0 = -d_0$ and $e'_0 = -e_0$. From these equations, we get $b_0 + c_0 + d_0 + 2e_0 \leq 0$ and then $b_0 + c_0 + d_0 + 2e_0 = 0$. Thus we have

$$l_0 = 1 + m_0 + n_0 = 1 - q_0. \tag{6}$$

From (4) and (5), we have

$$2l_0 + m_0 + n_0 - 2q_0 = 2^{-(a_0+b_0+c_0+e_0)}[((1+\omega^2)(1+\omega^3))^{c_0-b_0} + ((1+\omega^2)(1+\omega^3))^{b_0-c_0}]$$

which along with (6) shows that

$$q_0 = \frac{2 - 2^{-(a_0+b_0+c_0+e_0)}[((1+\omega^2)(1+\omega^3))^{c_0-b_0} + ((1+\omega^2)(1+\omega^3))^{b_0-c_0}]}{5}.$$

Since $q_0 \in \mathbb{Z}$ and $[(1+\omega^2)(1+\omega^3)]^{c_0-b_0} + [(1+\omega^2)(1+\omega^3)]^{b_0-c_0}$ is a positive integer, we have $a_0 + b_0 + c_0 + e_0 \leq 0$. Using the reverse-orientation as before, we get $a_0 + b_0 + c_0 + e_0 = 0$.

Thus we have the following equations

$$a_0 + 3b_0 + 3c_0 + 4d_0 + 5e_0 = 0, \tag{7}$$

$$a_0 + b_0 + c_0 + e_0 = 0, \tag{8}$$

$$b_0 + c_0 + d_0 + 2e_0 = 0, \tag{9}$$

from which we get

$$a_0 = e_0, b_0 = -c_0 - 2e_0, d_0 = 0.$$

Thus $\mathrm{Ind}_{A_5} D_X = a_0(\rho_0 - 2\rho_1 + \rho_4) + c_0(\rho_2 - \rho_1)$. This completes the proof of Theorem 1.

We can also study the G-Index of A_5 action on homotopy $\sharp_n S^2 \times S^2$ in the similar way, and get the following result.

Corollary 2 *Let X be a closed smooth 4-manifold which is homotopy equivalent to $\sharp_n S^2 \times S^2$. If X admits a spin alternating group A_5 action with $b_2^+(X/A_5) = b_2^+(X)$, and denote by* $\mathrm{Ind}_{A_5} D_X = a_0\rho_0 + b_0\rho_1 + c_0\rho_2 + d_0\rho_3 + e_0\rho_4$*, then* $|b_0 + c_0 + d_0 + 2e_0| \leq \frac{n-1}{2}$.

Notice that when X is homotopy equivalent to $\sharp_n S^2 \times S^2$, $b_2^+(X) = n$ and $k = 0$.

Theorem 3 *Let X be a closed smooth 4-manifold which is homotopy equivalent to $\sharp_n S^2 \times S^2$. Suppose X admit a smooth spin alternating group A_5 action and $b_2^+(X/A_5) = 0$, $b_2^+(X/ <s>) = 0$ and $b_2^+(X/ <t>) \neq 0$. Then as an element of $R(A_5)$,* $\mathrm{Ind}_{A_5} D$ *is of the form*

$$a_0\rho_0 + b_0(\rho_1 + \rho_2) + (a_0 + b_0)\rho_3 - (a_0 + 2b_0)\rho_4,$$

and $n \equiv 0 \bmod 4$.

Proof. Let X is homotopy equivalent to $\sharp_n S^2 \times S^2$. Next we assume $b_2^+(X/A_5) = 0$, $b_2^+(X/ <s>) = 0$ and $b_2^+(X/ <t>) \neq 0$, that is $a_1 = b_1 = c_1 = e_1 = 0$ and $d_1 \neq 0$. Then $b_2^+(X) = a_1 + 3b_1 + 3c_1 + 4d_1 + 5e_1 = 4d_1$. Since $d_1 \in \mathbb{Z}$, we have $n \equiv 0 \bmod 4$.

Considering the action of ϕs, we know the action of ϕs on $h, \rho_1 h, \rho_2 h, \rho_3 h, \rho_4 h$ and $\rho_3 \tilde{1}$ are all non-trivial but it acts on $1, \rho_1\tilde{1}, \rho_2\tilde{1}, \rho_4\tilde{1}$ all with a 1-dimensional invariant subspace. So

$$\dim(V(\rho_1,\rho_2,\rho_3,\rho_4)h)_{\phi s} - \dim(W(\rho_1,\rho_2,\rho_3,\rho_4)\tilde{1})_{\phi s} = -(a_1 + b_1 + c_1 + e_1) = 0,$$

and then $d(f^{\phi s}) = 1$. By tom Dieck formula, we have

$$\begin{aligned} \mathrm{tr}_{\phi s}\alpha &= \mathrm{tr}_{\phi s}[\lambda_{-1}(d_1\rho_3)\tilde{1} - \lambda_{-1}(a_0\rho_0 + b_0\rho_1 + c_0\rho_2 + d_0\rho_3 + e_0\rho_4)h] \\ &= [(1-\omega)(1-\omega^2)(1-\omega^3)(1-\omega^4)]^{d_1}[(1-\phi)(1-\phi^{-1})]^{-(a_0+b_0+c_0+e_0)} \\ &\quad [(1-\omega^2\phi)(1-\omega^2\phi^{-1})]^{-(c_0+d_0+e_0)}[(1-\omega^3\phi)(1-\omega^3\phi^{-1})]^{-(c_0+d_0+e_0)} \\ &\quad [(1-\omega\phi)(1-\omega\phi^{-1})]^{-(b_0+d_0+e_0)}[(1-\omega^4\phi)(1-\omega^4\phi^{-1})]^{-(b_0+d_0+e_0)}. \end{aligned}$$

Since $\mathrm{tr}_{s\bullet}\alpha$£ $U(1) \to \mathbb{C}$ is a C^0-function and ϕ is a generic element, then $a_0 + b_0 + c_0 + e_0 \leq 0$, $c_0 + d_0 + e_0 \leq 0$, $b_0 + d_0 + e_0 \leq 0$. Besides, $\dim(\mathrm{Ind}D_{A_5}) = a_0 + 3b_0 + 3c_0 + 4d_0 + 5e_0 = 0$. Then we have

$$a_0 + b_0 + c_0 + e_0 = c_0 + d_0 + e_0 = b_0 + d_0 + e_0 = 0, \tag{10}$$

which means $b_0 = c_0$, $d_0 = a_0 + b_0$ and $e_0 = -(a_0 + 2b_0)$. This completes the proof of Theorem.

4. An example of Theorem 1.

As we know there exist a smooth action of A_5 on the standard $S^2 \times S^2$ induced by the icosahedral action on each factor. Furthermore, the fixed points of this action for every non-trivial element is 4 isolated points. Next we compute $\mathrm{Ind}_{A_5} D_X$ as a virtual A_5-representation.

(1) When $g = 1$, $spin(g, X) = -\frac{sign(X)}{8} = 0$.

(2) When $g = t$, denote m_+, m_- the number of fixed points with local representation $(1, 2)$ and $(1, 1)$ respectively. Besides

$$\nu_+(P) = \frac{1}{(\zeta^{1/2} - \zeta^{-1/2})((\zeta^2)^{1/2} - (\zeta^2)^{-1/2})} = 1/3,$$

$$\nu_-(P) = \frac{1}{(\zeta^{1/2} - \zeta^{-1/2})(\zeta^{1/2} - \zeta^{-1/2})} = -1/3.$$

Since $sign(X/ < g >)$ is integer, we have $m_+ = m_- = 2$. Then $spin(g, X) = m_+\nu_+(P) + m_-\nu_-(P) = 0$.

(3) When $g = x$, two of the fixed points have $\nu(P) = -1/4$, and the other two have $\nu(P) = +1/4$. Then $spin(g, X) = -1$.

(4) When $g = s$, the local representation of the 4 fixed points may be of type $(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4)$ or $(4, 4)$. Besides we have

$$\nu_{(1,1)} = -\nu_{(1,4)} = \nu_{(4,4)} = \frac{1}{\zeta + \zeta^4 - 2},$$

$$\nu_{(1,2)} = \nu_{(3,4)} = \frac{1}{-2\zeta^2 - 2\zeta^3 - 1},$$

$$\nu_{(1,3)} = \nu_{(2,4)} = \frac{1}{-2\zeta - 2\zeta^4 - 1},$$

$$\nu_{(2,2)} = -\nu_{(2,3)} = \nu_{(3,3)} = \frac{1}{\zeta^2 + \zeta^3 - 2}.$$

Note that

$$\frac{1}{\zeta + \zeta^4 - 2} + \frac{1}{\zeta^2 + \zeta^3 - 2} = -1,$$

$$\frac{1}{-2\zeta^2 - 2\zeta^3 - 1} + \frac{1}{-2\zeta - 2\zeta^4 - 1} = 0.$$

If $spin(g, X)$ is rational, then $spin(g, X) = 0, \pm 1$ or ± 2.

(5) When $g = s^2$, the result is the same as above.

Then the coefficient a_0, b_0, c_0, d_0 and e_0 can be computed as follows.

$$a_0 = \frac{1 \times 1 \times 0 + 1 \times 20 \times 0 + 1 \times 15 \times 0 + 1 \times 12 \times spin(s, X) + 1 \times 12 \times spin(s^2, X)}{60}.$$

Since a_0 is an integer, the only possible case is $a_0 = 0$. Similarly, $b_0 = c_0 = d_0 = e_0 = 0$. Thus $\mathrm{Ind}_{A_5} D_X = 0$. This is consistent with Theorem 1.

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