Journal of Mathematics Research; Vol. 8, No. 1; 2016
ISSN 1916-9795 E-ISSN 1916-9809
Published by Canadian Center of Science and Education

Singular Values of Two Parameter Families $\lambda \frac{e^{az}-1}{z}$ and $\lambda \frac{z}{e^{az}-1}$

Mohammad Sajid[1]

[1] College of Engineering, Qassim University, Saudi Arabia

Correspondence: College of Engineering, Qassim University, Saudi Arabia. E-mail: msajid@qec.edu.sa

Received: November 25, 2015 Accepted: December 10, 2015 Online Published: January 7, 2016

doi:10.5539/jmr.v8n1p10 URL: http://dx.doi.org/10.5539/jmr.v8n1p10

Abstract

The singular values of two parameter families of entire functions $f_{\lambda,a}(z) = \lambda \frac{e^{az}-1}{z}$, $f_{\lambda,a}(0) = \lambda a$ and meromorphic functions $g_{\lambda,a}(z) = \lambda \frac{z}{e^{az}-1}$, $g_{\lambda,a}(0) = \frac{\lambda}{a}$, $\lambda, a \in \mathbb{R}\backslash\{0\}$, $z \in \mathbb{C}$, are investigated. It is shown that all the critical values of $f_{\lambda,a}(z)$ and $g_{\lambda,a}(z)$ lie in the right half plane for $a < 0$ and lie in the left half plane for $a > 0$. It is described that the functions $f_{\lambda,a}(z)$ and $g_{\lambda,a}(z)$ have infinitely many singular values. It is also found that all the singular values of $f_{\lambda,a}(z)$ are bounded and lie inside the open disk centered at origin and having radius $|\lambda a|$ and all the critical values of $g_{\lambda,a}(z)$ belong to the exterior of the disk centered at origin and having radius $|\frac{\lambda}{a}|$.

Keywords: Critical values, singular values

1. Introduction

Singular values are enormously applicable in the dynamics of functions for the description of the Julia sets and the Fatou sets. The dynamics of functions, which have finite singular values, are widely studied by many researchers. But, in the presence of infinite number of singular values, it is very crucial to investigate the dynamical properties of such functions [Nayak & Prasad (2010, 2014); Prasad (2005); Rottenfusser, Ruckert, Rempe & Schleicher (2011); Sajid & Kapoor (2004)]. Recently, the singular values of one parameter families of functions are found by Lim (2016) and Sajid (2014a, 2014b).

A point z^* is said to be a critical point of $f(z)$ if $f'(z^*) = 0$. The value $f(z^*)$ corresponding to a critical point z^* is called a critical value of $f(z)$. A point $w \in \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is said to be an asymptotic value for $f(z)$, if there exists a continuous curve $\gamma : [0, \infty) \to \hat{\mathbb{C}}$ satisfying $\lim_{t\to\infty} \gamma(t) = \infty$ and $\lim_{t\to\infty} f(\gamma(t)) = w$. A singular value of f is defined to be either a critical value or an asymptotic value of f. A function f is called critically bounded or it is said to be a function of bounded type if the set of all singular values of f is bounded, otherwise unbounded-type. The importance of singular values in the dynamics of a transcendental functions can be seen in [Morosawa, Nishimura, Taniguchi & Ueda (2000); Zheng (2010)].

In this work, the singular values of two parameter families of functions (i) $\lambda \dfrac{e^{az}-1}{z}$ for $a \in \mathbb{R}\backslash\{0\}$, are described which is a generalized family of functions $\lambda \dfrac{e^{z}-1}{z}$ [Kapoor & Prasad (1998)] and $\lambda \dfrac{b^{z}-1}{z}$ [Sajid (2015a)]; and (ii) $\lambda \frac{z}{e^{az}-1}$ are described which is a generalization of one parameter families of functions $\lambda \frac{z}{e^{z}-1}$ [Sajid (2015b)] and $\lambda \frac{z}{b^{z}-1}$ [Sajid (2015c)]. For this purpose, the following two parameter families of functions are considered

$$\mathcal{F} = \left\{ f_{\lambda,a}(z) = \lambda \frac{e^{az}-1}{z} \text{ and } f_{\lambda,a}(0) = \lambda a : \lambda, a \in \mathbb{R}\backslash\{0\}, z \in \mathbb{C} \right\}$$

$$\mathcal{G} = \left\{ g_{\lambda,a}(z) = \lambda \frac{z}{e^{az}-1} \text{ and } g_{\lambda,a}(0) = \frac{\lambda}{a} : \lambda, a \in \mathbb{R}\backslash\{0\}, z \in \mathbb{C} \right\}$$

The functions $f_{\lambda,a} \in \mathcal{F}$ and $g_{\lambda,a} \in \mathcal{G}$ are transcendental entire and meromorphic respectively; these are neither even nor odd and not periodic.

This paper is organized as follows: For $a < 0$, it is shown that, in Theorem 1, the functions $f'_{\lambda,a}(z)$ and $f'_{\lambda,a}(z)$ have no zeros in the left half plane; and $f_{\lambda,a}(z)$ and $g_{\lambda,a}(z)$ map the right half plane inside the open disk centered at origin and having radius $|\lambda a|$ and exterior of the open disk centered at origin and having radius $|\frac{\lambda}{a}|$ respectively. For $a > 0$, it is also found that, in Theorem 2, the functions $f'_{\lambda,a}(z)$ and $f'_{\lambda,a}(z)$ have no zeros in the right half plane; and $f_{\lambda,a}(z)$ and $g_{\lambda,a}(z)$ map the left half plane inside the open disk centered at origin and having radius $|\lambda|a$ and exterior of the open disk centered at origin and having radius $\frac{|\lambda|}{a}$ respectively. It is proved that the function $f_{\lambda,a} \in \mathcal{F}$ and $g_{\lambda,a} \in \mathcal{G}$ have infinitely many singular values in Theorem 3. In Theorem 4, it is seen that all the singular values of $f_{\lambda,a} \in \mathcal{F}$ are bounded and lie inside the open disk centered at origin and having radius $|\lambda a|$ and in Theorem 5, all the critical values of $g_{\lambda,a} \in \mathcal{G}$ belong to the exterior of the open disk centered at origin and having radius $|\frac{\lambda}{a}|$.

Let us denote $H^- = \{z \in \hat{\mathbb{C}} : Re(z) < 0\}$ and $H^+ = \{z \in \hat{\mathbb{C}} : Re(z) > 0\}$ the left half plane and the right half plane respectively.

Lemma 1 *The equation $e^{-w} = 1 - w$ has no zeros in H^+.*

The rigorous proof of this lemma can be seen in [Kapoor & Prasad (1998)] and a short proof is found in [Sajid (2015b)].

Lemma 2 *Let $h(z) = e^{az}$ for an arbitrary fixed $z \in \mathbb{C}$. Then,*

(i) for $a < 0$ and $z \in H^+$

(ii) for $a > 0$ and $z \in H^-$,

the following inequality holds

$$|e^{az} - 1| < |z||a| \tag{1}$$

Proof. Suppose that the line segment γ is defined by $\gamma(t) = tz$, $t \in [0, 1]$. Then,

$$\int_\gamma h(z)dz = \int_0^1 h(\gamma(t))\gamma'(t)dt = z\int_0^1 e^{atz}dt = \frac{1}{a}(e^{az} - 1)$$

(i) Since $M_1 \equiv \max_{t\in[0,1]} |h(\gamma(t))| = \max_{t\in[0,1]} |(e^a)^{tz}| < 1$ for $a < 0$ and $z \in H^+$, then

$$|e^{az} - 1| = \left|a\int_\gamma h(z)dz\right| \leq M_1|z||a| < |z||a|$$

(ii) Since $M_2 \equiv \max_{t\in[0,1]} |h(\gamma(t))| = \max_{t\in[0,1]} |(e^a)^{tz}| < 1$ for $a > 0$ and $z \in H^-$, then

$$|e^{az} - 1| = \left|a\int_\gamma h(z)dz\right| \leq M_2|z||a| < |z||a|$$

This completes the proof of lemma.

Lemma 3 *The equation $\frac{v}{\sin(v)} - e^{v\cot(v)-1} = 0$ has infinitely many solutions.*

The rigorous theoretical proof of this lemma is described in [Kapoor & Prasad (1998)] and the graphical proof is found in [Sajid (2015b)].

2. Singular Values of $f_{\lambda,a} \in \mathcal{F}$ and $g_{\lambda,a} \in \mathcal{G}$

Let $D_r(0)$ be the open disk centered at origin and radius r. The functions $f'_{\lambda,a}(z)$ and $f'_{\lambda,a}(z)$ have no zeros in the left half plane; and $f_{\lambda,a} \in \mathcal{F}$ and $g_{\lambda,a} \in \mathcal{G}$ map the right half plane inside the open disk centered at origin and having radius $|\lambda a|$; and exterior of the open disk centered at origin and having radius $|\frac{\lambda}{a}|$ respectively for $a < 0$, are found in the following theorem:

Theorem 1 *Let $f_{\lambda,a} \in \mathcal{F}$ and $g_{\lambda,a} \in \mathcal{G}$ for $a < 0$. Then,*

(a) $f'_{\lambda,a}(z)$ and $g'_{\lambda,a}(z)$ have no zeros in the left half plane H^-.

(b) $f_{\lambda,a}(z)$ and $g_{\lambda,a}(z)$ map the right half plane H^+ inside $D_{|\lambda a|}(0)$ and exterior of $D_{|\frac{\lambda}{a}|}(0)$ respectively.

Proof. (a) Since $f'_{\lambda,a}(z) = \lambda\frac{(az-1)e^{az}+1}{z^2} = 0$ and $g'_{\lambda,a}(z) = \lambda\frac{(1-az)e^{az}-1}{(e^{az}-1)^2} = 0$, these gives $e^{-az} = 1 - az$. Put $w = az$, then, $e^{-w} = 1 - w$. By Lemma 1, it follows that the functions $f'_{\lambda,a}(z)$ and $g'_{\lambda,a}(z)$ have no zeros in the left half plane H^- for $a < 0$ since $z = \frac{w}{a}$.

(b) Using Inequality (1) for Lemma 2 (i), we have

$$\left|\frac{e^{az} - 1}{z}\right| < |a| \text{ for all } z \in H^+.$$

It follows that

$$|f_{\lambda,a}(z)| = |\lambda\frac{e^{az} - 1}{z}| < |\lambda a| \text{ for all } z \in H^+.$$

Hence, $f_{\lambda,a}(z)$ maps H^+ inside $D_{|\lambda a|}(0)$.

Similarly, by Inequality (1) for Lemma 2 (i), we get

$$\left|\frac{z}{e^{az} - 1}\right| > \frac{1}{|a|} \text{ for all } z \in H^-.$$

It gives that

$$|g_{\lambda,a}(z)| = \left|\lambda \frac{z}{e^{az}-1}\right| > \left|\frac{\lambda}{a}\right| \text{ for all } z \in H^-.$$

This shows that $g_{\lambda,a}(z)$ maps H^- exterior of $D_{|\frac{\lambda}{a}|}(0)$. The proof of theorem is completed for $a < 0$.

In the following theorem, the similar results are shown as Theorem 1 for $a > 0$:

Theorem 2 *Let $f_{\lambda,a} \in \mathcal{F}$ and $g_{\lambda,a} \in \mathcal{G}$ for $a > 0$. Then,*

(i) $f'_{\lambda}(z)$ and $g'_{\lambda,a}(z)$ have no zeros in the right half plane H^+.

(ii) $f_{\lambda,a}(z)$ and $g_{\lambda,a}(z)$ map the left half plane H^- inside $D_{|\lambda|a}(0)$ and exterior of $D_{\frac{|\lambda|}{a}}(0)$ respectively.

For $a > 0$, the proof of the theorem can be similarly obtained as Theorem 1 by using Lemma 1 and Inequality (1) for Lemma 2 (ii).

The following theorem describes that the functions $f_{\lambda,a} \in \mathcal{F}$ and $g_{\lambda,a} \in \mathcal{G}$ have infinitely many singular values:

Theorem 3 *Let $f_{\lambda,a} \in \mathcal{F}$ and $g_{\lambda,a} \in \mathcal{G}$. Then, the functions $f_{\lambda,a}(z)$ and $g_{\lambda,a}(z)$ possess infinitely many singular values.*

Proof. For critical points, $f'_{\lambda,a}(z) = 0$ and $g'_{\lambda,a}(z) = 0$. It follows that $(az-1)e^{az} + 1 = 0$. Assume $w = az$, then $(w-1)e^{w} + 1 = 0$. Using the real and imaginary parts of this equation

$$\frac{v}{\sin(v)} - e^{v\cot(v)-1} = 0 \tag{2}$$

$$u = 1 - v\cot(v) \tag{3}$$

By Lemma 3, it is seen that Equation (2) has infinitely many solutions. Let $\{v_k\}_{k=-\infty,k\neq 0}^{k=\infty}$ be the solutions of Equation (2). Then, from Equation (3), $u_k = 1 - v_k\cot(v_k)$ for k nonzero integer. For $z_k = \frac{u_k+iv_k}{a}$, the critical values $f_{\lambda,a}(z_k)$ and $g_{\lambda,a}(z_k)$ are distinct for different k. It shows that the functions $f_{\lambda,a}(z)$ and $g_{\lambda,a}(z)$ have infinitely many critical values for both $a < 0$ and $a > 0$.

The finite asymptotic value of $f_{\lambda,a}(z)$ is 0 since $f_{\lambda,a}(z) \to 0$ as $z \to \infty$ along the positive real axis for $a < 0$ and negative real axis $a > 0$ respectively. Similarly, the finite asymptotic value of $g_{\lambda,a}(z)$ is 0 since $g_{\lambda,a}(z) \to 0$ as $z \to \infty$ along the negative real axis for $a < 0$ and positive real axis for $a > 0$.

Therefore, it follows that the functions $f_{\lambda,a} \in \mathcal{F}$ and $g_{\lambda,a} \in \mathcal{G}$ possess infinitely many singular values for both $a < 0$ and $a > 0$.

In the following theorem, it is proved that $f_{\lambda,a} \in \mathcal{F}$ has bounded singular values and lie inside the open disk:

Theorem 4 *Let $f_{\lambda,a} \in \mathcal{F}$. Then, all the singular values of $f_{\lambda,a}(z)$ are bounded and lie inside $D_{|\lambda a|}(0)$.*

Proof. For $a < 0$, by Theorem 1 (a), the function $f'_{\lambda,a}(z)$ has no zeros in the left half plane H^-. Hence, all the critical points lie in the right half plane H^+. By using Theorem 1 (b), the function $f_{\lambda,a}(z)$ maps the right half plane H^+ inside $D_{|\lambda|a}(0)$. It follows that all the critical values of $f_{\lambda,a}(z)$ are lying inside $D_{|\lambda a|}(0)$ for $a < 0$.

Similarly, for $a > 0$, using Theorem 2 (i) and (ii), it is easily deduce that all the critical values of $f_{\lambda,a}(z)$ lie inside $D_{|\lambda a|}(0)$.

Since $f_{\lambda,a}(z)$ has only one asymptotic value 0, so all the singular values of $f_{\lambda,a} \in \mathcal{F}$ are bounded and lie inside $D_{|\lambda a|}(0)$.

The following theorem shows that the function $g_{\lambda,a} \in \mathcal{G}$ has all the critical values belong to the exterior of the open disk:

Theorem 5 *Let $g_{\lambda,a} \in \mathcal{G}$. Then, all the critical values of $g_{\lambda,a}(z)$ belong to the exterior of $D_{|\frac{\lambda}{a}|}(0)$.*

Proof. For $a < 0$, by Theorem 1 (a), the function $g'_{\lambda,a}(z)$ has no zeros in the left half plane H^-. It follows that all the critical points lie in the right half plane H^+. But, by using Theorem 1 (b), the function $g_{\lambda,a}(z)$ maps H^+ onto the exterior of $D_{|\frac{\lambda}{a}|}(0)$. Consequently, all the critical values of $g_{\lambda,a} \in \mathcal{G}$ belong to the exterior of $D_{|\frac{\lambda}{a}|}(0)$ for $a < 0$.

For $a > 0$, using similar arguments as above, by Theorem 2 (i) and (ii), it deduce that all the critical values of $g_{\lambda,a} \in \mathcal{G}$ belong to the exterior of $D_{\frac{|\lambda|}{a}}(0)$.

References

Kapoor, G. P., & Prasad, M. G. P. (1998). Dynamics of $(e^z - 1)/z$: the Julia set and bifurcation. *Ergod. Th. & Dynam. Sys., 18 (6)*, 1363-1383. http://dx.doi.org/10.1017/S0143385798118011

Lim, D. (2016). Fixed points and dynamics on generating function of Genocchi numbers. *J. Nonlinear Sci. Appl., 9 (3)*, 933-939.

Morosawa, S., Nishimura, Y., Taniguchi, M., & Ueda, T. (2000). *Holomorphic Dynamics*. Cambridge University Press.

Nayak, T., & Prasad, M. G. P. (2010). Iteration of certain meromorphic functions with unbounded singular values. *Ergod. Th. & Dynam. Sys., 30 (3)*, 877-891. http://dx.doi.org/10.1017/S0143385709000364

Nayak, T., & Prasad, M. G. P. (2014). Julia sets of Joukowski-Exponential maps. *Complex Anal. Oper. Theory, 8 (5)*, 1061-1076. http://dx.doi.org/10.1007/s11785-013-0335-1

Prasad, M. G. P. (2005). Chaotic burst in the dynamics of $f_\lambda(z) = \lambda\frac{\sinh(z)}{z}$. *Regul. Chaotic Dyn., 10 (1)*, 71-80. http://dx.doi.org/10.1070/RD2005v010n01ABEH000301

Rottenfusser, G., Ruckert, J., L. Rempe, L., & Schleicher, D. (2011). Dynamics rays of bounded-type entire functions. *Ann. Math., 173 (1)*, 77-125. http://dx.doi.org/10.4007/annals.2011.173.1.3

Sajid, M. (2014a). Singular values and real fixed points of one parameter family of function $zb^z/(b^z - 1)$. *Journal of Mathematics and System Science, 4 (7)*, 486-490.

Sajid, M. (2014b). Singular values of a family of singular perturbed exponential map. *British Journal of Mathematics and Computer Science, 4 (12)*, 1678-1681. http://dx.doi.org/10.9734/BJMCS/2014/9598

Sajid, M. (2015a). Singular values of one parameter family $\lambda\frac{b^z-1}{z}$. *Journal of Mathematics and Computer Science, 15 (3)*, 204-208.

Sajid, M. (2015b). Singular values and fixed points of family of Generating Function of Bernoulli's numbers. *J. Nonlinear Sci. Appl., 8 (1)*, 17-22.

Sajid, M. (2015c). Singular values of one parameter family of Generalized Generating Function of Bernoulli's numbers. *Appl. Math. Inf. Sci., 9 (6)*, 2921-2924. http://dx.doi.org/10.12785/amis/090619

Sajid, M., & Kapoor, G. P. (2004). Dynamics of a family of non-critically finite even transcendental meromorphic functions. *Regul. Chaotic Dyn., 9 (2)*, 143-162. http://dx.doi.org/10.1070/RD2004v009n02ABEH000272

Zheng, J. H. (2010). On fixed-points and singular values of transcendental meromorphic functions. *Science China Mathematics, 53 (3)*, 887-894. http://dx.doi.org/10.1007/s11425-010-0036-4

Copyrights