Journal of Mathematics Research; Vol. 7, No. 4; 2015
ISSN 1916-9795 E-ISSN 1916-9809
Published by Canadian Center of Science and Education

A Strong Class of Lifted Valid Inequalities for the Shortest Path Problem in Digraphs with Negative Cost Cycles

Mamane Souleye Ibrahim[1]

Correspondence: Mamane Souleye Ibrahim. Université A. Moumouni, Faculté des Sciences et techniques, BP 10.622, Niamey, Niger. E-mail: ibrah_dz@yahoo.fr

Received: September 17, 2015 Accepted: October 13, 2015 Online Published: November 24, 2015

doi:10.5539/jmr.v7n4p162 URL: http://dx.doi.org/10.5539/jmr.v7n4p162

Abstract

In this paper, we address a strong class of lifted valid inequalities for the shortest path problem in digraphs with possibly negative cost cycles. We call these lifted inequalities the *incident lifted valid inequalities* (*ILI*) as they are based on the incident arcs of a given vertex. The *ILI* inequalities are close in spirit of the so-called *simple lifted valid inequalities* (*SLI*) and *cocycle lifted valid inequalities* (*CLI*) introduced in Ibrahim et al. (2015). However, as we will see the *ILI* inequalities are stronger than the first ones in term of linear relaxation strengthening. Indeed, contrary to *SLI* and *CLI* inequalities, consider the same instances, in a cutting plane algorithm, the computational results prove that the *ILI* inequalities provide the optimal integer solution for all the considered instances within no more than three iterations except one case for which after the first strengthening iteration, there exists no generated inequality.

Keywords: polytope, digraphs, shortest path, valid inequality, lifting

1. Introduction

Let $G = (V, A)$ be a general directed graph, where V represents the vertex set and A the set of arcs with an arc cost function $w : A \to \mathbb{R}$. Consider two given vertices $s \in V$ and $t \in V$. We define a *s*-*t* path as a sequence $(v_0, a_1, \ldots, a_k, v_k)$, where $k \geq 1$, $v_0, v_1, \ldots v_k$ are vertices, $v_0 = s$, $v_k = t$, and a_i is an arc connecting v_{i-1} and v_i $(i = 1, \ldots k)$. An elementary *s*-*t* path is a sequence $(v_0, a_1, \ldots, a_k, v_k)$ in which each vertex v_i, $(i = 0 \ldots k)$ appears once, ie, $v_i \neq v_j, \forall i \neq j$. In the sequel, we denote by $\mathcal{P}$ the polytope induced by all the $s - t$ elementary directed paths of the digraph G and p_i, $i \in \{1, \ldots, q\}$ is the i^{th} $s - t$ elementary path of G. That is, we assume that the digraph G contains q $s - t$ elementary paths.

We consider the problem that consists in searching an elementary shortest path from the source vertex s to the sink vertex t (*single origin - single destination shortest path problem*) in digraphs containing negative cost cycles. In addition to the *single origin - single destination shortest path problem*, notice that there exists others versions of shortest path problem, as the *single-source all-destinations shortest path problem* and the *all-pairs shortest paths problem*. In general digraphs containing negative cost cycles, the shortest path problem is well-known to be NP-hard as it includes the asymmetric traveling salesman problem as a special case (see also Garey and Johnson (1979)). We recall that without negative cost cycles, the *single-source all-destinations shortest path problem* and the *all-pairs shortest paths problem* can be solved easily by standard known algorithms such as Djikstra's algorithm or Bellman-Ford algorithm for the *single-source all-destinations shortest path problem* (see Bellman (1958), Djiskstra (1959), Ford and Fulkerson (1962)) and Floyd-Warshall algorithm for the *all-pairs shortest paths problem* (see Floyd (1962)).

In this paper, we continue our investigation begun in Ibrahim et al. (2015) about a cutting plane algorithm devised for the *single origin - single destination shortest path problem* with possibly negative cost cycles. We recall that the algorithm is based on a MIP formulation of the *single origin - single destination shortest path problem* in digraphs possibly containing negative cost cycles. Previously, we have used the so-called *simple lifted valid inequalities* (*SLI*) and *cocycle lifted valid inequalities* (*CLI*) (see, Ibrahim et al. (2015) to build strong linear relaxations. Here, we address a new class of lifted valid inequalities called the *incident lifted valid inequalities* (*ILI*) as they are based on the incident arcs of a given vertex. As we will see, the latter lifted inequalities, namely the *ILI* inequalities, are stronger than the first ones introduced in Ibrahim et al. (2015) in term of linear relaxation strengthening. Indeed, contrary to *SLI* and *CLI* inequalities, consider the same instances tested in Ibrahim et al. (2015), in a cutting plane process, the computational results prove that the *ILI* inequalities provide the optimal integer solution for all the considered instances within no more than three iterations except one case for which after the first strengthening iteration, there exists no inequality to be generated.

The paper is organized as follow, in section 2, we recall the MIP formulation of the *s*-*t* shortest path problem based on a non-simultaneous flow model used in the resolution process and the *SLI* and *CLI* inequalities. We also present a new class of lifted inequalities for $\mathcal{P}$. Section 3 is devoted to the computational results proving the superiority of the lifted valid inequalities presented in this paper comparatively to the ones introduced in Ibrahim et al. (2015).

2. A Strong Class of Lifted Valid Inequalities for $\mathcal{P}$

Consider the sub-digraph $G_X = (X, E)$ supporting the (fractional) optimal solution of the linear relaxation of the following $s-t$ elementary shortest path MIP model:

$$(P): \ \min \sum_{(i,j)\in A} w_{ij} y_{ij},$$

subject to:

$$\sum_{j\in\Gamma^+(s)} z_{sj}^k - \sum_{j\in\Gamma^-(s)} z_{js}^k = x_k, \ k \in V \setminus \{s\} \tag{1}$$

$$\sum_{j\in\Gamma^+(i)} z_{ij}^k - \sum_{j\in\Gamma^-(i)} z_{ji}^k = 0, \ k \in V - \{s\}, \ i \in V \setminus \{s,k\} \tag{2}$$

$$\sum_{j\in\Gamma^+(k)} z_{kj}^k - \sum_{j\in\Gamma^-(k)} z_{jk}^k = -x_k, \ k \in V \setminus \{s\} \tag{3}$$

$$z_{ij}^k \le y_{ij}, \ (i,j) \in E, \ k \in V \setminus \{s\} \tag{4}$$

$$\sum_{j\in\Gamma^+(i)} y_{ij} = x_i, \ i \in V \setminus \{s,t\} \tag{5}$$

$$\sum_{j\in\Gamma^-(i)} y_{ji} = x_i, \ i \in V \setminus \{s,t\} \tag{6}$$

$$\sum_{j\in\Gamma^+(s)} y_{sj} = 1 \tag{7}$$

$$\sum_{j\in\Gamma^-(t)} y_{jt} = 1 \tag{8}$$

$$z_{ij}^k \ge 0, \ k \in V \setminus \{s\}, \ (i,j) \in A \tag{9}$$

$$x_i \in \{0,1\}, \ y_{ij} \in \{0,1\}, \ i \in V, \ j \in A. \tag{10}$$

Where x_i is a binary variable associated to the vertex $i \in V$, y_{ij} is a binary variable associated to the arc $(i, j) \in A$ and the variable $z_{ij}^k \ge 0$ represents the flow that passes through the arc $(i, j) \in A$, from the source vertex s to the terminal vertex $k \in V \setminus \{s\}$. $\Gamma^+(i)$ and $\Gamma^-(i)$ denote the sets of arcs going out from the vertex i and coming into the vertex i, respectively. For further details about the above shortest path formulation, one can refer to Maculan et al. (2003), Ibrahim et al. (2009).

According to the results presented in Ibrahim et al. (2015), with respect to the sub-digraph $G_X = (X, E)$ supporting the (fractional) optimal solution of the linear relaxation, we have that

$$\sum_{w\in S_k} x_w + \sum_{(u,v)\in F_k} y_{u,v} \le k \tag{11}$$

is valid to the polytope $\mathcal{P}_X$ of all the $s-t$ elementary directed paths of the sub-digraph G_X.

Where (S_k, F_k) is k-subset pair. That is $S_k \subset X$, $F_k \subset E$ and $\forall \ p_i, \ i \in \{1, \ldots, q\}$, where p_i is a $s-t$ directed path of G_X, we have $|(S_k \cap V(p_i))| + |(F_k \cap E(p_i))| \le k$ and $|S_k| + |F_k| = k + 1$.

In Ibrahim et al. (2015), we have seen that the valid inequality (11) generated on G_X can be transformed into the following SLI inequality of *order k* for the whole digraph G.

$$\sum_{w\in S_k} x_w + \sum_{(u,v)\in F_k} y_{u,v} - \sum_{(u',v')\in A\setminus E} y_{u',v'} \le k \tag{12}$$

In the same way, the valid inequality (11) can also be transformed into the following *CLI* inequalities of *order k* for the whole digraph G.

$$\sum_{w\in S_k} x_w + \sum_{(u,v)\in F_k} y_{u,v} - \sum_{(u',v')\in w^+(S)} y_{u',v'} \leq k \tag{13}$$

$$\sum_{w\in S_k} x_w + \sum_{(u,v)\in F_k} y_{u,v} - \sum_{(u',v')\in w^-(S)} y_{u',v'} \leq k \tag{14}$$

Where $S = X$ is the set of vertices of the sub-digraph $G_S = (S, E_S)$ such that $E_S = \{(u', v') \in A : u', v' \in X\}$, $w^-(S)$ is a cutset having all its arcs outgoing from S and $w^+(S)$ is a cutset having all its arcs incoming to S.

Proposition 1. For the sake of simplicity, we assume that $|S_k| = k + 1$ and $F_k = \emptyset$. The following lifted valid inequality

$$\sum_{w\in S_k} x_w - \sum_{(u',w)\in A\setminus E} y_{u',w} - \sum_{(w,v')\in A\setminus E} y_{w,v'} \leq k \tag{15}$$

is valid for $\mathcal{P}$. We call the valid inequality (15) *the incident lifted valid inequality* of *order k* as it is based on the incident arcs of the vertices $w \in S_k$.

Proof. Consider any elementary $s - t$ path in G. If this path passes only through the nodes and arcs of G_X, then $\sum_{(u',w)\in A\setminus E} y_{u,v} = 0$ and $\sum_{(w,v')\in A\setminus E} y_{u,v} = 0$. So, the inequality (15) holds because (11) is valid for all elementary $s-t$ paths in G_X. Otherwise, under the hypothesis that it is passed at least a $s-t$ path through every vertex $w \in S_k$, such a path uses exactly one incoming arc to w, say (u', w), or exactly one outgoing arc (w, v') from w. So, $\sum_{(u',w)\in A\setminus E} y_{u',w} + \sum_{(w,v')\in A\setminus E} y_{w,v'} \leq k + 1$, and hence (15) holds. □

Remark. The above lifted inequality can be written by taking into account both the vertices of S_k and the arcs of F_k.

Proposition 2. The inequality (15) is stronger than the inequality (12).

Proof. Consider the inequality (15). Let $A^- = \{(u', w) \in A \setminus E : w \in S_k\}$ and $A^+ = \{(w, v') \in A \setminus E : w \in S_k\}$. It's obvious that the inequality (15) is stronger than the inequality (12), as $(A^- \cup A^+) \subset A \setminus E$. □

In the following section, the computational experiments confirm the theoretical results of the above proposition.

3. Computational Results

We report on computational experiments on the same instances of digraphs randomly generated and tested in Ibrahim and al. (2015). We recall that these instances feature large integrality gaps and contain negative cost cycles. Consider the linear relaxation of the *single origin - single destination shortest path problem* flow based linear formulation (1-10) presented in section 2, in a cutting plane framework, we use the incident lifted valid inequalities of order $k = 0$, to build strengthened linear relaxations of the above formulation (1-10). For more details on the generation of the instances that we deal with, one can refer to Ibrahim et al. (2015). To generate the subset pair (S_k, F_k), with $k = 0$, we resort to the enumerative procedure used in Ibrahim et al. (2015). Such an enumerative procedure is efficient as it is run with respect to the sub-digraph G_X which contains generally a few number of $s - t$ elementary paths.

Algorithms are implemented in C and all computations have been carried out on a computer equipped with a 1.50 GHz Intel (R) core (TM) 2 CPU. All instances are solved by open software *glpk* (see GNU Linear Programming Kit, 2007). In the following tables, the first ten selected problems (problem1 to problem10) have $100 = 10 \times 10$ vertices and the others (problem11 to problem18) have $200 = 20 \times 10$ vertices.
The meaning of the figures shown in the following table is as follows:

- The first column of table 1 presents the identifiers of the different instances;

- On the second, third and fourth columns of table 1, we display the relative gap improvements gsl (%), gcl (%) and gil (%) obtained by strengthening with simple lifted valid inequalities (SLI), cocycle-lifted valid inequalities (CLI) and incident lifted valid inequalities of order $k = 0$, respectively. The relative improvement is expressed in percents and is computed as $\frac{\bar{z}_{LI}-\bar{z}}{z_{spp}-\bar{z}} * 100$, where $\bar{z}_{LI}$ is the optimal value obtained after lifted inequalities strenghening, $\bar{z}$ is the optimal value of the linear relaxation of the considered problem and z_{spp} is the optimal value of the shortest path;

- On the fifth column, we have the number of iterations #*nbrit* that suffices to produce the optimal integer solution.
- On the sixth column, we show the number #*ILI* of generated incident lifted inequalities of order 0.

Table 1. Gap improvements obtained using *SLI*, *CLI* and *ILI*

name	gsl (%)	gcl (%)	gil (%)	#nbrit	#ILI
problem1	17.64	35.29	50	2	4
problem2	0.87	1.75	100	1	7
problem3	2.61	7.70	100	1	4
problem4	2.94	5.88	6.66	2	1
problem5	13.30	75	100	1	9
problem6	2.27	18.18	100	1	1
problem7	7.73	53.33	53.33	1	6
problem8	0.90	3.45	62.06	1	7
problem9	23.07	57.38	61.53	3	3
problem10	10.72	32	100	1	6
problem11	13.79	65.51	100	1	5
problem12	30	80	100	1	1
problem13	0.5	1	100	1	11
problem14	4.41	11.76	100	1	11
problem15	0	0	31.86	2	3
problem16	3.03	18.8	100	1	4
problem17	5.36	16.90	50	3	9
problem18	4.25	12.5	100	1	2

On all the lines of the above table, we observe that $gil > gcl \geq gsl$ except the line represented by the instance named problem 7 for which we have $gil = gcl$. That proves that the ILI inequalitied of order 0 are significantly stronger than the SLI and CLI studied in Ibrahim et al. (2015). Moreover for 11 instances among 18 considered, a strengthening with the ILI inequalities of order 0 directly displays the optimal integer solution (ie, the optimal $s-t$ directed path. This is characterized by the fact that for all these 11 instances, its corresponding values on the fourth and on the fifth columns are equals to 100% and 1, respectively. That shows that, in the cutting plane process, only one iteration suffices to display the optimal integer solution. So, 100% of gap have been closed. Nevertheless, for some of the considered instances, we observe on the fifth column that to obtain the optimal $s-t$ elementary path, we have to perform two or three iterations of strengthening by using some ILI inequalities of order 0. As example, consider the instance named problem1, the gap improvement obtained by the first strengthening iteration is equal to 50% (see the fourth column of the first line). The fifth column of the first line shows that (2) strengthening iterations have been performed in view to obtain the optimal $s-t$ elementary path. And (4) ILI inequalities have been generated, (see the cell of the last column corresponding to the first line).

4. Conclusion

We continue our investigation on the shortest path problem between two given vertices in digraphs possibly containing negative cycles. In previous papers, we present a flow based *single origin single destination* linear formulation of the shortest path problem with possibly negative cost cycles in the considered digraphs and we introduce some lifted valid inequalities used to perform a cutting plane algorithm in view to solve efficiently the problem. We have presented two types of lifted valid inequalities called the *simple lifted valid inequalities* and the *cocycle lifted valid inequalities*. In this paper, we address a new class of lifted valid inequalities named the incident lifted valid inequalities as they are based on the incident arcs of a given vertex. Computational results, carried out on the same digraphs with size up to 200 vertices show that by using the *incident lifted inequalities*, an iterative strengthening procedure provides the exact integer optimal solution in less than three iterations for all of the test examples except one instance for which after the first strengthening iteration, there exists no generated lifted valid inequalities. This results feature that the incident lifted valid inequalities are significantly stronger than the first lifted inqualities previously introduced, (we recall that, consider the same instances of digraphs and the *simple lifted valid inequalities*, the integrality gap is 100% closed in about half of our considered instances within ten iterations of cutting plane generation).

References

Bellman, R. (1958). On routing problem. *Quartely of Applied Mathematics, 16*, 87-90.

Dijkstra, E. W. (1959). A note on two problems in connexion with graphs. *Numerische Mathematik, 1,* 269-271. http://dx.doi.org/10.1007/BF01386390

Floyd, R. W. (1962). Algorithm 97 (Shortest Path). *Communications of the ACM, 5*(6), 345. http://dx.doi.org/10.1145/367766.368168

Ford, L. R., & Fulkerson, D. R. (1962). *Flows in networks*. Princeton University Press. Hillsdale, NJ: Erlbaum.

Garey, M., & Johnson, D. (1979). *Computer and Intractibility: a guide to the theory of NP-completeness*. Freeman, San Francisco.

Ibrahim, M. S., Maculan, N., & Minoux, M. (2009). Strong flow-based formulation for the shortest path problem in digraphs with negative cycles. *International Transaction in Operations Research (ITOR), 16*, 361-369. http://dx.doi.org/10.1111/j.1475-3995.2008.00681.x

Ibrahim, M. S., Maculan, N., & Minoux, M. (2015). *Problème du plus court chemin avec des longueurs négatives*. Editions universitaires européennes, Saarbrucken (Allemagne).

Ibrahim, M. S., Maculan, N., & Minoux, M. (2015). Valid inequalities and lifting procedures for the shortest path problem in digraphs with negative cycles. *Optimization letters, 9,* 345-357. http://dx.doi.org/10.1007/s11590-014-0747-5

Maculan, N., Plateau, G., & Lisser, A. (2003). Integer linear models with a polynomial number of variables and constraints for some classical combinatorial optimization problems. *Pesquisa Operacional, 23,* 161-168. http://dx.doi.org/10.1590/S0101-74382003000100012