Journal of Mathematics Research; Vol. 7, No. 4; 2015
ISSN 1916-9795 E-ISSN 1916-9809
Published by Canadian Center of Science and Education

On The Minimal Number of Elements of Markov Partitions for Pseudo-Anosov Homeomorpisms

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Received: October 31, 2015 Accepted: November 16, 2015 Online Published: November 24, 2015

doi:10.5539/jmr.v7n4p157 URL: http://dx.doi.org/10.5539/jmr.v7n4p157

Abstract

In the paper an estimation of the minimal number of elements of Markov partition for generalized pseudo-Anosov homeomorphism of closed non necessary orientable surface is given. It is formulated in terms of characteristic of invariant foliation of generalized pseudo-Anosov homeomorphism.

Keywords: pseudo-Anosov homeomorphism, Markov partition, invariant foliation, singular type

1. Introduction

Pseudo-Anosov homeomorphisms of orientable surfaces were introduced by W. Thurston (1988) in his research on J. Nielsen classification of surface homeomorphisms up to isotopy. Markov partitions are very useful tools for investigation of geometrical and dynamical properties of such homeomorphisms. The number of theirs elements is essential in particulary for combinatoric description of these partitions and for calculating the number of periodic points of homeomorphism and of entropy. It is well known that in particular case of hyperbolic homeomorphism of 2-torus there Markov partition onto two elements exists and of course there is no Markov partition with one element. We will consider the common case.

We begin with definitions.

We will mind rectangle $[0, 1] \times [0, 1] \subset \mathbb{R}^2$ with its partition onto intervals of horizontal lines under *non-singular coordinate neighborhood with nonsingular foliation*. Also, the *singular coordinate neighborhood with d-pronged ($d \in \mathbb{N}$, $d \neq 2$) singularity* will be the neighborhood of origin in $\mathbb{R}^2$ fibered onto subsets of two types: 1) intervals of d rays issuing out of origin (singular leaves); 2) arcs of convex curves lying in sectors between these rays and asymptotic to them (in the case $d = 1$ these are arcs of parabolas with singular leaf as the mirror symmetry axis). We will say that d is the *valency* of singularity.

Definition 1 Let M be a closed surface, $S \subset M$ its finite subset. *Singular foliation* of M with *singularities* in S is the family of linear connected subsets of M called *leaves* such that

(1) M is the union of all leaves;

(2) intersection of any two leaves either is empty, or is contained in S;

(3) for each $x \in M \setminus S$ (correspondingly $x \in S$) neighborhood U and its homeomorphism onto nonsingular (correspondingly singular of some valency d) coordinate neighborhood in $\mathbb{R}^2$ which maps x to origin and any linear connected component of intersection with U of any leaf onto leaf of corresponding coordinate foliation exists.

Definition 2 The homeomorphism $f : M \to M$ is said to *generalized pseudo-Anosov (GPA)* if it preserves two transversal foliations $\mathcal{W}^u$, $\mathcal{W}^s$ with common singularities expanding leaves of $\mathcal{W}^u$ with the factor $\lambda > 1$ and contacting leaves of $\mathcal{W}^s$ with the factor λ^{-1}. The number λ is called the *dilatation* of f.

Remarks.

1) Because there is no need in the smooth structure of surface and of its foliations we understand transversality in the following sense. Two arcs γ_1, γ_2 are transversal if their intersection points are isolated and for each intersection point x exists neighborhood $U \ni x$ and its homeomorphism φ to $\mathbb{R}^2$ which maps x to origin end parts of curves lying in U onto intervals of horizontal and vertical axis correspondingly.

2) Expanding and contracting in this definition are understood usually with respect to transversal invariant measures μ^s, μ^u. The transversal measure μ^s is the family of Borel measures defined on the arcs of leaves of $\mathcal{W}^u$ so that if two arcs are

fiber homotopic (with respect to $\mathcal{W}^u$), then their measures are equal. The transversal measure μ^u is defined by swapping symbols s and u. So the property of foliations in the definition 1 means $\mu^s(\gamma) = \lambda\mu^s(\gamma)$ and $\mu^u(\gamma) = \lambda^{-1}\mu^u(\gamma)$ for γ arc of leaf of $\mathcal{W}^u$ (of $\mathcal{W}^s$ correspondingly).

Definition 3 Let $f : M \to M$ be a generalized pseudo-Anosov homeomorphism. The *rectangle* is a closed set $\Pi \subset M$ that is the image of the map $\varphi : [0,1] \times [0,1] \to M$ so that it is one-to-one on the interior of square and maps each horizontal (vertical) interval onto arc of contracting (expanding) leaf. Let us denote by Π° image of interior of square. The images of horizontal (vertical) sides of this square are called *contracting (expanding) sides* of rectangle Π.

Definition 4 *Markov partition* for the generalized pseudo-Anosov homeomorphism $f : M \to M$ is finite family of rectangles $\mathcal{P} = \{\Pi_1, \ldots, \Pi_n\}$ such that $\Pi_i^\circ \cap \Pi_j^\circ = \varnothing$ (for $i \neq j$), $\cup_{i=1}^n \Pi_i = M$ and $f(\partial^s\mathcal{P}) \subset \partial^s\mathcal{P}$, $f(\partial^u\mathcal{P}) \supset \partial^u\mathcal{P}$. Here $\partial^s\mathcal{P}$ and $\partial^u\mathcal{P}$ are unions of contracting (expanding) sides of all rectangles of $\mathcal{P}$.

It is well known that for Anosov diffeomorphism of 2-torus (which essentially is the same as GPA-homeomorphism with no singularities) there exist Markov partition consisting of two rectangles. It is easy to see that there are no Markov partitions with one element for it. It is natural question on the minimal number of rectangles for arbitrary GPA-homeomorphism. In this paper we will establish en estimation from above for this number. We need two additional definitions to formulate the final result.

Definition 5 The *singular type* of generalized pseudo-Anosov homeomorphism f is the sequence of $\mathcal{S} = \mathcal{S}(f) := \{s_d : d \in \mathbb{N}\}$ element s_d of which is the number of d-pronged singularities of invariant foliations of f.

Evidently, only finite number of elements of the sequence $\mathcal{S}(f)$ are non-zero.

Definition 6 Let us say that the family $\mathbf{W} = W$ of leaves of f-invariant foliation (either $\mathcal{W}^s$ or $\mathcal{W}^u$) is *invariant* if $W \in \mathbf{W}$ implies $f(W) \in \mathbf{W}$. The leaf W is *periodic of period m* if $f^k(W) = W$ and $f^i(W) \neq W$ for $0 < i < k$.

Theorem *Let f is the generalized pseudo-Anosov homeomorphism of the closed surface. Let $\mathcal{S} = \{s_d : d \in \mathbb{N}\}$ is its singular type and m —the minimal period of periodic leaves of its contracting foliations. Then minimal number of elements of Markov partitions for f is*

$$n_{\min} \leq m + \frac{1}{2}\sum_d d s_d. \tag{1}$$

To be sure that this estimate is valid for the Anosov diffeomorphism, it is natural to assume that the latter has a unique singularity of valence 2 in its arbitrary fixed point. Note that there are other reasons for such point of view on invariant foliations of Anosov diffeomorphisms.

Let us note also that because f maps singular leaves onto singular leaves and singular points to singular points of the same valency, the minimal number of periodic leaves may be estimated by

$$m \leq \min_{s_d \neq 0}\{d s_d\}.$$

Consequently, the minimal number of the elements of Markov partitions may be estimated be means of the singular type of GPA-homeomorphism.

The proof of the theorem consists of the explicit constructing of Markov partition with the number of elements equal to the value in right side of (1).

2. The Proof of the Theorem

In what follows is assumed that $f : M \to M$ is GPA-homeomorphism, $\mathcal{W}^s$, $\mathcal{W}^u$ are its contracting and expanding foliations and $\mathcal{S} = \{s_d\}$ its singular type.

To prove the theorem we consider two cases:

1) The periodic contracting leaf with minimal period M is singular;

2) The periodic contracting leaf with minimal period M is nonsingular.

In the proof we will use well known properties of invariant foliations of GPA-homeomorphisms whose proves can be found in (Fathi, 1979). These properties are valid for both foliations $\mathcal{W}^s$ and $\mathcal{W}^u$.

1. There are no closed leaves and no leaves joining singular points.

2. Each singular leaf is dense in the surface. The same is true for the both linear connected components of every non-singular leaf onto which arbitrary point divide it.

3. Each periodic leaf contain periodic point.

Let us begin with the proof in the first case

Lemma 1. *Let W be a periodic singular leaf of period m of the contracting foliation of f and $W_k := f^k(W)$ ($0 \le k < m$). Then there exist arcs $w_k \subset W_k$ with the following properties:*
1) $f(w_k) \subset w_{k+1}$, $f(w_m) \subset w_0$;
2) one endpoint of w_k is singular point and another one belongs to expanding leaf beginning from some singular point;
3) for any singular point p whose arc of expanding leaf connects p with endpoint of some w_k this arc do not intersect none of arcs w_l ($0 \le l < m$) in their interior points.

Proof. Let p_k be a singular periodic point belonging to the leaf W_k. Note that some of these points may coincide (in the case that corresponding leaves coming out of the same singularity). Let $x_0 \neq p_0$ be an arbitrary point of the leaf W_0. Define points $x_k := f^k(x_0)$ ($0 < k < m$) each belongs to the leaf W_k. Denote by $(p_k, x_k)_s$ an open arc of W_k joining points p_k and x_k. Let p be some singular point (possibly one of p_k or not) and arbitrary expanding leaf coming out of this point. Denote by y_0 the point of first intersection of this leaf with some of arcs $(p_k, x_k)_s$ i.e. $y_0 \in (p_l, x_l)_s$ for some l and the arc $(p, y_0)_u$ of this leaf (joining p and y_0) do not intersect other arcs $(p_k, x_k)_s$. Such point y_0 exists because each leaf of expanding and contracting foliation is dense in the surface.

Let us suppose that the point y_0 belongs to the arc $(p_0, x_0)_s$. In other case we can to renumber leaves W_k and, correspondingly, points p_k and x_k so that $f(P_k, x_k)_s \subset (p_{k+1}, x_{k+1})s$ for $k < m-1$, $f(p_{m-1}, x_{m-1})s \subset (p_0, x_0)_s$ and $(p_k, x_k)_s \cap (p, y_0)_u = \varnothing$ for all $k \neq 0$.

Now we define points $y_k := f^K(y_0)$ for $1 \le k \le m-1$. Then $y_k \in (p_k, x_k)_s$ so that $f(p_k, y_k)_s = (p_{k+1}, y_{k+1})_s$ for $k < m-1$ and $f(p_{m-1}, y_{m-1})_s \subset (p_0, y_0)_s$. Consequently, the family arcs $(p_k, y_k)_s$ satisfy conditions 1,2 of the Lemma 1 but possibly do not satisfy to the condition 3.

We change this family to obtain the family satisfying this condition too. For each k consider the set Y_k which is the intersection with $(p_k, y_k]_s$ (it is semi-open arc) all arcs $(f^i p, y_i]_u$ ($0 \le i \le m-1$). Let q_k be such point of the set Y_k that the arc $(p_k, q_k)_s$ does not contain other points of this set. Let $w_k := (p_k, q_k)_s$. Then family of arcs w_k satisfy to the conditions 2 and 3 and we need to prove that the condition 1 remains to be true.

To do this consider the sets Y_k. First let us note that because $(p, y_0)_u \cap (p_k, y_k)_s = \varnothing$ for all $k \le m-1$, then for each $i \le k$ we have

$$(p_k, y_k)_s \cap (f^i p, y_i)_u = f^i((p_{k-i}, y_{k-i})_s \cap (p, y_0)_u) = \varnothing. \tag{2}$$

Hence

$$Y_k \setminus \{y_k\} = \bigcup_{i=k}^{m-1} (p_k, y_k)_s \cap (f^i p, y_i)_u.$$

It follows that $Y_m \setminus \{y_m\} = \varnothing$ i.e. $q_m = y_m$. For $k \geqslant 1$ we have

$$f^{-1}(Y_k \setminus \{y_k\}) = \bigcup_{i=k-1}^{m-1} (p_{k-1}, y_{k-1})_s \cap (f^{i-1} p, y_{i-1})_u = \bigcup_{i=k-2}^{m-2} (p_{k-1}, y_{k-1})_s \cap (f^i p, y_i)_u \subset Y_{k-1} \setminus \{y_{k-1}\}.$$

This means that $f^{-1}(q_k) \notin (p_{k-1}, q_{k-1})_s$ that is $f((p_{k-1}, q_{k-1})_s) \subset (p_k, q_k)_s$.

It remains to prove that $f((p_{m-1}, q_{m-1})_s) \subset (p_0, q_0)_s$. If $q_0 = y_0$ then

$$f((p_{m-1}, q_{m-1})_s) = f((p_{m-1}, y_{m-1})_s) = f^m((p_0, q_0)_s) \subset (p_0, q_0)_s.$$

In other case $q_0 \in (f^i p, y_i)_u$ for some i, $0 < i \le m-1$. Then $f^{-1}(q_0) \in (f^{-1} p, y_{-1})_u$. According to (2) the arc $(f^{-1} p, y_{i-1})_u$ do not intersect the arc $(p_{m-1}, y_{m-1})_s$. Hence the inclusion

$$(p_{m-1}, f^{-1}(q_0))_s \subset (p_{m-1}, y_{m-1})_s = p_{m-1}, q_{m-1\,s}$$

is false. Consequently $f((p_{m-1}, q_{m-1})_s) \subset (p_{m-1}, y_{m-1})_s = (p_{m-1}, q_{m-1})_s$ as required. □

Lemma 2 *If there exist the family of arcs $w_0, \ldots, w_{m-1}$ enabling the properties 1-3 of lemma 1 then*
1) there exists Markov partition $\mathcal{P}$ with $\partial^s \mathcal{P} = \bigcup_{k=1}^{m} w_k$;
2) the number of elements of $\mathcal{P}$ is $m + \frac{1}{2} \sum_d d s_d$.

Proof. Let us denote by $\Gamma^s := \bigcup_{k=0}^{m-2} w_k$. The property 1 of Lemma 2 implies that $f(\Gamma^s) \subset \Gamma^s$. Let p_k be the singular endpoint of w_k. Let q_k be another endpoint and of w_k and p'_k be the singular point arc $[p'_k, q_k]_u$ of the expanding leaf that joins p'_k and q_k. What is more, the arc $(p'_k, q_k]_u$ does not intersect w_k excepting point q_k and does not intersect other arcs w'_k. Let us remind that the points p_k, p'_k are not necessary distinct, possibly all of them coincide (for example, in the case when f has only one singular point). Let us continue each arc $(p'_k, q_k]_u$ to the first intersection with interior of one of the arcs w_l $(0 \le l \le m-1)$. Denote Γ^u_1 the union of all these arcs. Now consider each expanding singular leaf W^u which does not contain any arc in Γ^u_1 (if such leaf exists). Let it begins in the singular point p (again it may be that this is one of the points p_k, p'_k). Let x be such point of W^u that x belongs to the interior of one of arcs w_k and $(p, x)_u$ does not intersect any of w_k. Let Γ^u_2 be the union of all such arcs and $\Gamma^u := \Gamma^u_1 \cup \Gamma^u_2$. It is easy to see that image of each of arcs of which the set Γ^u is constructed contains one of these arcs. So $f(\Gamma^u) \supset \Gamma^u$.

Consider the set $\mathcal{P}^\circ$ of connected components of the set $M \setminus \Gamma^s \cup \Gamma^u$. Evidently it consists of the finite number of elements. We denote them $\Pi^\circ_1, \dots, \Pi^\circ_n$. Also denote their closures Π_i, define the family of subsets $\mathcal{P} := \{\Pi_i : 1 \le i \le n\}$ of M and show that it is Markov partition.

Let us begin with the checking that each Π_i is a rectangle. Because Π°_i does not contain singular point, each point $x \in \Pi^\circ_i$ enables of neighborhood U and homeomorphism φ of some rectangle in $\mathbb{R}^2$ mapping it onto U in such a way that horizontal and vertical intervals are mapped onto arcs of contracting and expanding foliations. Beginning with some point $x \in \Pi^\circ_i$ we can extend this neighborhood to the set U_ε $(0 < \varepsilon < 1)$ in such a way that the boundary of U_ε is close enough to the boundary of Π°_i and corresponding homeomorphism φ_ε maps onto it the rectangle $[\varepsilon, 1-\varepsilon] \times [\varepsilon, 1-\varepsilon]$ mapping horizontal (vertical) intervals onto arcs of contracting (expanding) leaves. Then the closure of $\bigcup_\varepsilon U_\varepsilon$ is Π_i and continuation of φ_ε onto Π_i satisfy the properties of the definition of rectangle.

Now it is easy to see that the constructed family of rectangles $\mathcal{P}$ is Markov partition. It follows from the fact that $\partial^s\mathcal{P} = \Gamma^s$, $\partial^u\mathcal{P} = \Gamma^u$ and $f(\Gamma^s) \subset \Gamma^s$, $f(\Gamma^u) \supset \Gamma^u$.

Now let us calculate the number n of the elements of $\mathcal{P}$. The contracting boundary of each rectangle $\Pi_i \in \mathcal{P}$ is the union of two arcs $u_{i,1}, u_{i,2}$ of contracting leaves (possibly of the same leaf). Each of these arcs is contained in some arc w_k $(0 \le k \le m-1)$. It is possible that both of them are contained in the same w_k. In this case they may be intersected but can not coincide because otherwise it would be exist closed curve consisting of expanding arcs. Consider the set $\mathcal{U}$ being the set of all such contracting arcs $u_{i,j}$. It consists of $2n$ elements. Each endpoint of each arc w_k is the common point of two arcs in $\mathcal{U}$. Then in $\bigcup_{k=0}^{m-1} w_k$ there are $2n - 2m$ endpoints of arcs of $\mathcal{U}$ which are the interior points of corresponding arcs w_k. The construction implies that each of such points belongs to some singular expanding leaves and each of such leaves contains such point. Because the number of all singular expanding leaves is $\sum_d ds_d$ it follows

$$2n - 2m = \sum_d ds_d. \tag{3}$$

This proves the lemma. □

All that is lacking is to prove the theorem for the second case.

Lemma 3 *Let there exists a non-singular contracting leaf of the period m then there exists Markov partition with $m + \frac{1}{2}\sum_d ds_d$ elements.*

Proof. Let W be a non-singular leaf of contracting foliation with period m. Because each non-singular leaf is periodic there exists Markov partition $\mathcal{P}' = \{\Pi'_1, \dots, \Pi'_{m'}\}$ constructed in the proof of the lemma 2 (here m' is the period of some singular contracting leaf). Let us denote all objects in the construction of $\mathcal{P}'$ by the same symbols but with an accent. In particulary, those arcs which union is $\partial^s\mathcal{P}'$ we denote as $w'_0, \dots, w'_{m'-1}$. Let $p_0 \in W$ be a periodic point. It follows from the construction of the partition $\mathcal{P}'$ that the set $\partial^s\mathcal{P}' \cup \partial^u\mathcal{P}'$ does not contain non-singular periodic points. So there exists element Π' of $\mathcal{P}'$ containing p_0. Let w_0 be an arc of W, containing Π', contained in Π' and having its endpoints on the contracting boundary of Π'. Then each arc $w_k = f^k(w_0)$ $(1 \le k \le m-1)$ lying in rectangles of Π' and has endpoints on contracting boundary of the corresponding rectangle. Denote $\Gamma^s := \bigcup_k w_k$. Evidently $f(\Gamma^s) \subset \Gamma^s$.

Now consider each singular leaf W^u beginning in some singular point p and define the first point of its intersection x_1 with the arcs w_k. If x_1 is an endpoint of sum arc of Γ^s let us continue the arc $[p, x_1]_u$ up to the the second intersection with these arcs and do so until we get an arc $[p, x]_u$ with x in interior of some w_k. Denote union of all these arcs as Γ^u. Evidently $f(\Gamma^u) \supset \Gamma^u$.

Just the same way as in the proof of Lemma 2 we see that the closure of each connected component of the set $M \setminus (\Gamma^s \cup \Gamma^u)$ is an rectangle, the set of all of them is Markov partition and the number n of its elements satisfy the equality (3). This finish the proof of the lemma. □

Thus, the statement of the theorem is proved in both cases above.

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