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Liquidity Premiums in a Lévy Market

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Abstract

This paper gives a theorem for the continuous time super-replication cost of European options where the stock price follows an exponential Lévy process. Under a mild assumption on the legend transform of the trading cost function, the limit of the sequence of the discrete super-replication cost is proved to be greater than or equal to an optimal control problem. The main tool is an approximation multinomial scheme based on a discrete grid on a finite time interval [0,1] for a pure jump Lévy model. This multinomial model is constructed similar to that proposed by (Szimayer & Maller, Stoch. Proce. & Their Appl., 117, 1422-1447, 2007). Furthermore, it is proved that the existence of a liquidity premium for the continuous-time model under a Lévy process. This paper concentrates on the Lévy processes with infinitely many jumps in any finite time interval. The approach overcomes some difficulties that can be encountered when the Lévy process has infinite activity.

Keywords: Lévy process, liquidity premium, optimal control

1. Introduction

In this paper, the problem of the continuous time super-replication cost of a European option in a one-dimensional Lévy model is studied.

Related problems were studied by (Cetin, Soner & Touzi, 2010) and (Gokay & Soner, 2012). Both of the two works showed that the continuous time super-replication cost for a binomial model exists. In (Gokay & Soner, 2012), it was shown that a stochastic optimal control problem could be seen as a dual of the super-replication problem. (Gokay & Soner, 2012) got the duality result implicitly through a partial differential equation and not by a straightforward argument. And, the proof given in it was limited to Markovian claims. (Dolinsky & Soner, 2013) took the one-dimensional binomial version of model proposed by (Cetin, Jarrow & Protter, 2004), which was also adopted in (Cetin & Roger, 2007) and (Gokay & Soner, 2012). Their paper was based on nonmarkovian claims and more general liquidity functions, which was an extension of (Gokay & Soner, 2012). In (Dolinsky & Soner, 2013), the dual of the discrete model was derived as an optimal control problem and the construction given in (Kusuoka, 1995) was applied to prove that a liquidity premium exists.

However, all the continuous time super-replication works so far have been based on a binomial model. That is, the conclusions are all under the condition that the stock price at time t is given by $S_t = s_0 exp(B_t)$, where B_t is a Brownian motion. However, the restriction to Brownian motion does not allow for infinite activity of the Lévy process that is actually frequently used for building stochastic models in finance, economics and many other fields.

In this present paper, the continuous time super-replication problem is extended to one-dimensional Lévy model. The existence of the liquidity premium is proved which should have the practical importance in the real world. For background of Lévy processes, the readers are referred to (Applebaum, 2004), (Bertoin, 1996) and (Sato, 1999). Special emphasis is placed on Lévy processes that have infinitely many jumps, almost surely, in any finite time interval.

The duality result of the discrete model given in Theorem 3.1 of (Dolinsky & Soner, 2013) remains correct for our multinomial model. To show the existence of the liquidity premium, the main tool is a multinomial approximation scheme that is similar to that proposed by (Maller, Solomon & Szimayer, 2006). This multinomial model is based on a discrete grid, in a finite time interval [0, 1], and having a finite number of states, for a Lévy process. In (Maller & Szimayer, 2007), each jump step of the multinomial scheme is actually the first jump with certain size in each subinterval. And, it is proved that the discrete multinomial approximation sequence converges to the continuous time Lévy model in mean under the Skorokhod J_1 topology. To show this, the uniformly boundedness of the discrete stock price has to be shown first. The Dominant convergence theorem will be adopted to complete the proofs.

The remainder of this paper is organized as follows. In section 2, the set-up is outlined. The main convergence results of this paper are presented and proved in Section 3. Section 4 gives the conclusions and discussions.

2. Setup and Preliminaries

In this present section, we will first construct a multinomial model and introduce the super-replication problem. And then, the Lévy model will be defined.

2.1 Super-replication Problem

First of all, a multinomial model is constructed. This multinomial scheme is similar to that proposed by (Maller et al., 2006). For each $n = 1, 2, \cdots$, the number of time steps in $[0, 1]$ is denoted by $N(n)$. For a fixed $M > 0$, $n \in \mathbb{N}$, let $\mathcal{M}(n) = \{\frac{i}{N(n)n}, i \in \mathbb{N}\} \cap (-M, M)$. Let $\Omega_n = \mathcal{M}(n)^{N(n)}$ be the space of $\omega(n) = (\omega_1(n), \omega_2(n), \cdots, \omega_{N(n)}(n))$ with the product probability, where $\omega_i(n) \in \mathcal{M}(n)$. Define the canonical sequence of random variables $X_i(n), i = 1, \cdots, N(n)$ by

$$X_i(n)(\omega(n)) = \omega_i(n),$$

and consider the natural filtration $\mathcal{F}_k(n) = \sigma\{X_1(n), X_2(n), \cdots, X_k(n)\}$, $1 \le k \le N(n)$, and let $\mathcal{F}_0(n)$ be trivial. For any n, suppose the $N(n)$-step multinomial model of a financial market which is active at times $0, \frac{1}{N(n)}, \frac{2}{N(n)}, \cdots, 1$. Assume that the discrete stock price at time $\frac{k}{N(n)}$ is given by

$$S_n(k) = S_0 \exp\left(\sum_{j=1}^{k} X_j(n)\right), \quad k = 0, 1, \cdots, N(n). \tag{1}$$

Let $\mathbb{D}[0, 1]$ be the space of all càdlàg functions from $[0, 1]$ to $\mathbb{R}$.

Secondly, I recall some definitions and preliminaries given in (Dolinsky & Soner, 2013). Let $F : \mathbb{D}[0, 1] \to \mathbb{R}_+$ be a continuous map. Assume that there exist constants $L, p > 0$ for which

$$F(y) \le L\left(1+ \| y \|_{Sk}^{p}\right), \quad \forall y \in \mathbb{D}[0, 1], \tag{2}$$

where $\| \cdot \|_{Sk}$ is the Skorokhod norm. For the Skorokhod J_1 topology, the readers are referred to (Karatzas & Shreve, 1991) and (Jacod & Shiryaev, 2003). Let the payoff function of a European claim with maturity $T = 1$ be

$$F_n := F(S_n). \tag{3}$$

Let $g(t, S, v)$ be the trading cost function at time t, where S is the stock price and v is the trading volume at time t. Assume that $g : [0, 1] \times \mathbb{D}[0, 1] \times \mathbb{R} \to [0, \infty)$ is nonnegative, adapted, convex for every $(t, S) \in [0, 1] \times \mathbb{D}[0, 1]$, and $g(t, S, 0) = 0$. The Legendre transform of g is given by

$$G(t, S, y) = \sup_{z \in \mathbb{R}}(yz - g(t, S, z)), \ \forall (t, S, y) \in [0, 1] \times \mathbb{D}[0, 1] \times \mathbb{R}. \tag{4}$$

Let $\pi = \left(x, \{v(k)\}_{k=0}^{N(n)}\right)$ be a self-financing portfolio strategy where x is the initial capital and $v(k)$ is the number of stocks the investor holds at time $k/N(n)$ before a transfer is made at this time. Assume that $v(0) = 0$ and $v(k)$ is an $\mathcal{F}_{k-1}$−measurable random variable. The portfolio value of π is defined given by

$$Y^{\pi}(k+1) = Y^{\pi}(k) + v(k+1)(S_n(k+1) - S_n(k)) - g\left(\frac{k}{N(n)}, S_n, v(k+1) - v(k)\right)$$

for $k = 0, 1, \cdots, N(n) - 1$ with initial value $Y^{\pi}(0) = x$.

The super-replication price of a European claim with payoff F_n is defined as

$$V_n := V_n(g, F_n) = \inf\{x | \ \exists \pi \in \Pi_n^x \ with \ Y^{\pi}(n : g) \ge F_n \ \mathbb{P} - a.s.\},$$

where Π_n^x is the set of all portfolios with initial capital x.

At last, the Theorem 3.1 in (Dolinsky & Soner, 2013) is recalled in the following remark. It claims that a dual of the super-replication price is an optimal control problem in which the controller is allowed to choose any probability measure on $(\Omega_n, \mathcal{F}(n))$ in their binomial model. This conclusion is also true for the multinomial model in this present paper.

Remark 1 Let Q_n be the set of all probability measures on $(\Omega_n, \mathcal{F}(n))$. Then

$$V_n = \sup_{\mathbb{P} \in Q_n} \mathbb{E}^{\mathbb{P}}\left(F_n - \sum_{k=0}^{N(n)-1} G\left(\frac{k}{N(n)}, S_n, \mathbb{E}^{\mathbb{P}}(S_n(N(n)) | \mathcal{F}_k(n)) - S_n(k)\right)\right),$$

where $\mathbb{E}^{\mathbb{P}}$ denotes the expectation with respect to a probability measure $\mathbb{P} \in Q_n$.

2.2 Lévy Model

Let $L = (L_t,\ 0 \le t \le 1)$ be a Lévy process with càdlàg paths defined on a completed probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathcal{F}$ is as in the Definition 2 below. Let $\mathbb{F}^L = (\mathcal{F}_t^L)_{0\le t\le 1}$ be the natural filtration generated by $(L_t,\ 0 \le t \le 1)$. Assume that the Lévy triplet of $(L_t,\ 0 \le t \le 1)$ is $(\gamma,\ \sigma,\ \nu) = (0,\ 0,\ \nu)$, where γ is the drifting term, σ is the volatility of the Brownian motion part, and the Lévy measure ν is the intensity of the jump process of the Lévy process. Thus, this Lévy process is determined by the Lévy measure, ν. By the Lévy-Ito Decomposition,

$$L_t = \int_0^t \int_{|x|<1} x\tilde{N}(dt, dx) + \int_0^t \int_{|x|\ge 1} xN(dt, dx), \tag{5}$$

where $N(\cdot,\ \cdot)$ is the associated independent Poisson random measure process on $\mathbb{R}^+ \times (\mathbb{R}\backslash\{0\})$ with intensity ν. And, for $t \in [0,\ 1]$ and $A \in \mathcal{B}(\mathbb{R}\backslash\{0\})$, $\tilde{N}(t, A) = N(t, A) - t\nu(A)$ is the compensate Poisson process.

Definition 1 *Let θ be a product measure taking the form $\theta((0,\ t] \times A) = t\nu(A)$ for each $t \ge 0$, $A \in \mathcal{B}(\mathbb{R})$. Let $E \in \mathcal{B}(\mathbb{R})$. Let $\mathcal{H}_2(1,\ E)$ be the linear space of all equivalent classes of mappings $H : [0,\ 1] \times E \times \Omega \to \mathbb{R}$ which coincide almost everywhere with respect to $\theta \times \mathbb{P}$ and which satisfy the following conditions:*
(1) H is predictable;
(2) $\|H\|^2 \triangleq \int_0^1 \int_E \mathbb{E}(H(t,\ x))^2 \nu(dx)dt < \infty$.
Especially, write $\mathcal{H}_2(1,\ \{0\}) = \mathcal{H}_2(1)$.

Remark 2 Naturally, $\mathcal{H}_2(1,\ \mathbb{R})$ is isomorphic to $L^2(\mathbb{R},\ \nu)\otimes\mathcal{H}_2(1)$, where $L^2(\mathbb{R},\ \nu)$ is the set of square $\nu-$integral functions.

For any $H \in \mathcal{H}_2(1,\ \mathbb{R})$. By Remark 2, there exists $f(t) \in \mathcal{H}_2(1)$ and $g(x) \in L^2(\mathbb{R},\ \nu)$ such that $H(t,\ x) = f(t)g(x)$. Let $\mathbb{F}^f = (\mathcal{F}_t^f)_{t\in[0,1]}$ be the natural filtration generated by $(f(t),\ t \in [0, 1])$.

Definition 2 *Define that $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,1]}$, where for any $t \in [0, 1]$, $\mathcal{F}_t$ is the smallest $\sigma-$algebra containing $\mathcal{F}_t^f$ and $\mathcal{F}_t^L$. Suppose that $\mathcal{F}_0$ contains all $\mathbb{P}-$null sets and that $\mathcal{F}_1 = \mathcal{F}$.*

Definition 3 *For any $H(t,\ x) \in \mathcal{H}_2(1, \mathbb{R})$, let $S_H(t) \triangleq s_o \exp(\varepsilon_t)$, where*

$$\varepsilon_t = \int_0^t \int_{\mathbb{R}} H(s,\ x)N(ds,\ dx) - \int_0^t \int_{\mathbb{R}} \left[e^{H(s,\ x)} - 1\right] \nu(dx)ds. \tag{6}$$

By Corollary 5.2.2 of (Applebaum, 2004), $S_H(t)$ is a local martingale.

3. Main Results

In this section, the existence of liquidity premium is showed. The convergence results are presented and proved.

3.1 Existence of Liquidity Premium

Assumption 1 Assume that G defined in (4) satisfies the following growth and scaling conditions:

(a) There are constants $C,\ p$, and $\beta \ge 2$ such that

$$G(t,\ S,\ y) \le Cy^\beta(1+ \|\ S\ \|_\infty)^p,\ \forall(t,\ S,\ y) \in [0,\ 1] \times \mathbb{D}[0,\ 1] \times \mathbb{R}.$$

(b) There exists $m(n) \downarrow 0$ and a continuous function

$$\hat{G} :\ [0,\ 1] \times \mathbb{D}[0,\ 1] \times \mathbb{R} \to [0,\ \infty]$$

such that $\lim_{n\to\infty} \frac{\bar{\nu}(m(n))^2}{N(n)} = 0$ and, for any bounded sequence $\{x_n\}$ and convergent sequences $t_n \to t,\ S_n \to S$ in the Skorohod norm,

$$\lim_{n\to\infty} \left| N(n)G\left(t_n,\ S_n,\ \frac{x_n\bar{\nu}(m(n))^2}{N(n)} S_n(t)\right) - \hat{G}(t,\ S,\ x_nS(t))\right| = 0.$$

The main result of this paper is stated in the following theorem. The proof will be given in subsection 3.3.

Theorem 1 *Let G be a dual function satisfying Assumption 1(a), and let $\widehat{G}$ be as in Assumption 1(b). Then,*

$$\lim_{n\to\infty} V_n \ge \sup_{H\in\mathcal{A}_M} J(S_H),$$

where

$$J(S_H) := \mathbb{E}\left\{F(S_H) - \int_0^1 \widehat{G}(t,\ S_H,\ \frac{(t-1)S_H(t)}{2})dt\right\},$$

$\mathcal{A}_M := \{H(t,\ x) \in \mathcal{H}_2(1,\mathbb{R}) :\ |H| < M\ and\ \mathbb{E}(S_H(t) = 1)\ for\ any\ t \in [0,\ 1]\}.$

Remark 3 It follows from the Theorem 5.2.4 in (Applebaum, 2004) that $S_H(t)$ is a martingale if $\mathbb{E}(S_H(t) = 1)$ for any $t \in [0,\ 1]$.

3.2 Multinomial Approximation Schemes and Lemmas

To show Theorem 1, we need the following constructions of multinomial schemes and lemmas.

Fix $H(t,\ x) \in \mathcal{A}_M$. Suppose that $H(t,\ x) = f(t)g(x)$ for some $f(t) \in \mathcal{H}_2(1)$ and $g(x) \in L^2(\mathbb{R},\ \nu)$. Since $|H| < M$ almost surely, it is harmless to assume that $|f| < \alpha$ and $|g| < \beta$ for some positive finite numbers, α and β, with probability 1. By the proof of Lemma 4.1.4 of (Applebaum, 2004), a sequence of simple processes, $f_n(t)$, can be constructed such that, as $n \to \infty$,

$$\int_0^1 \mathbb{E}|f_n - f|^2 dt \to 0. \tag{7}$$

Actually,

$$f_n(t) = f_n(t_{j-1}(n)) \tag{8}$$

if $t \in [t_{j-1}(n),\ t_j(n))$, $j = 1,\ \cdots,\ N(n)$, where $t_j(n)$'s are the grid points of an equal partition of the time interval $[0,\ 1]$. Let $H_n(t,\ x) = f_n(t)g_n(x)$, where $g_n(x) := g(x)1_{\{|x|>m(n)\}}$. Moreover, $|f_n| < \alpha$ and $|g_n| < \beta$ for all $n \in \mathbb{N}$.

Let $\tilde{S}_n(t) := s_0 \exp(\tilde{\varepsilon}_n(t))$, where

$$\tilde{\varepsilon}_n(t) := \int_0^t \int_{\mathbb{R}} H_n(s,\ x)N(ds,\ dx) - \int_0^t \int_{\mathbb{R}} \left[e^{H_n(s,\ x)} - 1\right] \nu(dx)ds. \tag{9}$$

Lemma 2 *We have*

$$\mathbb{E}\left(\sup_{t\in[0,\ 1]} \left|\tilde{S}_n(t) - S_H(t)\right|\right) \longrightarrow 0.$$

That is, $\tilde{S}_n \xrightarrow{L^1} S_H$ *as* $n \to \infty$. *Here and later, "*$\xrightarrow{L^1}$*" denotes convergence in mean.*

Proof. By the definition of $\varepsilon_n(t)$ and ε_t, we have

$$\begin{aligned}
&\mathbb{E}\left(\sup_{t\in[0,1]} |\tilde{\varepsilon}_n(t) - \varepsilon_t|\right)\\
\leq\ &\mathbb{E}\left(\sup_{t\in[0,1]} \left|\int_0^t \int_{\mathbb{R}} (H_n(s,x) - H(s,x))\tilde{N}(ds,\ dx)\right|\right)\\
&+\mathbb{E}\left(\sup_{t\in[0,1]} \left|\int_0^t \int_{\mathbb{R}} \left[\left(e^{H_n(s,x)} - 1 - H_n(s,x)\right) - \left(e^{H(s,x)} - 1 - H(s,x)\right)\right] \nu(dx)ds\right|\right).
\end{aligned}$$

First of all, consider that $\int_0^t \int_{\mathbb{R}} (H_n(s,x) - H(s,x))\tilde{N}(ds,\ dx)$ is a martingale for $H,\ H_n \in \mathcal{H}_2(1,\mathbb{R})$. Thus, by the Doob's martingale inequality, we have

$$\begin{aligned}
&\mathbb{E}^2\left(\sup_{t\in[0,1]} \left|\int_0^t \int_{\mathbb{R}} (H_n(s,x) - H(s,x))\tilde{N}(ds,\ dx)\right|\right)\\
\leq\ &\mathbb{E}\left(\sup_{t\in[0,1]} \left|\int_0^t \int_{\mathbb{R}} (H_n(s,x) - H(s,x))\tilde{N}(ds,\ dx)\right|^2\right)\\
\leq\ &4\mathbb{E}\left(\left|\int_0^1 \int_{\mathbb{R}} (H_n(s,x) - H(s,x))\tilde{N}(ds,\ dx)\right|^2\right)\\
\leq\ &4\mathbb{E}\left(\int_0^1 \int_{\mathbb{R}} (H_n(s,x) - H(s,x))^2 \nu(dx)ds\right) \overset{n\to\infty}{\longrightarrow} 0.
\end{aligned}$$

Here, the convergence follows from the construction of H_n.

Secondly, by the Taylor expansion,

$$\begin{aligned}
&\mathbb{E}\left(\sup_{t\in[0,1]}\left|\int_0^t\int_{\mathbb{R}}\left[\left(e^{H_n(s,x)}-1-H_n(s,x)\right)-\left(e^{H(s,x)}-1-H(s,x)\right)\right]\nu(dx)ds\right|\right)\\
=\ &\mathbb{E}\left(\sup_{t\in[0,1]}\left|\int_0^t\int_{\mathbb{R}}\sum_{k=2}^{\infty}\frac{H_n^k-H^k}{k!}\nu(dx)ds\right|\right)\\
=\ &\mathbb{E}\left(\sup_{t\in[0,1]}\left|\sum_{k=2}^{\infty}\left[\int_0^t\int_{|x|\le m(n)}\frac{-H^k}{k!}\nu(dx)ds+\int_0^t\int_{|x|>m(n)}\frac{H_n^k-H^k}{k!}\nu(dx)ds\right]\right|\right)\\
\le\ &\mathbb{E}\left(\sum_{k=2}^{\infty}\int_0^1\int_{|x|\le m(n)}\frac{|H|^k}{k!}\nu(dx)ds\right)+\mathbb{E}\left(\sum_{k=2}^{\infty}\int_0^1\int_{|x|>m(n)}\frac{|H_n^k-H^k|}{k!}\nu(dx)ds\right).
\end{aligned}$$

Since $H\in\mathcal{H}_2(1,\mathbb{R})$, $\mathbb{E}\int_0^1\int_{|x|\le m(n)}H^2\nu(dx)ds\to 0$ as $n\to\infty$.

$$\begin{aligned}
\mathbb{E}\left(\sum_{k=2}^{\infty}\int_0^1\int_{|x|\le m(n)}\frac{|H|^k}{k!}\nu(dx)ds\right)&\le\mathbb{E}\left(\sum_{k=2}^{\infty}\int_0^1\int_{|x|\le m(n)}\frac{H^2M^{k-2}}{k!}\nu(dx)ds\right)\\
&\le\sum_{k=2}^{\infty}\frac{M^{k-2}}{k!}\left(\mathbb{E}\int_0^1\int_{|x|\le m(n)}H^2\nu(dx)ds\right)\\
&\to 0.
\end{aligned}$$

Recall that, for any $t\in[0,1]$, $x\in\mathbb{R}$, $H_n(t,x)=f_n(t)g_n(x)$, where $g_n(x)=g(x)1_{|x|>m(n)}$, and that $|f_n|,\ |f|<\alpha$ and $|g|<\beta$. So,

$$\begin{aligned}
&\mathbb{E}\left(\sum_{k=2}^{\infty}\int_0^1\int_{|x|>m(n)}\frac{|H_n^k-H^k|}{k!}\nu(dx)ds\right)\\
=\ &\mathbb{E}\left(\sum_{k=2}^{\infty}\int_0^1\int_{|x|>m(n)}\frac{|g|^k|f_n^k-f^k|}{k!}\nu(dx)ds\right)\\
=\ &\sum_{k=2}^{\infty}\left[\int_{|x|>m(n)}\frac{|g|^k}{k!}\nu(dx)\mathbb{E}\left(\int_0^1|f_n^k-f^k|ds\right)\right]\\
\le\ &\sum_{k=2}^{\infty}\left[\frac{\beta^{k-2}}{k!}\int_{|x|>m(n)}g^2\nu(dx)\mathbb{E}\left(\int_0^1|f_n-f||f_n^{k-1}+f_n^{k-2}f+\cdots+f^{k-1}|ds\right)\right]\\
\le\ &\sum_{k=2}^{\infty}\left[\frac{k\beta^{k-2}\alpha^{k-1}}{k!}\int_{|x|>m(n)}g^2\nu(dx)\mathbb{E}\left(\int_0^1|f_n-f|ds\right)\right]\longrightarrow 0,
\end{aligned}$$

as $n\to\infty$, where the convergence follows from the boundedness of $\sum_{k=2}^{\infty}\frac{k\beta^{k-2}\alpha^k}{k!}$, $g\in L^2(\mathbb{R},\ \nu)$ and that $f_n\xrightarrow{L^2}f$.

Above all, $\varepsilon_n(t)\xrightarrow{L^1}\varepsilon_t$ under the uniform topology.

At last, consider that $\tilde{\varepsilon}_n(t)$ is a martingale. So, $\exp(\tilde{\varepsilon}_n(t))$ is a submartingale. By the Doob's martingale inequality, we have

$$\max_{n\ge1}\mathbb{E}^2\left(\sup_{t\in[0,1]}e^{\tilde{\varepsilon}_n(t)}\right)\le\max_{n\ge1}\mathbb{E}\left(\sup_{t\in[0,1]}e^{\tilde{\varepsilon}_n(t)}\right)^2\le\max_{n\ge1}4\mathbb{E}\left(e^{2\tilde{\varepsilon}_n(1)}\right).$$

Hence, we only need to show $\max_{n\ge1}\mathbb{E}\left(e^{2\tilde{\varepsilon}_n(1)}\right)<\infty$. Then, by the Dominant Convergence Theorem, the lemma follows immediately.

By (9),

$$e^{2\tilde{\varepsilon}_n(1)}=\exp\left\{\int_0^1\int_{\mathbb{R}}2H_n(s,\ x)N(ds,\ dx)-\int_0^1\int_{\mathbb{R}}2\left[e^{H_n(s,\ x)}-1\right]\nu(dx)ds\right\}.$$

Consider that

$$\begin{aligned}
& \mathbb{E}\left(\exp\left[\int_0^1 \int_{\mathbb{R}} 2H_n(s,\ x)N(ds,\ dx)\right]\right) \\
= \ & \mathbb{E}\left(\exp\left[\sum_{j=1}^{N(n)} \int_{\mathbb{R}} 2H_n(t_{j-1}(n),\ x)N([t_{j-1}(n), t_j(n)),\ dx)\right]\right) \\
= \ & \mathbb{E}\left(\prod_{j=1}^{N(n)} \exp\left[\int_{\mathbb{R}} 2H_n(t_{j-1}(n),\ x)N([t_{j-1}(n), t_j(n)),\ dx)\right]\right) \\
= \ & \mathbb{E}\left(\prod_{j=1}^{N(n)} \exp\left[(t_j(n) - t_{j-1}(n)) \int_{\mathbb{R}} (e^{2H_n(t_{j-1}(n),\ x)} - 1)\nu(dx)\right]\right) \\
= \ & \mathbb{E}\left(\exp\left[\sum_{j=1}^{N(n)} (t_j(n) - t_{j-1}(n)) \int_{\mathbb{R}} (e^{2H_n(t_{j-1}(n),\ x)} - 1)\nu(dx)\right]\right) \\
= \ & \mathbb{E}\left(\exp\left[\int_0^1 \int_{\mathbb{R}} (e^{2H_n(s,\ x)} - 1)\nu(dx)ds\right]\right),
\end{aligned}$$

where the third equality follows from the generating function of a compound Poisson process and the independence of increment for the Poisson process $N([t_{j-1}(n), t_j(n)),\ \cdot)$, $j = 1, 2, \cdots, N(n)$.

Then,

$$\begin{aligned}
& \mathbb{E}\left(e^{2\tilde{\varepsilon}_n(1)}\right) \\
= \ & \mathbb{E}\left(\exp\left[\int_0^1 \int_{\mathbb{R}} (e^{2H_n(s,\ x)} - 2e^{H_n(s,\ x)} + 1)\nu(dx)ds\right]\right) \\
= \ & \mathbb{E}\left(\exp\left[\int_0^1 \int_{\mathbb{R}} (e^{H_n(s,\ x)} - 1)^2\nu(dx)ds\right]\right) \\
\leq \ & e^{2M}\mathbb{E}\left(\exp\left[\int_0^1 \int_{\mathbb{R}} H_n(s,\ x)^2\nu(dx)ds\right]\right) \\
= \ & e^{2M}\mathbb{E}\left(\exp\left[\int_0^1 f_n(s)^2 ds \int_{\mathbb{R}} g_n(x)^2\nu(dx)\right]\right) \\
\leq \ & e^{2M}\left(e^{\alpha^2 \int_{\mathbb{R}} g_n(x)^2\nu(dx)}\right) \\
< \ & \infty,
\end{aligned}$$

where the first inequality follows from the assumption that $|H_n| < M$, the second from $|f_n| < \alpha$ and the last from $g_n \in L^2(\mathbb{R},\ \nu)$. Thus, the lemma follows.

Suppose the Lévy process given in (5) has infinitely many jumps in any finite time interval. Similar to the idea in (Szimayer & Maller, 2007), only the first jump in each subinterval with certain magnitude is taken.

Definition 4 *Let $\triangle L_t$ be the jump of the Lévy process at time $t \in [0,\ 1]$. For $j = 1,\ \cdots,\ N(n)$, let $\tau_j(n) = \inf\{t \in (t_{j-1}(n),\ t_j(n)] \mid |\triangle L_t| > m(n)\}$ and $Y_j^1(n)$ be the size of the first such jump occurs at $\tau_j(n)$ if there is such a jump. Define $\bar{S}_n(t) = s_0 \exp(\bar{\varepsilon}_n(t))$, where*

$$\begin{aligned}
& \bar{\varepsilon}_n(t) = \tilde{\varepsilon}_{n,1}(t) \\
:= \ & \sum_{j=1}^{N(n)} 1_{\{\tau_j(n)\leq t\}} \left\{ H_n(\tau_j(n),\ Y_j^1(n)) - \ln\left[1 + \frac{\int_{|x|>m(n)} (e^{H_n(\tau_j(n),\ x)} - 1)\nu(dx)}{\bar{\nu}(m(n))}\right]\right\}.
\end{aligned}$$

Here $\bar{\nu}(m(n)) = \nu([-m(n),\ m(n)]^c)$.

The following lemma shows that the sequence $\bar{S}_n$ is uniformly bounded and $|\tilde{S}_n - \bar{S}_n| \xrightarrow{L^1} 0$.

Lemma 3 *We have*

(i) $\max_{n\geq 1} \mathbb{E}\left(\sup_{t\in[0,1]} \bar{S}_n(t)\right) \leq \infty$*;*

(ii) $\mathbb{E}\left(\sup_{t\in[0,\ 1]}\left|\tilde{S}_n(t)-\bar{S}_n(t)\right|\right)\longrightarrow 0$ *as* $n\to\infty$.

Proof. (i)By definition 4. $\bar{S}_n(t)=s_0e^{\bar{\varepsilon}_n(t)}$ is a martingale. Indeed, let $\vartheta_j(n)$ be the number of jumps of L_t in the subinterval, $(t_{j-1}(n),\ t_j(n)]$, with magnitude greater than $m(n)$, and $Y_{j,k}(n)$ be the size of the kth such jump that occurs at time $t_j^k(n)$, $k=1,2,3,\cdots,\vartheta_j(n)$. Take expectation of $\exp\left[\sum_{k=1}^{\vartheta_j(n)}H_n(t_j^k(n),\ Y_{j,k}(n))\right]$ with respect to $Y_{j,k}(n)$'s:

$$\begin{aligned}
&\mathbb{E}_Y\left(\exp\left[\sum_{k=1}^{\vartheta_j(n)}H_n(t_j^k(n),\ Y_{j,k}(n))\right]\right)\\
=\ &\mathbb{E}_Y\left(\exp\left[\int_{\mathbb{R}}H_n(t_{j-1}(n),\ x)N([t_{j-1}(n),t_j(n)),\ dx)\right]\right)\\
=\ &\exp\left[(t_j(n)-t_{j-1}(n))\int_{\mathbb{R}}\left(e^{H_n(t_{j-1}(n),\ x)}-1\right)\nu(dx)\right].
\end{aligned}$$

On the other hand, consider that $\sum_{k=1}^{\vartheta_j(n)}H_n(t_j^k(n),\ Y_{j,k}(n))$ is a compound Poisson random variable with $\mathbb{E}(\vartheta_j(n))=\bar{\nu}(m(n))/N(n)$. By the generating function of a compound Poisson random variable,

$$\mathbb{E}_Y\left(e^{\left[\sum_{k=1}^{\vartheta_j(n)}H_n(t_j^k(n),\ Y_{j,k}(n))\right]}\right)=\exp\left[\frac{\bar{\nu}(m(n))}{N(n)}\left(\mathbb{E}_Y(e^{H_n(t_{j-1}(n),Y_{j,k}(n))})-1\right)\right],$$

for each $k=1,2,3,\cdots,\vartheta_j(n)$. Hence,

$$\mathbb{E}_Y(e^{H_n(t_{j-1}(n),Y_{j,k}(n))})=1+\frac{\int_{\mathbb{R}}\left(e^{H_n(t_{j-1}(n),\ x)}-1\right)\nu(dx)}{\bar{\nu}(m(n))}.$$

And so, $\mathbb{E}\exp\left\{H_n(\tau_j(n),\ Y_j^1(n))-\ln\left[1+\frac{\int_{|x|>m(n)}(e^{H_n(\tau_j(n),x)}-1)\nu(dx)}{\bar{\nu}(m(n))}\right]\right\}=1$ for each $j=1,2,\cdots,N(n)$. Therefore, $\bar{S}_n(t)=s_0e^{\bar{\varepsilon}_n(t)}$ is a martingale.

Again, by the Doob's martingale inequality,

$$\mathbb{E}^2\left(\sup_{t\in[0,1]}\bar{S}_n(t)\right)\le\mathbb{E}\left(\sup_{t\in[0,1]}\bar{S}_n(t)\right)^2\le 4\mathbb{E}\left(\bar{S}_n(1)\right)^2\le 4s_0\mathbb{E}\left(e^{2\bar{\varepsilon}_{n,1}(1)}\right).$$

Since all of the terms of $\bar{\varepsilon}_n(1)$ are mutually independent with respect to $(\Omega,\mathcal{F},\mathbb{P})$,

$$\begin{aligned}
&\mathbb{E}\left(e^{2\bar{\varepsilon}_n(1)}\right)\\
=\ &\mathbb{E}\left\{\prod_{j=1}^{N(n)}\exp\left[1_{\{\tau_j(n)\le1\}}\left(2H_n(\tau_j(n),Y_j^1)-2\ln\left(1+\frac{\int_{\mathbb{R}}(e^{H_n(t_{j-1}(n),x)}-1)\nu(dx)}{\bar{\nu}(m(n))}\right)\right)\right]\right\}\\
=\ &\mathbb{E}\left\{\prod_{j=1}^{N(n)}\left[\mathbb{P}(\vartheta_j(n)=0)+\mathbb{P}(\vartheta_j(n)>0)\cdot\frac{1+\frac{\int_{\mathbb{R}}(e^{2H_n(t_{j-1}(n),x)}-1)\nu(dx)}{\bar{\nu}(m(n))}}{\left(1+\frac{\int_{\mathbb{R}}(e^{H_n(t_{j-1}(n),x)}-1)\nu(dx)}{\bar{\nu}(m(n))}\right)^2}\right]\right\}.
\end{aligned}$$

Note that

$$\begin{aligned}
&\frac{1+\frac{\int_{\mathbb{R}}(e^{2H_n(t_{j-1}(n),x)}-1)\nu(dx)}{\bar{\nu}(m(n))}}{\left(1+\frac{\int_{\mathbb{R}}(e^{H_n(t_{j-1}(n),x)}-1)\nu(dx)}{\bar{\nu}(m(n))}\right)^2}\\
=\ &1+\frac{\frac{\int_{\mathbb{R}}(e^{H_n(t_{j-1}(n),x)}-1)^2\nu(dx)}{\bar{\nu}(m(n))}-\left(\frac{\int_{\mathbb{R}}(e^{H_n(t_{j-1}(n),x)}-1)\nu(dx)}{\bar{\nu}(m(n))}\right)^2}{\left(1+\frac{\int_{\mathbb{R}}(e^{H_n(t_{j-1}(n),x)}-1)\nu(dx)}{\bar{\nu}(m(n))}\right)^2}\\
\le\ &1+\frac{\frac{\int_{\mathbb{R}}(e^{H_n(t_{j-1}(n),x)}-1)^2\nu(dx)}{\bar{\nu}(m(n))}}{\left(1+\frac{\int_{\mathbb{R}}(e^{H_n(t_{j-1}(n),x)}-1)\nu(dx)}{\bar{\nu}(m(n))}\right)^2}.
\end{aligned}$$

Since $|H_n| < M$, $\left(1 + \frac{\int_{\mathbb{R}}(e^{H_n(t_{j-1}(n),x)}-1)\nu(dx)}{\bar{\nu}(m(n))}\right)^2 \in (e^{-2M}, e^{2M})$. Because $H_n \in \mathcal{H}_2(1,\mathbb{R})$,

$$\frac{\int_{\mathbb{R}}(e^{H_n(t_{j-1}(n),x)} - 1)^2\nu(dx)}{\bar{\nu}(m(n))} < \frac{e^{2M}\int_{\mathbb{R}}(H_n(t_{j-1}(n), x)^2\nu(dx)}{\bar{\nu}(m(n))} = O\left(\frac{1}{\bar{\nu}(m(n))}\right).$$

Hence,

$$\begin{aligned}
0 &\leq \mathbb{E}\left(e^{2\tilde{\varepsilon}_{n,1}(1)}\right)\\
&\leq \mathbb{E}\left\{\prod_{j=1}^{N(n)}\left[e^{-\frac{\bar{\nu}(m(n))}{N(n)}} + \left(1 - e^{-\frac{\bar{\nu}(m(n))}{N(n)}}\right)\left(1 + O\left(\frac{1}{\bar{\nu}(m(n))}\right)\right)\right]\right\}\\
&= \mathbb{E}\left\{\prod_{j=1}^{N(n)}\left[1 + \left(1 - e^{-\frac{\bar{\nu}(m(n))}{N(n)}}\right)O\left(\frac{1}{\bar{\nu}(m(n))}\right)\right]\right\}\\
&= \mathbb{E}\left\{\prod_{j=1}^{N(n)}\left[1 + O\left(\frac{1}{N(n)}\right)\right]\right\}.
\end{aligned}$$

Moreover, $\max_{n\geq 1}\mathbb{E}\left(e^{2\tilde{\varepsilon}_{n,1}(1)}\right) < \infty$. Thus, part (i) is obtained.

(ii) Let

$$\begin{aligned}
&\tilde{\varepsilon}_{n,2}(t)\\
:= &\sum_{j=1}^{N(n)} 1_{\{\vartheta_j(n)\geq 2\}}\sum_{k=2}^{\vartheta_j(n)} 1_{\{t_j^k(n)\leq t\}}\left[H_n(t_j^k(n), Y_{j,k}(n)) - ln\left(1 + \frac{\int_{\mathbb{R}}(e^{H_n(t_{j-1}(n),x)} - 1)\nu(dx)}{\bar{\nu}(m(n))}\right)\right],
\end{aligned}$$

Obviously, $\tilde{\varepsilon}_{n,2}(t) + \tilde{\varepsilon}_{n,1}(t) = \tilde{\varepsilon}_n(t)$ for any $t \in [0,1]$. Actually, $\tilde{\varepsilon}_{n,1}(t)$ collects all of the first jump in each subinterval with magnitude greater than $m(n)$, whereas $\tilde{\varepsilon}_{n,2}(t)$ collects over all subintervals such jumps except for the first one.

Consider that

$$\begin{aligned}
&\mathbb{E}\left(\sup_{t\in[0,1]}\left|\tilde{S}_n(t) - \bar{S}_n(t)\right|\right) = \mathbb{E}\left(\sup_{t\in[0,1]}\left|\bar{S}_n(t)\right|\left|e^{\tilde{\varepsilon}_{n,2}(t)} - 1\right|\right)\\
\leq\ &\mathbb{E}\left(\sup_{t\in[0,1]}\left|\bar{S}_n(t)\right|\sup_{t\in[0,1]}\left|e^{\tilde{\varepsilon}_{n,2}(t)} - 1\right|\right)\\
=\ &\mathbb{E}\left(\sup_{t\in[0,1]}\left|\bar{S}_n(t)\right|\right)\mathbb{E}\left(\sup_{t\in[0,1]}\left|e^{\tilde{\varepsilon}_{n,2}(t)} - 1\right|\right),
\end{aligned}$$

where the last equality follows from the independence of $\left(\bar{S}_n(t)\right)_t$ and $(\varepsilon_{n,2}(t))_t$.

Part (i) showed $\max_{n\geq 1}\mathbb{E}\left(\sup_{t\in[0,1]}\bar{S}_n(t)\right) < \infty$. So, to show $\tilde{S}_n(t) \xrightarrow{L^1} \bar{S}_n(t)$ as $n \to \infty$, we only need to show that

$$\mathbb{E}\left(\sup_{t\in[0,1]}\left|e^{\tilde{\varepsilon}_{n,2}(t)} - 1\right|\right) \to 0.$$

Similar to the arguments for $\bar{\varepsilon}_n(t)$ in part (i), $e^{\tilde{\varepsilon}_{n,2}(t)}$ is also a martingale. Then, by the Doob's martingale inequality, we have

$$\begin{aligned}
& \mathbb{E}^2\left(\sup_{t\in[0,1]}\left|e^{\tilde{\varepsilon}_{n,2}(t)}-1\right|\right) \le \mathbb{E}\left(\sup_{t\in[0,1]}\left|e^{\tilde{\varepsilon}_{n,2}(t)}-1\right|\right)^2 \\
\le\ & 4\mathbb{E}\left(\left|e^{\tilde{\varepsilon}_{n,2}(1)}-1\right|^2\right) \\
=\ & 4\mathbb{E}\left(e^{2\tilde{\varepsilon}_{n,2}(1)}-2e^{\tilde{\varepsilon}_{n,2}(1)}+1\right) \\
=\ & 4\left[\mathbb{E}\left(e^{2\tilde{\varepsilon}_{n,2}(1)}\right)-1\right] \\
=\ & 4\left\{\prod_{j=1}^{N(n)}\left[\mathbb{P}(\vartheta_j(n)<2)+\mathbb{E}\left(1_{(}\vartheta_j(n)\ge 2)\cdot\prod_{k=2}^{\vartheta_j(n)}\frac{1+\frac{\int_{\mathbb{R}}(e^{2H_n(t_{j-1}(n),x)}-1)\nu(dx)}{\bar{\nu}(m(n))}}{\left(1+\frac{\int_{\mathbb{R}}(e^{H_n(t_{j-1}(n),x)}-1)\nu(dx)}{\bar{\nu}(m(n))}\right)^2}\right)\right]-1\right\},
\end{aligned}$$

where the second equality follows from $\mathbb{E}\left(e^{\tilde{\varepsilon}_{n,2}(1)}\right)=1$ and the last equality from the definition of $\tilde{\varepsilon}_{n,2}(t)$ and the independence of $\left(\vartheta_j(n),\ H_n(t_j^k(n),Y_{j,k}(n))\right)_{j,k}$. From $|H_n|<M$, it follows that $|\eta_{n,j}|=\left|\frac{1+\frac{\int_{\mathbb{R}}(e^{2H_n(t_{j-1}(n),x)}-1)\nu(dx)}{\bar{\nu}(m(n))}}{\left(1+\frac{\int_{\mathbb{R}}(e^{H_n(t_{j-1}(n),x)}-1)\nu(dx)}{\bar{\nu}(m(n))}\right)^2}\right|$ is bounded, uniformly for $n\in\mathbb{N}$ and $j=1,2,\cdots,N(n)$, by a positive finite number, say η. So,

$$\mathbb{E}\left(1_{(}\vartheta_j(n)\ge 2)\cdot\prod_{k=2}^{\vartheta_j(n)}\eta_{n,j}\right)\le\sum_{l=2}^{\infty}\eta^{l-1}\ \mathbb{P}(\vartheta_j(n)=l).$$

Since $\vartheta_j(n)$ is a Poisson r.v. with expectation $\lambda(n)=\frac{\bar{\nu}(m(n))}{N(n)}$, $\mathbb{P}(\vartheta_j(n)=k)=e^{-\lambda(n)}\frac{\lambda(n)^k}{k!}$ for any $k=0,1,2,\cdots$. By Assumption 1 (b), $\lim_{n\to\infty}\frac{\bar{\nu}(m(n))^2}{N(n)}=0$. Hence,

$$\mathbb{E}\left(1_{(}\vartheta_j(n)\ge 2)\cdot\prod_{k=2}^{\vartheta_j(n)}\eta_{n,j}\right)\le\sum_{k=2}^{\infty}\frac{\eta^k\lambda(n)^k}{k!}e^{-\lambda(n)}=O(\lambda(n)^2).$$

Above all,

$$\begin{aligned}
\frac{1}{4}\mathbb{E}^2\left(\sup_{t\in[0,1]}\left|e^{\tilde{\varepsilon}_{n,2}(t)}-1\right|\right) &\le \prod_{j=1}^{N(n)}\left(e^{-\lambda(n)}+\lambda(n)e^{-\lambda(n)}+O(\lambda(n)^2)\right)-1 \\
&= \prod_{j=1}^{N(n)}\left(1+O(\lambda(n)^2)\right)-1 \\
&= \left[1+O\left(\frac{\bar{\nu}(m(n))}{N(n)}\right)^2\right]^{N(n)}-1 \\
&= O\left(\frac{\bar{\nu}(m(n))^2}{N(n)}\right)\overset{n\to\infty}{\longrightarrow}0,
\end{aligned}$$

where the convergence also follows from the assumption that $\lim_{n\to\infty}\frac{\bar{\nu}(m(n))^2}{N(n)}=0$. Thus, we got part (ii).

Next, the definition of S_n is given.

Definition 5 *For any $t\in[0,1]$, let $\hat{S}_n(t)=s_0\exp(\hat{\varepsilon}_n(t))$, where $\hat{\varepsilon}_n(t)=\sum_{j=1}^{N(n)}1_{\{\tau_j(n)\le t\}}\frac{\lfloor\Delta\tilde{\varepsilon}_{n,1}(\tau_j(n))N(n)n\rfloor}{N(n)n}$ and $\Delta\tilde{\varepsilon}_{n,1}(\tau_j(n))=\tilde{\varepsilon}_{n,1}(\tau_j(n))-\tilde{\varepsilon}_{n,1}(\tau_j(n)^-)$. For any $k=0,1,2,\cdots,N(n)$, let $S_n(k)=s_0\exp(\varepsilon_n(k))$, where $\varepsilon_n(k)=\hat{\varepsilon}_n(t_k(n))$.*

Remark 4 Each $S_n(k)$ defined above is of the form (1). Indeed, by Definition 5, $\Delta\tilde{\varepsilon}_{n,1}(\tau_j(n))=H_n(\tau_j(n),\ Y_j^1(n))-\ln\left[1+\frac{\int_{|x|>m(n)}(e^{H_n(\tau_j(n),x)}-1)\nu(dx)}{\bar{\nu}(m(n))}\right]$. It is known that $|H|<M$ almost surely. For any given $\omega\in\Omega$, let $M-|H|>c_1$ for some constant $c_1>0$. And, then $M-|H_n|>c_1$ by the definition of H_n. Set d be a positive number such that both $|ln(1+d)|<c_1$ and $|ln(1-d)|<c_1$. Since $g\in L^2(\mathbb{R},\nu)$, there exists $c_2>0$ such that $|g_n|\le|g|<\frac{ln(1+d)}{2\alpha}$ for all $n\in\mathbb{N}$ whenever $|x|<c_2$. Let

n be large enough such that $\left|\frac{\int_{|x|>c_2}(e^{H_n(\tau_j(n),x)}-1)\nu(dx)}{\bar{\nu}(m(n))}\right| < \frac{d}{2}$. Thus, $\left|\ln\left[1+\frac{\int_{|x|>m(n)}(e^{H_n(\tau_j(n),x)}-1)\nu(dx)}{\bar{\nu}(m(n))}\right]\right| < c_1$. Hence, $\Delta\tilde{\varepsilon}_{n,1}(\tau_j(n)) < M$ for all $j = 1, 2, \cdots, N(n)$ and $n \in \mathbb{N}$.

Lemma 4 *We have*
(i) $\mathbb{E}\left(\rho(S_n(\lfloor N(n)t\rfloor),\ S_H(t))\right) \to 0$ *as* $n \to \infty$*, where* ρ *is the Skorokhod distance;*
(ii) $\max_{n\geq 1}\mathbb{E}\left(\max_{0\leq k\leq N(n)} S_n(k)\right) \leq \infty$.

Proof. (i) Consider that

$$\begin{aligned}
&\sup_{t\in[0,1]}\left|\hat{S}_n(t)-\bar{S}_n(t)\right| \\
\leq\ & \sup_{t\in[0,1]}\left|\bar{S}_n(t)\right|\cdot\sup_{t\in[0,1]}\left|\exp\left[\sum_{j=1}^{N(n)}\left(1_{\{\tau_j(n)\leq t\}}\frac{\lfloor\Delta\tilde{\varepsilon}_{n,1}(\tau_j(n))N(n)n\rfloor}{N(n)n}-\Delta\tilde{\varepsilon}_{n,1}(\tau_j(n))\right)\right]-1\right| \\
\leq\ & \sup_{t\in[0,1]}\left|\bar{S}_n(t)\right|\cdot\left|e^{\frac{1}{n}}-1\right|.
\end{aligned}$$

By Lemma 3(i), $\max_{n\geq 1}\mathbb{E}\left(\sup_{t\in[0,1]}\bar{S}_n(t)\right)\leq\infty$. So, $\mathbb{E}\left(\sup_{t\in[0,1]}\left|\hat{S}_n(t)-\bar{S}_n(t)\right|\right)\to 0$ as $n\to\infty$.

Since $S_n(\lfloor N(n)t\rfloor) = \hat{S}_n(t_j(n))$ if $t \in [t_j(n), t_{j+1}(n))$,

$$\mathbb{E}[\rho(S_n(\lfloor N(n)t\rfloor),\ \hat{S}_n(t_j(n)))] \leq \Delta t(n) \xrightarrow{a.s} 0.$$

Together with Lemma 2 and Lemma 3(ii), we get $\mathbb{E}\left(\rho(S_n,\ S_H)\right)\to 0$ as $n\to\infty$, as required.

(ii) By Lemma 3(i) and the proof of part (i) above, it is easy to see that

$$\begin{aligned}
&\max_{n\geq 1}\mathbb{E}\left(\max_{0\leq k\leq N(n)} S_n(k)\right) = \max_{n\geq 1}\mathbb{E}\left(\sup_{t\in[0,1]}\left|\hat{S}_n(t)\right|\right) \\
\leq\ & \max_{n\geq 1}\mathbb{E}\left(\sup_{t\in[0,1]}\left|\hat{S}_n(t)-\bar{S}_n(t)\right|+\sup_{t\in[0,1]}\left|\bar{S}_n(t)\right|\right) \\
\leq\ & \infty.
\end{aligned}$$

Then, the proof is finished.

Lemma 5 *Let* $M_n(k) = \mathbb{E}(S_n(N(n)|\mathcal{F}_k(n)))$, $k = 1, 2, \cdots, N(n)$*, where* $\mathcal{F}_k(n)$ *is the natural filtration generated by* $\varepsilon_n(k)$*. Then,*

$$M_n(k) - S_n(k) = \frac{(\frac{k}{N(n)}-1)\bar{\nu}(m(n))^2}{2N(n)}(1+o(1))S_n(k).$$

Proof. By the mutually independence of $\{\tau_j(n)\}_j$ and $\{Y_j^1(n)\}_j$, $M_n(k)$ can be rewritten as

$$M_n(k) = \mathbb{E}(S_n(N(n))|\mathcal{F}_k(n)) = S_n(k)\prod_{j=k+1}^{N(n)}\mathbb{E}\left(\exp\left(\frac{\lfloor 1_{\{\vartheta_j(n)\geq 1\}}\delta_j(n)N(n)n\rfloor}{N(n)n}\right)|\mathcal{F}_k(n)\right),$$

where

$$\delta_j(n) = H_n(\tau_j(n), Y_j^1(n)) - \ln\left(1+\frac{\int_{\mathbb{R}}(e^{H_n(\tau_j(n),x)}-1)\nu(dx)}{\bar{\nu}(m(n))}\right).$$

Since $\mathbb{E}\left(e^{\delta_j(n)}\right) = 1$ when $\vartheta_j(n) \geq 1$, $\mathbb{E}\left(e^{\frac{\lfloor\delta_j(n)N(n)n\rfloor}{N(n)n}}\right) = 1 + O\left(\frac{1}{N(n)}\right)$.

$$\begin{aligned}
& \prod_{j=k+1}^{N(n)} \mathbb{E}\left(\exp\left(\frac{\lfloor\delta_j(n)N(n)n\rfloor}{N(n)n}\right)|\mathcal{F}_k(n)\right) \\
= & \prod_{j=k+1}^{N(n)} \left[\mathbb{P}(\vartheta_j(n) = 0) + \mathbb{P}(\vartheta_j(n) \geq 1)\mathbb{E}\left(e^{\frac{\lfloor\delta_j(n)N(n)n\rfloor}{N(n)n}}|\mathcal{F}_k^n\right)\right] \\
= & \prod_{j=k+1}^{N(n)} \left[e^{-\bar{\nu}(m(n))/N(n)} + \frac{\bar{\nu}(m(n))}{N(n)}e^{-\bar{\nu}(m(n))/N(n)}\left(1 + O\left(\frac{1}{N(n)n}\right)\right)\right] \\
= & \prod_{j=k+1}^{N(n)} \left[1 - \frac{\bar{\nu}(m(n)^2}{2N(n)^2} + o\left(\frac{\bar{\nu}(m(n)^2}{N(n)^2}\right)\right] \\
= & \left[1 - \frac{\bar{\nu}(m(n)^2}{2N(n)^2} + o\left(\frac{\bar{\nu}(m(n)^2}{N(n)^2}\right)\right]^{N(n)-k} \\
= & 1 - \frac{(N(n) - k)\bar{\nu}(m(n)^2}{2N(n)^2} + o\left(\frac{(N(n) - k)\bar{\nu}(m(n)^2}{2N(n)^2}\right).
\end{aligned}$$

Then, the lemma follows directly.

Now, we are ready to show the existence of the liquidity premium.

3.3 Proof of Theorem 1

The proof idea for the lower bound part of Theorem 3.5 in (Dolinsky & Soner, 2013) is used in the following proof of Theorem 1.

Proof. Fix $H \in \mathcal{A}_M$. From Theorem 3.1 of (Donlinsky & Soner, 2013), it follows that

$$V_n \geq \mathbb{E}\left(F(S_n) - \sum_{k=0}^{N(n)-1} G\left(\frac{k}{N(n)},\ S_n,\ M_n(k) - S_n(k)\right)\right).$$

By Lemma 4(i), $S_n \longrightarrow S_H$ in the mean under the Skorohod topology. So, by the Skorohod representation theorem, there exists a probability space, $(\tilde{\Omega},\ \tilde{\mathbb{F}},\ \tilde{\mathbb{P}})$, on which

$$S_n \longrightarrow S_H \ \ \tilde{P} - a.s.$$

on the space $\mathbb{D}[0,\ 1]$.

By the growth assumption on F and Lemma 4(ii), the sequence $F(S_n)$ is uniformly integrable. Then,

$$\lim_{n\to\infty} \mathbb{E}(F(S_n)) = \tilde{\mathbb{E}}(F(S_H)),$$

where $\tilde{\mathbb{E}}$ is the expectation with respect to $\tilde{\mathbb{P}}$. So, $M_n(k)$ can be redefined on the new space, $(\tilde{\Omega},\ \tilde{\mathbb{F}},\ \tilde{\mathbb{P}})$, as $M_n(k) = \tilde{\mathbb{E}}[S_n(N(n))|\tilde{\mathcal{F}}_k(n)]$. The joint distribution of M_n and S_n remains as before. The Assumption 1(b), Lemma 4 and Lemma 5 imply the sequence $N(n)G\left(\frac{\lfloor nt\rfloor}{N(n)}, S_n, M_n(\lfloor nt\rfloor) - S_n(\lfloor nt\rfloor)\right)$ is uniformly integrable in $\mathcal{L} \times \tilde{P}$, where $\mathcal{L}$ is the Lesbegue measure on $[0,\ 1]$. Since $\hat{G}$ is continuous, by Fubini's theorem, Assumption 1(a) and Lemma 5,

$$\begin{aligned}
& \lim_{n\to\infty} \mathbb{E}\left(\sum_{k=0}^{N(n)-1} G\left(\frac{k}{N(n)},\ S_n,\ M_n(k) - S_n(k)\right)\right) \\
= & \lim_{n\to\infty} \tilde{\mathbb{E}}\left(\int_{[0,\ 1]} N(n)G\left(\frac{\lfloor nt\rfloor}{n}, S_n, M_n(\lfloor nt\rfloor) - S_n(\lfloor nt\rfloor)\right)dt\right) \\
= & \tilde{\mathbb{E}}\left(\int_{[0,\ 1]} \hat{G}\left(t,\ S_H,\ \frac{(t-1)S_H(t)}{2}\right)dt\right).
\end{aligned}$$

Therefore, the theorem is obtained.

4. Conclussion and Discussion

As stated in the introduction, all the super-replication cost works so far have been based on the binomial model and Brownian motion. However, the restriction to Brownian motion does not allow for the Lévy process with infinite activity that is actually frequently used for building stochastic models in finance, economics and many other fields.

In this paper, the problem of continuous time super-replication cost of a European option in a one-dimensional Lévy model is studied. Special emphasis is placed on Lévy processes that have infinitely many jumps, almost surely, in any finite time interval. Under a mild assumption, the continuous time super-replication cost is proved to be greater than or equal to an optimal control problem. The existence of the liquidity premium is proved which should have the practical importance in the real world. So, the result in this paper is strong enough to fulfill the practical need.

The main tool is a multinomial approximation scheme that is based on a discrete grid, on a finite time interval $[0, 1]$, and having a finite number of states, for a Lévy process. The approach overcomes some difficulties that can be encountered when the Lévy process has infinite activity.

This paper showed the continuous time super-replication cost is greater than or equal to an optimal control problem. It would be interesting to know the condition under which the equal sign holds. However, due to the large jumps of Lévy process, research in the less than or equal to part is quite challenge. So, I will leave it in a future study.

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