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Pairs of Comparable Relations for Complete Homogeneous Symmetric Polynomial

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Abstract

Complete homogeneous symmetric polynomial has connections with binomial coefficient, composition, elementary symmetric polynomial, exponential function, falling factorial, generating series, odd prime and Stirling numbers of the second kind by different summations. Surprisingly the relations in the context are comparable in pairs.

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Keywords: recurrence, symmetric polynomial, binomial coefficient, generating series, falling factorial, Stirling numbers of the second kind, composition, odd prime

1. Introduction

For each nonnegative integer *n*, complete homogeneous symmetric polynomial $h_n(x_1, \dots, x_k)$ or in brief $h_n\{x_k\}$ is the sum of all distinct monomials of degree *n* in the variables: $x_1, \dots, x_k$. Formally

$$h_n\{x_k\} \quad = \sum_{1 \le i_1 \le \dots \le i_n \le k} x_{i_1} x_{i_2} \dots x_{i_n}.$$

Some special values of $h_n\{x_k\}$ are: $h_0\{x_k\} = 1$; $h_n\{0\} = 0$; $h_1\{x_k\} = x_1 + \dots + x_k$ and $h_n\{x_1\} = x_1^n$.

The monomials again belong to the distinct symmetric polynomials. More precisely, $h_n\{x_k\}$ is the sum of all distinct monomial symmetric polynomials of degree *n* in *k* variables: $x_1, \dots, x_k$.

Example: $h_4\{x_3\} = m_{(4)}\{x_3\} + m_{(3,1)}\{x_3\} + m_{(2,2)}\{x_3\} + m_{(2,1,1)}\{x_3\}$.

[$m_{(a,b,c,\ \dots)}(x_1, \dots, x_k)$, or in brief, $m_{(a,b,c,\ \dots)}\{x_k\}$ = Distinct monomial symmetric polynomial of *k* variables: $x_1, \dots, x_k$ in degree *n* such that *n* = *a* + *b* + *c* + …]

$$= (x_1^4 + x_2^4 + x_3^4) + (x_1^3 x_2 + x_1 x_2^3 + x_1^3 x_3 + x_1 x_3^3 + x_2^3 x_3 + x_2 x_3^3)$$
$$+ (x_1^2 x_2^2 + x_1^2 x_3^2 + x_2^2 x_3^2) + (x_1^2 x_2 x_3 + x_1 x_2^2 x_3 + x_1 x_2 x_3^2).$$

$h_n\{x_k\}$ involves with the recurrences of many patterns. From its formal definition, it is easy to find a fundamental identity or a recurrence relation which yields further the recurrence relations of two kinds. Some combinatorial relationships are the consequences of a pair of characterization formulas for the polynomial. A recurrence function for the polynomial, which is analogous with a function for falling factorial, helps to find the relation of the polynomial with elementary symmetric polynomial. The relation between the polynomials of two kinds is useful to establish divisibility of the polynomial-summations by an odd prime. It is a curious fact that many relations involving $h_n\{x_k\}$ are comparable in pairs, and we show the pairs in all topics of the paper.

2. Recurrence Relations of Two Kinds for $h_n\{x_k\}$

Letting that x_m is a definite variable in the set: $x_1, \dots, x_{k+1}$, we denote other *k* variables by $y_1, \dots, y_k$. From the formal definition of $h_n\{x_k\}$, it then follows that a term of $h_{n+1}\{y_k\}$ is also a term of $h_{n+1}\{x_{k+1}\}$, which does not contain x_m as a factor; and if x_m is multiplied with a term of $h_n\{x_{k+1}\}$ then the product is a term of $h_{n+1}\{x_{k+1}\}$, which contains x_m as a factor. This implies that $h_{n+1}\{y_k\}$ is the sum of some terms of $h_{n+1}\{x_{k+1}\}$ where none of these terms has a factor x_m; and $x_m\, h_n\{x_{k+1}\}$ is the sum of some other terms of $h_{n+1}\{x_{k+1}\}$ where x_m is a common factor of these terms. We know that the number of terms of $h_n\{x_k\}$ is $\binom{k+n-1}{n}$. Hence $h_{n+1}\{y_k\}$ is the sum of $\binom{k+n}{n+1}$ among $\binom{k+n+1}{n+1}$ terms of $h_{n+1}\{x_{k+1}\}$ and $x_m\, h_n\{x_{k+1}\}$ is the sum of remaining $\binom{k+n}{n}$ terms of $h_{n+1}\{x_{k+1}\}$. Clearly $h_{n+1}\{x_{k+1}\}$ is the sum of these two parts. That is, we have the following fundamental identity or recurrence relation for the polynomial.

$$h_{n+1}\{x_{k+1}\} = h_{n+1}\{y_k\} + x_m\, h_n\{x_{k+1}\} \ . \tag{1}$$

When $x_m = x_{k+1}$ then other k variables are: $x_1, \ldots, x_k$ for which evidently we can write $\{x_k\}$ instead of $\{y_k\}$ in (1). Hence from (1), we have:

$$h_{n+1}\{x_{k+1}\} = h_{n+1}\{x_k\} + x_{k+1}\, h_n\{x_{k+1}\}. \tag{1.1}$$

(1.1) can yield the recurrence relations of two kinds. We show other applications of (1.1) also in the subsequent topics.

2.1 Kind 1 with a Series of k Terms

From (1.1),

$$\sum_{i=1}^{k} [\, h_{n+1}\{x_{i+1}\} - h_{n+1}\{x_i\}\,] = \sum_{i=1}^{k} x_{i+1}\, h_n\{x_{i+1}\}\, .$$

$$\Rightarrow h_{n+1}\{x_{k+1}\} = \sum_{i=1}^{k+1} x_i\, h_n\{x_i\}\, .$$

$$\Rightarrow h_n\{x_k\} = \sum_{i=1}^{k} x_i\ h_{n-1}\{x_i\}\, . \tag{2}$$

(2) is the recurrence relation of kind 1 whose applications are shown in Topic 3.3 and Topic 4. The modified form of (2) is (2.1) below. We can write:

$$h_1\{x_k\} = \sum\nolimits_{i_1=1}^{k} x_{i_1} \ .$$

Then from (2),

$$h_2\{x_k\} = \sum_{i=1}^{k} x_i\ h_1\{x_i\} = \sum_{i_2=1}^{k} x_{i_2} \sum_{i_1=1}^{i_2} x_{i_1}\, .$$

In this way $h_n\{x_k\}$ is a recurrence such that

$$h_n\{x_k\} = \sum_{i_n=1}^{k} x_{i_n} \ldots \sum_{i_3=1}^{i_4} x_{i_3} \sum_{i_2=1}^{i_3} x_{i_2} \sum_{i_1=1}^{i_2} x_{i_1}\, . \tag{2.1}$$

(2.1) has specialty to generate all $\binom{k+n-1}{n}$ terms of $h_n\{x_k\}$ that involve with $\binom{k+n-1}{n}$ ordered integers. Let (2.1) be decomposed for k = 3 and $n \in$ (1, 2, 3) in succession.

$$\begin{aligned}
h_1\{x_3\} &= x_3 + x_2 + x_1. \\
h_2\{x_3\} &= x_3\,(x_3 + x_2 + x_1) + x_2\,(x_2 + x_1) + x_1 . x_1 \\
&= x_3 . x_3 + x_3 . x_2 + x_3 . x_1 + x_2 . x_2 + x_2 . x_1 + x_1 . x_1 \ . \\
h_3\{x_3\} &= x_3\,(x_3 . x_3 + x_3 . x_2 + x_3 . x_1 + x_2 . x_2 + x_2 . x_1 + x_1 . x_1) \\
&\quad + x_2\,(x_2 . x_2 + x_2 . x_1 + x_1 . x_1) + x_1 . x_1 . x_1 \\
&= x_3 . x_3 . x_3 + x_3 . x_3 . x_2 + x_3 . x_3 . x_1 + x_3 . x_2 . x_2 + x_3 . x_2 . x_1 + x_3 . x_1 . x_1 \\
&\quad + x_2 . x_2 . x_2 + x_2 . x_2 . x_1 + x_2 . x_1 . x_1 + x_1 . x_1 . x_1.
\end{aligned}$$

We can find the integer-sequences with respect to the bottom indices of x_3, x_2 and x_1 in the decompositions of $h_n\{x_3\}$ for n = 1, 2 and 3. The sequence for $h_1\{x_3\}$ is: 3 > 2 > 1. Omitting the multiplication dots (·), the sequence for $h_2\{x_3\}$ is: 33 > 32 > 31 > 22 > 21 > 11; and for $h_3\{x_3\}$ is: 333 > 332 > 331 > 322 > 321 > 311 > 222 > 221 > 211 > 111 respectively. Each sequence is the immediate consequence of the previous one. Thus the sequence for $h_4\{x_3\}$ is: 3333 > 3332 > 3331 > 3322 > 3321 > 3311 > 3222 > 3221 > 3211 > 3111 > 2222 > 2221 > 2211 > 2111 > 1111; and so on.

We further notice that the number of terms of $h_1\{x_k\}$ is k; this of $h_2\{x_k\}$ is: $\sum_{i=1}^{k} i$ or $\binom{k+1}{2}$; this of $h_3\{x_k\}$ is: $\sum_{i=1}^{k}\binom{i+1}{2}$or $\binom{k+2}{3}$; and so on. In other way the number of terms of $h_1\{x_k\}$ is $\sum_{i_1=1}^{k}\binom{i_1}{0}$; then this of $h_2\{x_k\}$ is: $\sum_{i_2=1}^{k}\sum_{i_1=1}^{i_2}\binom{i_1}{0}$; and so on. Thus $\binom{k+n-1}{n}$ or the number of terms of $h_n\{x_k\}$ is a recurrence, similar to (2.1) such that

$$\binom{k+n-1}{n} = \sum_{i_n=1}^{k} \cdots \sum_{i_3=1}^{i_4} \sum_{i_2=1}^{i_3} \sum_{i_1=1}^{i_2} \binom{i_1}{0}. \tag{2.2}$$

2.2. Kind 2 with a Series of n + 1 Terms

Again from (1.1),

$$\sum_{j=1}^{n} x_{k+1}^{n-j} \left[h_{j+1}\{x_{k+1}\} - x_{k+1} h_j\{x_{k+1}\} \right] = \sum_{j=1}^{n} x_{k+1}^{n-j} h_{j+1}\{x_k\}.$$

$$\Rightarrow h_{n+1}\{x_{k+1}\} = x_{k+1}^{n+1} + \sum_{j=1}^{n+1} x_{k+1}^{n-j+1} h_j\{x_k\}.$$

$$[\ h_1\{x_{k+1}\} = x_{k+1} + h_1\{x_k\}\ .]$$

$$\Rightarrow h_n\{x_{k+1}\} = x_{k+1}^{n} + \sum_{j=1}^{n} x_{k+1}^{n-j} h_j\{x_k\}.$$

$$\Rightarrow h_n\{x_{k+1}\} = \sum_{j=0}^{n} x_{k+1}^{n-j} h_j\{x_k\}. \tag{3}$$

(3) is the recurrence relation of kind 2. Substituting 1, … , $k + 1$ for x_1 , … , x_{k+1}, (3) is reduced to

$$h_n\{k+1\} = \sum_{j=0}^{n} (k+1)^{n-j} h_j\{k\}. \tag{3.1}$$

A consequence of (3.1) is:

$$h_n\{k\} = \sum_{i=0}^{k-1} (-1)^{\mathrm{i}} \frac{(k-i)^{n+k-1}}{i!\ (k-1-i)!}. \tag{3.2}$$

in a process of recursive substitution as shown.

$h_n\{1\} = 1 = \frac{1}{0!\,0!}$; then from (3.1):

$$h_n\{2\} = \sum_{j=0}^{n} 2^{n-j} h_j\{1\}$$

$$= \frac{2^{n+1}-1}{2-1} \frac{1}{0!\ 0!} = \frac{2^{n+1}}{0!\ 1!} - \frac{1}{1!\ 0!}$$, which is (3.2) for $k = 2$.

$$h_n\{3\} = 3^n + \sum_{j=1}^{n} 3^{n-j} h_j\{2\}$$

$$= 3^n + \sum_{j=1}^{n} 3^{n-j} \left(\frac{2^{j+1}}{0!\ 1!} - \frac{1}{1!\ 0!} \right)$$

$$= 3^n + \frac{2^2}{0!\ 1!} \sum_{j=1}^{n} 3^{n-j}\, 2^{j-1} - \frac{1}{1!\ 0!} \sum_{j=1}^{n} 3^{n-j}$$

$$= 3^n + \frac{2^2\,(3^n - 2^n)}{0!\ 1!\ (3-2)} - \frac{3^n - 1}{1!\ 0!\ (3-1)}$$

$$= 3^n \left(1 + \frac{2^2}{1!\ 1!} - \frac{1}{2!\ 0!}\right) - \frac{2^{n+2}}{1!\ 1!} + \frac{1}{2!\ 0!}$$

$$= \frac{3^{n+2}}{0!\ 2!} - \frac{2^{n+2}}{1!\ 1!} + \frac{1}{2!\ 0!}$$, which is (3.2) for $k = 3$.

… …

To find the successive results from (3.1), we use the identity:

$$\frac{(k+1)^k}{0!\ k!} - \frac{k^k}{1!\,(k-1)!} + \ldots + (-1)^k \frac{1}{k!\ 0!} = 1. \qquad (3.3)$$

Its derivation is shown in the next topic.

In general the above process of recursion runs in the following way.

$$h_n\{k+1\} = (k+1)^n + (k+1)^{n-1} h_1\{k\} + (k+1)^{n-2} h_2\{k\} + \ldots + h_n\{k\}$$

$$= (k+1)^n + (k+1)^{n-1} \sum_{i=0}^{k-1} (-1)^i \frac{(k-i)^k}{i!\ (k-1-i)!}$$

$$+ (k+1)^{n-2} \sum_{i=0}^{k-1} (-1)^i \frac{(k-i)^{k+1}}{i!\ (k-1-i)!} + \ldots + \sum_{i=0}^{k-1} (-1)^i \frac{(k-i)^{k+n-1}}{i!\ (k-1-i)!}.$$

$$= (k+1)^n + \frac{k^k}{0!\,(k-1)!} \left[(k+1)^{n-1} + (k+1)^{n-2} k + \ldots + k^{n-1}\right]$$

$$- \frac{(k-1)^k}{1!\ (k-2)!} \left[(k+1)^{n-1} + (k+1)^{n-2} (k-1) + \ldots + (k-1)^{n-1}\right] + \ldots$$

$$+ (-1)^{k-1} \frac{1}{(k-1)!\ 0!} \left[(k+1)^{n-1} + (k+1)^{n-2} + \ldots + 1\right].$$

$$= (k+1)^n + \frac{k^k}{0!\,(k-1)!} \frac{(k+1)^n - k^n}{(k+1) - k} - \frac{(k-1)^k}{1!\,(k-2)!} \frac{(k+1)^n - (k-1)^n}{(k+1) - (k-1)} + \ldots$$

$$+ (-1)^{k-1} \frac{1}{(k-1)!\,0!} \frac{(k+1)^n - 1}{(k+1) - 1}.$$

$$= (k+1)^n \left[1 + \frac{k^k}{1!\,(k-1)!} - \frac{(k-1)^k}{2!\,(k-2)!} + \ldots + (-1)^{k-1} \frac{1}{k!\ 0!}\right]$$

$$- \frac{k^{n+k}}{1!\ (k-1)!} + \frac{(k-1)^{n+k}}{2!\ (k-2)!} - \ldots + (-1)^k \frac{1}{k!\ 0!}$$

$$= \frac{(k+1)^{n+k}}{0!\ k!} - \frac{k^{n+k}}{1!\ (k-1)!} + \frac{(k-1)^{n+k}}{2!\ (k-2)!} - \ldots + (-1)^k \frac{1}{k!\ 0!}.$$

[By (3.3)]

$$= \sum_{i=0}^{k} (-1)^i \frac{(k+1-i)^{n+k}}{i!\ (k-i)!}.$$

2.3 Relation between $h_n\{k\}$ & Exponential Function

A pair of expansions of $(e^t - 1)^k$ helps to find the relation of the function with $h_n\{k\}$. Pairing in the relation is remarkable.

(a) Expansion 1: By the Binomial Theorem,

$$(e^t-1)^k = e^{tk} - \binom{k}{1} e^{t\,(k-1)} + \ldots + (-1)^{k-1}\binom{k}{k-1} e^t + (-1)^k$$

$$= \binom{k}{0}\left[1 + tk + \frac{t^2k^2}{2!} + \ldots\right] - \binom{k}{1}\left[1 + t(k-1) + \frac{t^2(k-1)^2}{2!} + \ldots\right] + \ldots$$

$$+ (-1)^{k-1}\binom{k}{k-1}\left[1 + t + \frac{t^2}{2!} + \ldots\right] + (-1)^k\binom{k}{k}.$$

$$\Rightarrow (e^t-1)^k = \sum_{n=0}^{k}(-1)^n\binom{k}{n}\sum_{m=0}^{\infty}(k-n)^m\frac{t^m}{m!} \quad [0^0 = 1]. \tag{4.1}$$

(b) Expansion 2: Again,

$$(e^t-1)^k = \left(t + \frac{t^2}{2!} + \frac{t^2}{3!} + \ldots\right)^k$$

$$= t^k + \frac{k}{2}\,t^{k+1} + \quad \text{the terms containing the higher powers of } t. \tag{4.2}$$

Equating the coefficients of t^m for $0 \leq m \leq k$, from (4.1) and (4.2),

$$\frac{1}{m!}\sum_{n=0}^{k}(-1)^n\binom{k}{n}(k-n)^m = \begin{cases} 0, \text{if } m \in \{0, 1, \ldots, (k-1)\}. & (5.1) \\ 1, \text{if } m = k. & (5.2) \end{cases}$$

(5.2) is (3.3) with a minor variation.

Excluding the terms for $0 \leq m \leq k-1$ whose values are all 0, (4.1) can be written:

$$(e^t-1)^k = \sum_{n=0}^{k}(-1)^n\binom{k}{n}\sum_{m=k}^{\infty}(k-n)^m\frac{t^m}{m!}. \tag{6.1}$$

From (3.2) and (6.1),

$$(e^t-1)^k = k!\sum_{n=0}^{\infty}\frac{h_n\{k\}\,t^{k+n}}{(k+n)!}. \tag{6.2}$$

It is interesting that the expression under $\sum$ on the right of (6.2) contains three pairs of n & k.

The Stirling number of the second kind: $S(n,\ k)$ is the number of ways of partitioning a set of n elements into exactly k nonempty subsets. A relation between e and $S(n,k)$ is given by:

$$(e^t-1)^k = k!\sum_{n=0}^{\infty}\frac{S(n,\ k)\,t^n}{n!}. \tag{6.3}$$

$h_n\{k\}$ and $S(n,\ k)$ are comparable. From the definition of $h_n\{x_k\}$ in the introduction, it follows that $h_n\{k\}$ or $h_n(1,2,\ldots,k)$ are the positive integers for $n \gtreqless k$. The special case is: $h_0\{k\} = 1$. On the other hand $S(n,\ k)$ are the positive integers for $n \geq k$. $S(n,\ k) = 0$ for $n < k$. Then excluding the terms whose values are all 0, the modified form of (6.3) is:

$$(e^t-1)^k = k!\sum_{n=0}^{\infty}\frac{S(n+k,\ k)\,t^{n+k}}{(n+k)!}. \tag{6.4}$$

From (6.2) and (6.4), we get the relation:

$$h_n\{k\} = S(n+k,k). \tag{7}$$

The familiar counting formula for $S(n,\ k)$ is:

$$S(n,k) = \frac{1}{k!}\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^n. \tag{8}$$

(7) is the consequence (3.2) and (8) also.

2.4 Link between the Recurrences for $h_n\{x_k\}$ When $x_1, \dots, x_k = 1, \dots, k$

When $x_1, x_2, \dots = 1, 2, \dots$ then from (3.2) and (2), we get:

$$\sum_{i=0}^{k-1}(-1)^{i}\frac{(k-i)^{n+k-1}}{i!\,(k-1-i)!} = k\sum_{i=0}^{k-1}(-1)^{i}\frac{(k-i)^{n+k-2}}{i!\,(k-1-i)!}$$

$$+ (k-1)\sum_{i=0}^{k-2}(-1)^{i}\frac{(k-1-i)^{n+k-3}}{i!\,(k-2-i)!} + \dots + 2\sum_{i=0}^{1}(-1)^{i}\frac{(2-i)^{n}}{i!\,(1-i)!} + 1. \tag{9.1}$$

The left hand side of (9.1) is a finite series of k terms and the right a finite series of (k + … + 1) or ½ k (k + 1) terms. We can write both the series in the order of $k^n, \dots, 1^n$. Then the modified form of (9.1) is:

$$c_1\,k^n + c_2\,(k-1)^n + \cdots + c_k\,1^n = d_1\,k^n + d_2\,(k-1)^n + \cdots + d_k\,1^n . \tag{9.2}$$

where $d_1 k^n$, $d_2\,(k-1)^n, \dots,$ $d_k\,1^n$ are in succession one, the sum of two, … , the sum of k terms among ½ k (k + 1) terms of the series on the right of (9.2). Equating the coefficients of like powers, we get: $d_1 = c_1$; $d_2 = c_2$; … ; $d_k = c_k$. The general form of these equalities is the following identity.

$$\sum_{i=0}^{m}(-1)^{i}\frac{(k+i)\,k^{i-1}}{i!} = (-1)^m\frac{k^m}{m!}. \tag{10}$$

(10) is the link between the recurrences of two kinds for $h_n\{k\}$ and can be established further by induction on m easily.

3. Comparable Pair of Characterization Formulas for $h_n\{x_k\}$ and Pairs of Summations

We can characterize $h_n\{x_k\}$ by a similar pair of generating series with a difference of same and alternating signs in the summations as shown.

$$\sum_{i=0}^{n-1} h_i\{x_{n+1-i}\}\prod_{j=1}^{n-i}(x-x_j) + x_1^n = x^n. \tag{11.1}$$

$$\sum_{i=0}^{n-1}(-1)^i\,h_i\{x_{n+1-i}\}\prod_{j=1}^{n-i}(x+x_j) + (-1)^n\,x_1^n = x^n. \tag{11.2}$$

Sameness in the proposed series is owing to the same sequence of equal number of variables in equal numbers of terms of the summations both of which are equal to x^n. Applying (1.1), we can prove both (11.1) and (11.2) in the same process of induction on n. Here we give the proof of (11.1).

Proof of (11.1): The proposition is trivial for n = 1. Consider the proposition holds for any given n. Then we deduce that

$$\sum_{i=0}^{n} h_i\{x_{n+2-i}\}\prod_{j=1}^{n+1-i}(x-x_j) + x_1^{n+1}$$

$$= \prod_{j=1}^{n+1}(x-x_j) + h_1\{x_{n+1}\}\prod_{j=1}^{n}(x-x_j) + h_2\{x_n\}\prod_{j=1}^{n-1}(x-x_j) + \dots$$

$$+ h_{n-1}\{x_3\}\prod_{j=1}^{2}(x-x_j) + h_n\{x_2\}\prod_{j=1}^{1}(x-x_j) + x_1^{n+1}.$$

$$= \prod_{j=1}^{n+1}(x-x_j) + [x_{n+1} + h_1\{x_n\}]\prod_{j=1}^{n}(x-x_j) + [x_n\,h_1\{x_n\} + h_2\{x_{n-1}\}]\prod_{j=1}^{n-1}(x-x_j)$$

$$+ \ \ldots \ + [\, x_2 \, h_{n-1}\{x_2\} \, + \, h_n\{x_1\} \,] \prod_{j=1}^{1}(x - x_j) \ + \ x_1^{n+1} .$$

[By (1.1)]

$$= \left[\prod_{j=1}^{n+1}(x - x_j) \, + x_{n+1} \prod_{j=1}^{n}(x - x_j) \right] \ + \ h_1\{x_n\} \left[\prod_{j=1}^{n}(x - x_j) \, + \, x_n \prod_{j=1}^{n-1}(x - x_j) \right] + \ \ldots$$

$$+ \ h_{n-1}\{x_2\} \left[\prod_{j=1}^{2}(x - x_j) \, + x_2 \prod_{j=1}^{1}(x - x_j) \right] \ + \ \left[h_n\{x_1\} \prod_{j=1}^{1}(x - x_j) \ + \ x_1^{n+1} \right].$$

$$= \ x \prod_{j=1}^{n}(x - x_j) \ + \ x \, h_1\{x_n\} \prod_{j=1}^{n-1}(x - x_j) \ + \ \ldots \ + \ x \, h_{n-1}\{x_2\} \prod_{j=1}^{1}(x - x_j) \ + x \, x_1^{n} .$$

$$= \ x . x^n \ = \ x^{n+1} .$$

The proposition follows.▮

3.1 Pair of Expressions for $\sum_{i=1}^{x}\binom{x+m-i}{m} i^n$ *from (11.1) and (11.2)*

Letting the power series: $1^n + 2^n + \cdots + x^n$ as the initial condition or zero order recurrence series: $S_0(x^n)$, we define the m^{th} order recurrence series: $S_m(x^n)$ by the recurrence relation: $S_m(x^n) = \sum_{j=1}^{x} S_{m-1}(j^n)$.

The initial condition is:

$$S_0(x^n) \ = \ \sum_{j=1}^{x} j^n = 1^n + 2^n + \ \ldots \ + x^n .$$

Then

$$S_1(x^n) \ = \ \sum_{j=1}^{x} S_0(j^n) \ = \ 1^n \ + \ (1^n + 2^n) + \ \ldots \ + (1^n + 2^n + \cdots + x^n)$$

$$= \sum_{i=1}^{x} (x + 1 - i) \, i^n \ ;$$

$$S_2(x^n) \ = \ \sum_{j=1}^{x} S_1(j^n) \ = \ \sum_{i=1}^{x} \ \ \sum_{j=1}^{x} (j \ + \ 1 - i) \, i^n$$

$$= \ \sum_{i=1}^{x} \binom{x + 2 - i}{2} i^n .$$

In general,

$$S_m(x^n) = \sum_{i=1}^{x} \binom{x + m - i}{m} i^n . \tag{12}$$

We can derive a pair of combinatorial formulas for $S_m(x^n)$ from (11.1) and (11.2).

(a) $S_m(x^n)$ from (11.1)

Substituting 1, ... , n for $x_1, \ldots, x_n$ in (11.1),

$$x^n = \sum_{i=0}^{n} h_i\{n+1-i\}\,(x-1)_{n-i}\,. \tag{13}$$

[Falling factorial: $(x)_n$ = $x\,(x-1)\,(x-2)\ldots(x-n+1)$.]

$$\Rightarrow x^n = \sum_{i=0}^{n} h_{n-i}\,\{i+1\}(x-1)_i. \tag{13.1}$$

When $x-1 < n$ then the values of the last $n-x+1$ terms on the right are all 0 owing to the law: ${}_aP_b$ or $(a)_b = 0$ for $a < b$; and in this case, x^n is equal to the sum of the first x terms. By the usual notation of combination, the modified form of (13.1) is:

$$x^n = \sum_{i=0}^{n} i!\;h_{n-i}\,\{i+1\}\binom{x-1}{i}. \tag{13.2}$$

Then

$$S_0(x^n) = \sum_{j=1}^{x} j^n = \sum_{i=0}^{n} i!\;h_{n-i}\,\{i+1\}\sum_{j=1}^{x}\binom{j-1}{i}\,.$$

$$\Rightarrow S_0(x^n) = \sum_{i=0}^{n} i!\;h_{n-i}\,\{i+1\}\binom{x}{i+1}. \tag{14}$$

Consequently,

$$\Rightarrow S_m\,(x^n) = \sum_{i=0}^{n} i!\;h_{n-i}\,\{i+1\}\,\binom{x+m}{m+i+1}. \tag{15}$$

(b) $S_m(x^n)$ from (11.2)

Substituting $1, \ldots, n$ for $x_1, \ldots, x_n$ in (11.2),

$$x^n = \sum_{i=0}^{n}(-1)^i\,h_i\{n+1-i\}\,(x+n-i)_{n-i}. \tag{16.1}$$

$$\Rightarrow x^n = \sum_{i=0}^{n}(-1)^i\,(n-i)!\;h_i\{n+1-i\}\,\binom{x+n-i}{n-i}\,. \tag{16.2}$$

Then

$$S_0(x^n) = \sum_{j=1}^{x} j^n = \sum_{i=0}^{n}(-1)^i\,(n-i)!\;h_i\{n+1-i\}\sum_{j=1}^{x}\binom{j+n-i}{n-i}.$$

$$\Rightarrow S_0(x^n) = \sum_{i=0}^{n}(-1)^i\,(n-i)!\;h_i\{n+1-i\}\,\binom{x+n+1-i}{x}. \tag{17}$$

Consequently,

$$S_m(x^n) = \textstyle\sum_{i=0}^{n}(-1)^i\;(n-i)!\;h_i\{n+1-i\}\,\binom{x+m+n+1-i}{x}. \tag{18}$$

3.2 Expressions for $\sum_{j=1}^{x} j^n (j+r)_r$ and $\sum_{j=1}^{x} j^n (j-1)_{r-1}$ from (13.1) and (16.1)

(a) $\sum_{j=1}^{x} j^n (j+r)_r$ from (13.1)

Multiplying (13.1) throughout by $x\,(x+r)_r$ and then substituting $n-1$ for n,

$$x^n(x+r)_r = \sum_{i=1}^{n} h_{n-i}\{i\}\,(x+r)_{r+i}\,.$$

$$\Rightarrow \sum_{j=1}^{x} j^n\,(j+r)_r = \sum_{i=1}^{n} h_{n-i}\{i\} \sum_{j=1}^{x} (j+r)_{r+i}$$

$$= \sum_{i=1}^{n} h_{n-i}\{i\}\,\tfrac{1}{r+i+1}\,(x+r+1)_{r+i+1}.$$

$$\Rightarrow \sum_{j=1}^{x} j^n\,(j+r)_r = (x+r+1)_{r+2} \sum_{i=1}^{n} h_{n-i}\{i\}\,\frac{1}{r+i+1}\,(x-1)_{i-1}. \qquad (19)$$

(b) $\sum_{j=1}^{x} j^n (j-1)_{r-1}$ from (16.1)

Multiplying (16.1) throughout by $x\,(x-1)_{r-1}$ and then substituting $n-1$ for n,

$$x^n(x-1)_{r-1} = \sum_{i=0}^{n-1} (-1)^i\,h_i\{n-i\}(x+n-1-i)_{r+n-1-i}\,.$$

$$\Rightarrow \sum_{j=1}^{x} j^n\,(j-1)_{r-1} = \sum_{i=0}^{n-1} (-1)^i\,h_i\{n-i\} \sum_{j=1}^{x} (j+n-1-i)_{r+n-1-i}$$

$$= \sum_{i=0}^{n-1} (-1)^i\,h_i\{n-i\}\,\frac{1}{r+n-i}\,(x+n-i)_{r+n-i}.$$

$$\Rightarrow \sum_{j=1}^{x} j^n\,(j-1)_{r-1} = (x+1)_{r+1} \sum_{i=0}^{n-1} (-1)^i\,\frac{1}{r+n-i}\,h_i\{n-i\}\,(x+n-i)_{n-1-i}\,. \qquad (20)$$

3.3 Applications of (19) and (20)

(a) Application of (19) to find $h_1\{k\}$, $h_2\{k\}$, ...

We can apply (2) and (19) to count $h_n\{k\}$ for $n = 1, 2,\dots$ Substituting $1, \dots, k$ for $x_1, \dots, x_k$ in (2), we get: $h_n\{k\} = \sum_{i=1}^{k} i\,h_{n-1}\{i\}$. Simply we write: $h_n\{k\} = \sum k\,h_{n-1}\{k\}$ or the summation series whose k^{th} term is $k\,h_{n-1}\{k\}$; and then obtain the counting formulas for $h_1\{k\}$, $h_2\{k\}$, … as shown.

$$h_1\{k\} = \sum k = \tfrac{1}{2}\,k\,(k+1)\ .$$

Applying (2) and (19),

$$h_2\{k\} = \sum k\,h_1\{k\}$$

$$= \tfrac{1}{2}\sum k^2\,(k+1)_1$$

$$= \tfrac{1}{2}\,(k+2)_3\,[\tfrac{1}{3} + \tfrac{1}{4}\,(k-1)_1\,]$$

$$= \tfrac{1}{4}\binom{k+2}{3}(3k+1)\,;$$

$$h_3\{k\} = \sum k\,h_2\{k\}$$

$$= \tfrac{1}{4}\sum \binom{k+2}{3}\,(3k^2+k\,)$$

$$= \frac{1}{24} \sum (k+2)_2 \, (3k^3 + k^2)$$

$$= \frac{1}{24}(k+3)_4 \left[\frac{1}{4}(3+1) + \frac{1}{5}(k-1)_1(3h_1\{2\}+1) + \frac{1}{6}(k-1)_2 \cdot 3 \right]$$

$$= \frac{1}{24}(k+3)_4 \left[1 + 2\,(k-1) + \frac{1}{2}(k-1)(k-2) \right]$$

$$= \binom{k+3}{4}\binom{k+1}{2} .$$

In this way,

$$h_4\{k\} = \frac{1}{48}\binom{k+4}{5}(15k^3 + 30k^2 + 5k - 2)\,;$$

$$h_5\{k\} = \frac{1}{8}\binom{k+5}{6}\binom{k+1}{2}(3k^2 + 7k - 2)\,;$$

$$h_6\{k\} = \frac{1}{576}\binom{k+6}{7}(63k^5 + 315\,k^4 + 315\,k^3 - 91k^2 - 42k + 16);$$

$$h_7\{k\} = \frac{1}{72}\binom{k+7}{8}\binom{k+1}{2}(9k^4 + 54k^3 + 51k^2 - 58k + 16).$$

From (3.2), we get: $h_n\{1\} = 1$ and $h_n\{2\} = 2^{n+1} - 1$; and hence primarily it is easy to verify the above successive results by putting k = 1 and k = 2.

(b) Application of (20) to find $e_1\{k\}$, $e_2\{k\}$, ...

Implementing the method of derivation of (1), we can also derive a fundamental identity for elementary symmetric polynomial. For n of k variables: $x_1, \dots, x_k$, elementary symmetric polynomial $e_n(x_1, \dots, x_k)$ or in brief

$$e_n\{x_k\} = \sum_{1 \le i_1 < i_2 < \dots < i_n \le k} x_{i_1} x_{i_2} \dots x_{i_n} .$$

Some special values of $e_n\{x_k\}$ are: $e_0\{x_k\}$ = 1; for $n > k$, $e_n\{x_k\}$ = 0; $e_1\{x_k\} = x_1 + \dots + x_k$; $e_k\{x_k\} = x_1 x_2 \dots x_k$.

Letting that x_m is a definite variable in the set $(x_1, \dots, x_{k+1})$, we denote other k variables by $y_1, \dots, y_k$. Then by the definition of $e_n\{x_k\}$, the following laws hold:

The number of terms of $e_n\{y_k\}$ is $\binom{k}{n}$. A term of $e_{n+1}\{y_k\}$ for $k \ge n + 1$ is also a term of $e_{n+1}\{x_{k+1}\}$, which does not contain x_m as a factor; and if x_m is multiplied with a term of $e_n\{y_k\}$ then the product is a term of $e_{n+1}\{x_{k+1}\}$, which contains x_m as a factor. This implies that $e_{n+1}\{y_k\}$ is the sum of $\binom{k}{n+1}$ terms of $e_{n+1}\{x_{k+1}\}$ where none of these terms has a factor x_m; and $x_m\, e_n\{y_k\}$ is the sum of other $\binom{k}{n}$ terms of $e_{n+1}\{x_{k+1}\}$ where x_m is a common factor of these terms. Then we have the following identity from the basic laws.

$$e_{n+1}\{x_{k+1}\} = e_{n+1}\{y_k\} + x_m\, e_n\{y_k\} . \tag{21}$$

We use (21) in the next topic. When $x_m = x_{k+1}$ then other k variables are: $x_1, \dots, x_k$ for which we can write $\{x_k\}$ instead of $\{y_k\}$. Hence from (21),

$$e_{n+1}\{x_{k+1}\} = e_{n+1}\{x_k\} + x_{k+1}\, e_n\{x_k\} . \tag{21.1}$$

$$\Rightarrow \sum_{i=1}^{k} [\, e_{n+1}\{x_{i+1}\} - e_{n+1}\{x_i\}\,] = \sum_{i=1}^{k} x_{i+1}\, e_n\{x_i\} .$$

$$\Rightarrow e_{n+1}\{x_{k+1}\} = \sum_{i=1}^{k} x_{i+1}\, e_n\{x_i\}. \tag{21.2}$$

Writing 1, …, k +1 for $x_1, \dots, x_{k+1}$, (21.2) is reduced to

$$e_{n+1}\{k+1\} = \sum_{i=1}^{k} (i+1) e_n\{i\}.$$

$$\Rightarrow e_n\{k\} = \sum_{i=1}^{k} i\, e_{n-1}\{i-1\}. \tag{21.3}$$

$$[e_0\{k\} = 1; \text{ for } k < n,\ e_n\{k\} = 0.]$$

To obtain the values of $e_1\{k\}, e_2\{k\}$, ... in succession, we write (21.3) simply as: $e_n\{k\} = \sum k\, e_{n-1}\{k-1\}$ or the summation series whose k^{th} term is $k\, e_{n-1}\{k-1\}$. Applying (20) and (21.3), we find the successive counting formulas as shown.

$$e_1\{k\} = \sum k = \frac{1}{2} k\,(k+1).$$

Applying (21.3) and (20),

$$\begin{aligned} e_2\{k\} &= \sum k\, e_1\{k-1\} \\ &= \frac{1}{2} \sum k^2\ (k-1)_1 \\ &= \frac{1}{2}\ (k+1)_3\ [\frac{1}{4}\ (k+2)_1 - \frac{1}{3}] \\ &= \frac{1}{4} \binom{k+1}{3} (3k+2); \end{aligned}$$

$$\begin{aligned} e_3\{k\} &= \sum k\, e_2\{k-1\} \\ &= \frac{1}{4} \sum k \binom{k}{3} (3k-1) \\ &= \frac{1}{24} \sum (k-1)_2 (3k^3 - k^2) \\ &= \frac{1}{24} (k+1)_4 \left[\frac{3}{6}\ (k+3)_2 - \frac{1}{5}\ (3h_1\{2\}+1)\ (k+2)_1 + \frac{1}{4}(3+1)\right] \\ &= \binom{k+1}{4} \left[\frac{1}{2}\ (k+3)(k+2) - 2\,(k+2) + 1\right] \\ &= \binom{k+1}{4} \binom{k+1}{2}. \end{aligned}$$

In this way,

$$e_4\{k\} = \frac{1}{48} \binom{k+1}{5} (15\,k^3 + 15\,k^2 - 10k - 8);$$

$$e_5\{k\} = \frac{1}{8} \binom{k+1}{6} \binom{k+1}{2} (3k^2 - k - 6);$$

$$e_6\{k\} = \frac{1}{576} \binom{k+1}{7} (63k^5 - 315k^3 - 224k^2 + 140k + 96);$$

$$e_7\{k\} = \frac{1}{72} \binom{k+1}{8} \binom{k+1}{2} (9k^4 - 18k^3 - 57k^2 + 34k + 80).$$

Since $e_n\{k\} = k!$ for $n = k$, primarily one can verify the above results by putting $n = k$.

Remark 1: Comparable forms of $h_n\{k\}$ and $e_n\{k\}$

The general form of the formulas for $h_1\{k\}$, $h_2\{k\}$, $h_3\{k\}$, …; and this for $e_1\{k\}$, $e_2\{k\}$, $e_3\{k\}$, …, can be written:

$$h_n\{k\} = \frac{1}{m} \binom{k+n}{n+1} (p_1\, k^{n-1} + \ldots + p_n). \tag{22.1}$$

$$e_n\{k\} \;=\; \frac{1}{m}\binom{k+1}{n+1}\,(q_1 k^{n-1} + \dots + q_n)\,. \tag{22.2}$$

Remark 2: Common property of divisibility of $h_n\{k\}$ and $e_n\{k\}$

For $n = 1$, $h_n\{k\} = e_n\{k\} = 1 + \dots + k = \binom{k+1}{2}$. We also notice that $\binom{k+1}{2}$ occurs as a common factor in the successive formulas for both $h_n\{k\}$ and $e_n\{k\}$ when n = 1, 3, 5 and 7. Different integer-values of $h_n\{k\}$ and $e_n\{k\}$ which can be obtained from their counting formulas lead to guess their common property of divisibility as stated in Conjectur1.1.

Conjecture 1.1: *Both $h_n\{k\}$ and $e_n\{k\}$ are exactly divisible by 1 + ... + k if n is odd.*

Conjecture 1.2 can be the basic rule with respect to Conjecture 1.1.

Conjecture 1.2: *A monomial symmetric polynomial of degree n in 1, 2, ..., k is exactly divisible by 1 + ... + k if n is odd.*

$h_n\{x_k\}$ is the sum of some distinct monomial symmetric polynomials and $e_n\{x_k\}$ is a special monomial symmetric polynomial. Therefore if it is possible to establish Conjecture 1.2 then Conjecture 1.1 is established at once.

Examples for Conjecture 1.2: Partitions of 5 are: 5, 4 + 1, 3 + 2, 3 + 1 + 1, 2 + 2 + 1, 2 + 1 + 1 + 1, 1 + 1 + 1 + 1 + 1. We present below two polynomial summations with respect to the partitions: 3 + 1 + 1 and 2 + 2 + 1.

$m_{(3,1,1)}(1,2,3) = 1^3\cdot 2\cdot 3 + 1\cdot 2^3\cdot 3 + 1\cdot 2\cdot 3^3 = 84.$

$m_{(2,2,1)}(1,2,3,4) = 1^2\cdot 2^2\cdot 3 + 1^2\cdot 2\cdot 3^2 + 1\cdot 2^2\cdot 3^2 + 1^2\cdot 2^2\cdot 4 + 1^2\cdot 2\cdot 4^2 + 1\cdot 2^2\cdot 4^2$

$+ 1^2\cdot 3^2\cdot 4 + 1^2\cdot 3\cdot 4^2 + 1\cdot 3^2\cdot 4^2 + 2^2\cdot 3^2\cdot 4 + 2^2\cdot 3\cdot 4^2 + 2\cdot 3^2\cdot 4^2 = 1030.$

84 and 1030 are divisible by (1 + 2 + 3) and (1 + 2 + 3 + 4) respectively.

4. Comparable Recurrence Functions for $(k)_n$ and $e_n\{x_k\}$ and a Relation between $h_n\{x_k\}$ and $e_n\{x_k\}$

(a) Recurrence function for $(k)_n$

Letting the initial condition: $F(1, k) = (k)_1$, we define an $(n + 1)^{\text{th}}$ order recurrence function $F(n + 1, k)$ by the recurrence relation:

$$\begin{aligned} F(n+1,k) &= (n+k)_1\,F(n,k) - (n+k)_2\,F(n-1,k) + \dots \\ &\quad + (-1)^{n-1}(n+k)_n\,F(1,\ k) + (-1)^n(n+k)_{n+1}. \end{aligned} \tag{23.1}$$

The solution of the recurrence function is Proposition1.

Proposition 1: $F(n + 1,\ k) = 0.$

Proof: The sum of the last two terms of the relation is:

$$\begin{aligned} &(-1)^{n-1}(n+k)_n\,F(1,\ k) + (-1)^n(n+k)_{n+1} \\ &= (-1)^{n-1}\{(n+k)_n\,k - (n+k)_{n+1}\} \\ &= (-1)^{n-1}\{(n+k)(n+k-1)\dots(k+1)\,k - (n+k)(n+k-1)\dots k\} = 0\,. \end{aligned}$$

It is then easy to prove the proposition by induction on n. We have: $F(2, k) = 0$. Assuming that the proposition is true for the first n natural numbers for any given n, we deduce that

$$\begin{aligned} F(n+2,\ k) &= (n+k+1)_1\,F(n+1,\ k) - (n+k+1)_2\,F(n,\ k) + \dots \\ &\quad + (-1)^{n-1}(n+k+1)_n\,F(2,\ k) + (-1)^n(n+k+1)_{n+1}\,F(1,\ k) + (-1)^{n+1}(n+k+1)_{n+2}\,. \\ &= (n+k+1)_1\cdot 0 - (n+k+1)_2\cdot 0 + \dots + (-1)^{n-1}(n+k+1)_r\cdot 0 + 0 \\ &= 0. \end{aligned}$$

The proposition follows. ▌

(b) Recurrence function for $e_n\{x_k\}$

Replacing the coefficients: $(n+k)_1,\ (n+k)_2,\ \dots,(n+k)_{n+1}$ from (23.1) by $e_1\{x_{n+k}\},\ e_2\{x_{n+k}\},\ \dots,e_{n+1}\{x_{n+k}\}$ respectively, we consider the recurrence relation (23.2) with the initial condition: $F(1, k) = e_1\{x_k\}$ as shown.

$$\begin{aligned} F(n+1,k) &= e_1\{x_{n+k}\}\,F(n,k) - e_2\{x_{n+k}\}\,F(n-1,k) + \dots \\ &\quad + (-1)^{n-1}e_n\{x_{n+k}\}\,F(1,\ k) + (-1)^n e_{n+1}\{x_{n+k}\}\,. \end{aligned} \tag{23.2}$$

(23.1) and (23.2) can yield the similar recurrence expressions that we show in Topic 4.2. The solution of the recurrence function: $F(n+1,\ k)$ is Proposition 4 which is the consequence of Proposition 2 and proposition 3.

Proposition 2: $F(n,1) = x_1^n$.

Proof: We give the proof by the method of induction.

$$F(1,1) = x_1 \ ;$$
$$F(2,1) = e_1\{x_2\}\, F(1{,}1) - e_2\{x_2\} = (x_1 + x_2)\, x_1 - x_1\, x_2 = x_1^2.$$

Hence the proposition holds for $n = 1$ and for $n = 2$. To complete the proof, we assume that the proposition holds for all $n \in \mathbb{N}$ with $1 \le n \le m$.

Then we deduce that

$$F(m+1,\ 1) = e_1\{x_{m+1}\}\, F(m,\ 1) - \ldots + (-1)^{m-1}\, e_m\{x_{m+1}\}\, F(1,\ 1) + (-1)^m\, e_{m+1}\{x_{m+1}\}\,.$$
$$= e_1\{x_{m+1}\}\, x_1^m - \ldots + (-1)^{m-1}\, e_m\{x_{m+1}\}\, x_1 + (-1)^m\, e_{m+1}\{x_{m+1}\}.$$

[By inductive assumption]

$$= [x_1 + e_1\{y_m\}]\, x_1^m - [x_1\, e_1\{y_m\} + e_2\{y_m\}]\, x_1^{m-1} + \ldots$$
$$+ (-1)^{m-1}\, [x_1\, e_{m-1}\{y_m\} + e_m\, \{y_m\}]\, x_1 + (-1)^m\, x_1\, e_m\{y_m\}\,.$$

[By (21)]

$$= x_1^{m+1} + [e_1\{y_m\}\, x_1^m - e_1\{y_m\}\, x_1^m] - [e_2\{y_m\}\, x_1^{m-1} - e_2\{y_m\}\, x_1^{m-1}] + \ldots$$
$$+ (-1)^{m-1}[e_m\{y_m\}\, x_1 - e_m\{y_m\}\, x_1]\,.$$
$$= x_1^{m+1} \ .$$

The proposition follows.▮

Corollary 1: $F(n,\ 1) = h_n\, \{x_1\}$.

Proposition 3: $F(n+1,\ k+1) = F(n+1,\ k) + x_{k+1}\, F(n,\ k+1)$.

Proof: The proof is inductive on n. When $n = 1$ and k is a fixed positive integer, we find:

$$F(2,\ k+1) = e_1\{x_{k+2}\}\, F(1,\ k+1) - e_2\, \{x_{k+2}\}$$
$$= [e_1\{x_{k+1}\} + x_{k+2}]\, F(1,\ k+1) - [e_2\{x_{k+1}\} + x_{k+2}\ e_1\{x_{k+1}\}]$$

[By (21.1)]

$$= e_1\{x_{k+1}\}\, [F(1,\ k) + x_{k+1}] + x_{k+2}\, e_1\{x_{k+1}\} - e_2\{x_{k+1}\} - x_{k+2}\, e_1\{x_{k+1}\}$$
$$= e_1\{x_{k+1}\}\, F(1,\ k) - e_2\{x_{k+1}\} + x_{k+1}\, e_1\{x_{k+1}\}$$
$$= F(2,\ k) + x_{k+1}\, F(1,\ k+1) \ .$$

Hence the proposition holds for $n = 1$ and a fixed k. We assume that the proposition holds for $n \in \mathbb{N}$ with $1 \le n \le m$ and a fixed k. Then we shall show that the proposition holds for $n = m + 1$ and a fixed k. We deduce that $F(m+2,\ k+1)$

$$= e_1\{x_{k+m+2}\}\, F(m+1,\ k+1) - e_2\{x_{k+m+2}\}\, F(m,\ k+1) + \ldots$$
$$+ (-1)^m\, e_{m+1}\{x_{k+m+2}\}\, F(1,\ k+1) + (-1)^{m+1}\, e_{m+2}\{x_{k+m+2}\}\,.$$
$$= [e_1\{x_{k+m+1}\} + x_{k+m+2}]\, F(m+1,\ k+1)$$
$$- [e_2\{x_{k+m+1}\} + x_{k+m+2}\, e_1\{x_{k+m+1}\}]\, F(m,\ k+1) + \ldots$$
$$+ (-1)^m\ [e_{m+1}\{x_{k+m+1}\} + x_{k+m+2}\ e_m\, \{x_{k+m+1}\}]\, F(1,\ k+1)$$
$$+ (-1)^{m+1}\ [e_{m+2}\{x_{k+m+1}\} + x_{k+m+2}\ e_{m+1}\{x_{k+m+1}\}] \ .$$
$$= e_1\{x_{k+m+1}\}\, F(m+1,\ k+1) - e_2\{x_{k+m+1}\}\, F(m,\ k+1) + \ldots$$
$$+ (-1)^m\, e_{m+1}\{x_{k+m+1}\}\, F(1,\ k+1) + (-1)^{m+1}\, e_{m+2}\{x_{k+m+1}\}$$
$$+ x_{k+m+2}\, F(m+1,\ k+1) - x_{k+m+2}\, F(m+1,\ k+1).$$
$$= e_1\{x_{k+m+1}\}\, [F(m+1,\ k) + x_{k+1} F(m,\ k+1)]$$
$$- e_2\{x_{k+m+1}\}\, [F(m,\ k) + x_{k+1}\, F(m-1,\ k+1)] + \ldots$$
$$+ (-1)^m\, e_{m+1}\{x_{k+m+1}\}\, [F(1,k) + x_{k+1}] + (-1)^{m+1}\, e_{m+2}\{x_{k+m+1}\}\,.$$

[By inductive assumption]

$= F(m + 2,\ k\ ) + x_{k+1}\, F(m + 1,\ k\ + 1).$

Thus we have the proposition by induction on *n*. Yet *k* can be given any positive integer-value to obtain the above result. It follows that the proposition holds for all $n, k \in \mathbb{N}$. ▌

Proposition 4: $F(n,\ k) = h_n\,\{x_k\}$.

Proof: From Proposition3,

$$\sum_{i=1}^{k} [F(n+1, i+1) - F(n+1, i\,)] = \sum_{i=1}^{k} x_{i+1}\, F(n, i+1)\,.$$

$$\Rightarrow F(n+1, k+1) - F(n+1, 1\,) = \sum_{i=1}^{k} x_{i+1}\, F(n, i+1)\,.$$

By Proposition 2,

$$F(n+1,\ \ k+1) = \sum_{i=1}^{k+1} x_i\, F(n, i\,)\,.$$

Then

(i) $F(2,\ k+1) = x_{k+1}\, F(1,\ k+1) + \ldots + x_1\, F(1,\ 1)$

$= x_{k+1}\, h_1\,\{x_{k+1}\} + \ldots + x_1\, h_1\,\{x_1\}\,.$

[The initial condition for (23.2) is: $F(1,\ k) = e_1\,\{x_k\} = h_1\,\{x_k\}$.]

$= h_2\,\{x_{k+1}\}$. [By (2)]

(ii) $F(3,\ k+1) = x_{k+1}\, F(2,\ k+1) + \ldots + x_1\, F(2,\ 1)$

$= x_{k+1}\, h_2\,\{x_{k+1}\} + \ldots + x_1\, h_2\,\{x_1\}$

$= h_3\{x_{k+1}\}$.

… …

Thus

$$F(n+1,\ k+1) = h_{n+1}\{x_{k+1}\}\,.$$

By the initial condition and Corollary 1,

$$F(n, k) = h_n\{x_k\}\,.$$

This completes the proof. ▌

4.1 Relation between $h_n\{x_k\}$ *and* $e_n\{x_k\}$

From (23.2) and Proposition 4, $h_{n+1}\{x_k\}$

$= e_1\,\{x_{k+n}\}\, h_n\{x_k\} - \ldots + (-1)^{n-1}\, e_n\{x_{k+n}\}\, h_1\{x_k\} + (-1)^n\, e_{n+1}\{x_{k+n}\}$.

Since $h_0\,\{x_k\} = e_0\,\{x_k\} = 1$, the above recurrence relation can be written:

$$\sum_{i=0}^{n} (-1)^i\, e_i\,\{x_{k+n-1}\}\, h_{n-i}\{x_k\} = 0\,. \tag{24}$$

We use (24) in Topic 5.

4.2 Common Occurrences of the Compositions in the Recurrence Expressions from (23.1) and (23.2)

(a) The initial condition of (23.1) is:

$$F(1, k) = (k)_1. \tag{25.1}$$

From Proposition1,

$$0 = F(2, k) = F(3, k) = F(4, k) = \ldots$$

That is,

$$0 = (k+1)_1\,(k)_1 - (k+1)_2\,. \tag{25.2}$$

$$= (k+2)_1(k+1)_1(k)_1 - (k+2)_1\,(k+1)_2 - (k+2)_2\,(k)_1 + (k+2)_3\,. \tag{25.3}$$

$$\begin{aligned} &= (k+3)_1(k+2)_1\,(k+1)_1\,(k)_1 - (k+3)_1\,(k+2)_1(k+1)_2 \\ &\quad - (k+3)_1\,(k+2)_2\,(k)_1 + (k+3)_1\,(k+2)_3 - (k+3)_2\,(k+1)_1(k)_1 \\ &\quad + (k+3)_2\,(k+1)_2 + (k+3)_3\,(k)_1 - (k+3)_4\,. \end{aligned} \tag{25.4}$$

… …

The process to find the successive results is recursive substitution. We get: (i) (25.2) from (23.1) and (25.1); (ii) (25.3) from (23.1), (25.1) and (25.2); (iii) (25.4) from (23.1), (25.1), (25.2) and (25.3); and so on. (25.1) contains only one bottom index: 1. Two sets of bottom indices in two terms of (25.2) are: (1, 1) and 2 such that 1 + 1 = 2. Four sets of bottom indices in four terms of (25.3) are: (1, 1, 1), (1, 2), (2, 1) and 3 such that 1 + 1 + 1 = 1 + 2 = 2 + 1 = 3. Eight sets of bottom indices in eight terms of (23.4) are: (1, 1, 1, 1), (1, 1, 2), (1, 2, 1), (1, 3), (2, 1, 1), (2, 2), (3, 1) and 4 such that 1 + 1 + 1 + 1 = 1 + 1 + 2 = 1 + 2 + 1 = 1 + 3 = 2 + 1 + 1 = 2 + 2 = 3 + 1 = 4. Thus 1, 2, 4 and 8 sets of bottom indices in 1, 2, 4 and 8 terms of (25.1), (25.2), (25.3) and (25.4) involve with 1, 2, 4 and 8 compositions of 1, 2, 3 and 4 respectively. In this way 2^{n-1} terms of (25.n) can involve with 2^{n-1} compositions of *n*.

(b) The initial condition of (23.2) is: $F(1, k) =$

$$h_1\{x_k\} = e_1\{x_k\}. \tag{26.1}$$

Then from (24) which is the consequence of (23.2) and Proposition 4, the recurrence expressions are:

$$h_2\{x_k\} = e_1\{x_{k+1}\}\, e_1\{x_k\} - e_2\{x_{k+1}\}. \tag{26.2}$$

$$h_3\{x_k\} = e_1\{x_{k+2}\}\, e_1\{x_{k+1}\}\, e_1\{x_k\} - e_1\{x_{k+2}\}\, e_2\{x_{k+1}\} \\ -e_2\{x_{k+2}\}\, e_1\{x_k\} + e_3\{x_{k+2}\}. \tag{26.3}$$

$$h_4\{x_k\} = e_1\{x_{k+3}\}\, e_1\{x_{k+2}\}\, e_1\{x_{k+1}\}\, e_1\{x_k\} \\ - e_1\{x_{k+3}\}\, e_1\{x_{k+2}\}\, e_2\{x_{k+1}\} - e_1\{x_{k+3}\}\, e_2\{x_{k+2}\}\, e_1\{x_k\} \\ + e_1\{x_{k+3}\}\, e_3\{x_{k+2}\} - e_2\{x_{k+3}\}\, e_1\{x_{k+1}\}\, e_1\{x_k\} \\ + e_2\{x_{k+3}\}\, e_2\{x_{k+1}\} + e_3\{x_{k+3}\}\, e_1\{x_k\} - e_4\{x_{k+3}\}. \tag{26.4}$$

$$\dots \quad \dots$$

1, 2, 4 and 8 sets of bottom indices of the notation: *e* in 1, 2, 4 and 8 terms of (26.1), (26.2), (26.3) and (26.4) involve with 1, 2 , 4 and 8 compositions of 1, 2, 3 and 4 respectively. In this way 2^{n-1} terms of (26.n) can involve with 2^{n-1} compositions of *n*.

We further notice that '*k*' occurs inside the parenthesis of (25.1); and as the bottom index of *x* in (26.1). Two sets of parenthesis in two terms of (25.2) contain two sets of integers: (*k* + 1, *k*); and *k* + 1; which are also two sets bottom indices of the notation: *x* in two terms of (26.2) in the same order. Four sets of parenthesis in four terms of (25.3) contain four sets of integers: (*k* + 2, *k* + 1, *k*); (*k* + 2, *k* + 1); (*k* + 2, *k*); and *k* + 2; which are also four sets bottom indices of the notation: *x* in four terms of (26.3) in the same order; and so on.

5. Divisibility of $h_n\{k\}$ by an Odd Prime

We establish here two theorems regarding divisibility of $h_n\{k\}$ by an odd prime *p*. The famous Lagrange's Theorem in classical number theory is important to prove the theorems.

Lagrange's Theorem: *Let p be an odd prime and x an integer, and let*

$$(x+1)\,(x+2)\,\dots\,(x+p-1) = x^{p-1} + c_1 x^{p-2} + \dots + c_{p-2}x + (p-1)!$$

Then the coefficients $c_1, \ \dots, \ c_{p-2}$ *are all divisible by p.*

Lagrange's Theorem in short form: *If p is an odd prime and n an integer with* $1 \le n \le p - 2$ *then* $e_n\{p-1\} \equiv 0 \ (mod\ p)$.

Theorem 1: *An odd prime p divides* $h_n\{k\}$ *if* $k + n = p$, $n \ge 1$, $k \ge 2$.

Proof: Substituting 1, 2, …, *k* for $x_1, \ x_2, \dots, \ x_k$ respectively, we get the reduced form of (24):

$$h_n\{k\} = \sum_{i=1}^{n} (-1)^{i-1}\, e_i\{k+n-1\}\, h_{n-i}\{k\}. \tag{27}$$

When $k + n = p$, $n \ge 1$, $k \ge 2$ then $1 \le n \le p - 2$. By Lagrange's Theorem, the coefficients: $e_1\{k+n-1\}$, …, $e_n\{k+n-1\}$ on the right of (27) are all divisible by *p* if $k + n = p$, $1 \le n \le p - 2$. The theorem follows at once.▌

We can enunciate Theorem1 in an alternative form.

Alternative form of Theorem 1: *If p is an odd prime then* $h_1\{p-1\}$, $h_2\{p-2\}$, … , $h_{p-2}\{2\}$ *are all divisible by p.*

Theorem 2: *An odd prime p divides* $h_n\{k\}$ if $k + n = p + 1$, $n \ge 1$, $k \ge 3$.

Proof. From (1.1),

$$h_n\{k\} = h_n\{k-1\} + k\, h_{n-1}\{k\}. \tag{28}$$

Let us verify divisibility of the right hand side of (28) by *p*. By Theorem1, *p* divides:

(i) $h_n\{k-1\}$ if $n + (k \quad -1) = p$, $\quad n \geq 1$, $\quad k - 1 \geq 2$ and

(ii) $h_{n-1}\{k\}$ if $(n \quad -1) + k = p$, $\quad n - 1 \geq 1$, $\quad k \geq 2$.

p divides $kh_{n-1}\{k\}$ if n =1 and $k = p$.

It follows that p divides both the terms on the right of (28); and hence divides $h_n\{k\}$ if $k + n = p + 1$, $n \geq 1$, $k \geq 3$. This completes the proof. ▌

Alternative form of Theorem 2: *If p is an odd prime then* $h_1\{p\}$, $h_2\{p-1\}$, $\ldots, h_{p-2}\{3\}$ *are all divisible by p.*

(1) and (21); (2) and (3); (2.1) and (2.2); (3.2) and (8); (11.1) and (11.2); (15) and (18); (19) and (20); (22.1) and (22.2); (25.n) and(26.n); Theorem 1 and Theorem 2 in pairs are comparable or analogous. The math formulas or laws under each pair are alike in some ways. In fact the polynomial, which is specified by the pair of adjectives: homogeneous and symmetric, is rhythmical for its involvement with many pairs of comparable relations.

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Annexure

The following inequalities may be relevant when we speak of comparable relations for $h_n\{k\}$.

6. Inequalities between $h_n\{k\}$ and $h_k\{n\}$

We have: $1^n < n^1$ where $n > 1$; $2^3 < 3^2$; $2^4 = 4^2$; and for other positive integer-values of n and k, the general rule is: $k^n > n^k$ for $n > k$. These are the inequalities between the monomials of single variables: k^n and n^k for $n, k \in \mathbb{N}$. The inequalities between $h_n\{k\}$ and $h_k\{n\}$ can be comparable with the inequalities between k^n and n^k. We have: $h_n\{1\} = 1^n$; and $h_1\{n\} = 1 + \ldots + n$. Hence $h_n\{1\} < h_1\{n\}$ for $n > 1$. We can get different integer-values for $h_n\{k\}$ by using the counting formulas: (3.2) & (2). Surveying the values, it is found that there is no single formula for the inequalities between $h_n\{k\}$ and $h_k\{n\}$ when $n \in (2, 3)$; or when $k \in (2, 3)$; and rather exists a general rule when $n, k \geq 4$, $n \neq k$. The rules for the inequalities are shown in Conjecture 2.

Conjecture 2: *For all n, k* $\in \mathbb{N}$,

$$h_n\{2\} < h_2\{n\} \text{ if } 3 \leq n \leq 9,$$

$$h_n\{3\} < h_3\{n\} \text{ if } 4 \leq n \leq 5,$$

and in all other cases, the general rule is: $h_n\{k\} > h_k\{n\}$*; more precisely:*

$$h_n\{2\} > h_2\{n\} \text{ if } n \geq 10,$$

$$h_n\{3\} > h_3\{n\} \text{ if } n \geq 6,$$

$$h_n\{k\} > h_k\{n\} \text{ if } n > k \geq 4.$$

The inequalities between the monomials of single variables give an idea about existence of the flowing math problem.

Let $n = a_1b_1 = a_2b_2 = a_3b_3 = \ldots$, where $n, a_1, b_1, a_2, b_2, \ldots$ are all positive real numbers. Which one among the monomials of single variables: $a_1^{b_1}$, $b_1^{a_1}$, $a_2^{b_2}$, $b_2^{a_2}, \ldots$ is the greatest? A math rule with respect to the problem is guessed and stated in Conjecture 3.

Conjecture 3: *Let* $n = a \cdot b$ *where n, a, and b are all positive real numbers. Then the greatest value of* a^b *for any given n is* $3^{n/3}$ *; that is, a = 3 and b =* $\frac{n}{3}$ *for the desired greatest value.*

Conjecture 3 further gives an idea about existence of the following math problem.

Let a positive integer *n* has *m* partitions: $n = a_1 + a_2 + \ldots = b_1 + b_2 + = \ldots = \ldots$. If we replace the plus signs (+) by the multiplication dots (·) then we get the products: $a_1 . a_2 . \ldots ;$ $b_1 . b_2 . \ldots ;$ and so on. Now the problem is: Which product is the greatest? A math rule is guessed regarding the problem and stated in Conjecture 4.

Some integers are of the kind: $3k$ for $k = 1$ then others are of two kinds: $3k - 1$ and $3k - 2$. Excluding the preliminary case of the first three natural numbers: 1, 2 and 3; Conjecture 4 is stated below for other natural numbers.

Conjecture 4: *(i) If the number is of the kind:* $3k$ *for* $k \geq 2$*, then the greatest product is* 3^k *with respect to the partition:* 3 + 3 + 3 + …; *(ii) if the number is of the kind:* $3k - 1$ *for* $k \geq 2$*, then the greatest product is* $2 \cdot 3^{k-1}$ *with respect to the partition:* 2 + 3 + 3 + 3 + …; *and (iii) if the number is of the kind:* $3k - 2$ *for* $k \geq 2$*, then the greatest product is* $2^2 \cdot 3^{k-2}$ *with respect to the partition:* 2 + 2 + 3 + 3 + 3 + …

The roles of 3 in Conjecture 3 and Conjecture 4 are remarkable.

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