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Global Attractor for Caginalp Hyperbolic Field-phase System with Singular Potential

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Abstract

This article is devoted to the study of the Caginalp hyperbolic phase-field system with singular potentials. We first prove the existence and uniqueness of solutions for Caginalp hyperbolic phase-field system with logarithmic potential. We then prove the existence of global attractor. One of main difficulties is to prove that the solutions are strictly separated from singular values of the potential.

Keywords: Caginalp hyperbolic phase-field system, logarithmic potential, Dirichlet boundary conditions, phase space, bounded absorbing set, dissipativity, global attractor

Subject Classification: Primary: 35D05, 35D10, 35D20, 35D25,35D30, Secondary: 46F30

1. Introduction

The global attractor is a compact invariant and smallest set which attracts the bounded sets of phase space. It presents two major default: it can attract the trajectories slowly and it can be sensitive to perturbations. In this article, we are interested in the study of the following Caginalp hyperbolic phase-field system in a smooth and bounded domain $\Omega \subset \mathbb{R}^n$, $1 \le n \le 3$,

$$\epsilon\partial_t^2 u + \partial_t u - \Delta u + f(u) = \partial_t \alpha \tag{1}$$

$$\partial_t^2 \alpha - \partial_t \Delta\alpha - \Delta\alpha = -\partial_t u, \tag{2}$$

with homogenous Dirichlet conditions

$$u = \alpha = 0 \text{ on } \partial\Omega, \tag{3}$$

and initial conditions

$$u|_{t=0} = u_0 \quad \partial_t u|_{t=0} = u_1 \qquad \alpha|_{t=0} = \alpha_0 \quad \partial_t \alpha|_{t=0} = \alpha_1, \tag{4}$$

where $\epsilon > 0$ is a relaxation parameter, $u = u(t, x)$, the order parameter and $\alpha = \alpha(t, x)$ are unknown functions, f is a given singular potential function.
Consider the following logarithmic potential function

$$f(s) = -k_0 s + k_1 \ln \tfrac{1+s}{1-s}, \quad s \in (-1, 1),\ 0 < k_1 < k_0.$$

The function f satisfies the following properties

$$f \in C^2(-1, 1), \quad \lim_{s\to\pm1} f(s) = \pm\infty, \quad \lim_{s\to\pm1} f'(s) = +\infty,$$

$$-c_0 \le F(s) \le f(s)s + c_0 \text{ where } F(s) = \int_0^s f(\tau)\, d\tau \text{ and } c_0 > 0, \tag{5}$$

$$-c_1 \le f'(s), \quad c_1 > 0 \quad \forall s \in [0, 1). \tag{6}$$

The system (1) – (2) has a sense only if

$$-1 < u(t, x) < 1$$

for almost all $(t, x) \in \mathbb{R}_+ \times \Omega$. This reason leads us to introduce the following quantity

$$\mathbb{D}[u(t)] = (1 - \|u(t)\|_{L^\infty})^{-1}.$$

We introduce also the standard energy norm of the initial and boundary value problem for singularly perturbed damped hyperbolic equation (1)

$$\|\zeta_u\|^2_{\varepsilon^\kappa(\epsilon)} = \|u\|^2_{H^{\kappa+1}} + \epsilon\|\partial_t u\|^2_{H^\kappa} + \|\partial_t u\|^2_{H^{\kappa-1}},$$

where $\varepsilon^\kappa(\epsilon)$ coincides with $[H^{\kappa+1}(\Omega)\times H^\kappa(\Omega)]\cap\{\zeta|_{\partial\Omega}=0\}$. This standard energy norm is used in (Fabrie, Galusinski, Miranville & Zelik, 2014), (Grasselli, Miranville, Pata & Zelik, 2007), (Moukoko 2014, 2015).

The Caginalp parabolic system, with various types of boundary conditions and for a regular or singular potentials f, has been extensively studied, (see, e.g.,(Cherfils, Gatti & Miranville, 2008), (Conti, Gatti & Miranville, 2012), (Cherfils & Miranville, 2009), (Efendiev, Miranville & Zelik, 2003 , 2004), (Fabrie, Galusinski, Miranville & Zelik, 2014), (Miranville & Quintanilla, 2009). There are few studies concerned the Caginalp parabolic-hyperbolic system.

Recently, above Caginalp hyperbolic phase-field system endowed with homogenous Dirichlet boundary conditions with a regular potentials, is studied in (Moukoko 2014, 2015), in order to prove the existence and uniqueness of solutions, existence of: global attractor, exponential attractors and the robust family of exponential attractors.

Here we are interested in the Caginalp hyperbolic system with homogenous Dirichlet boundary conditions and logarithmic potential. We prove the existence and uniqueness of solutions, as well as regularity. The main difficulties in this article is to prove that the order parameter u is strictly separated from the singular values of the potential.

In this article, we denote by $\|.\|$ and $(.,.)$ (or $\|.\|_\phi$) the norm and the scalar product in $L^2(\Omega)$ (in Φ).

2. Method

In this section we brief on method needed to prove our two main results of the next section.

We first prove the existence of the solution which are separated from the singular values ± 1 of the singular potential f. We replace the logarithmic potential by a regular function and prove that the solution of the resulting system is also the solution of initial system. We define two phase spaces. in order to end, we first prove the existence of the bounded absorbing set in each phase space, and owing to (Miranville & Zelik, 2008), we prove the existence of global attractor. Although in our study, we have to use classical methods of functional analysis applied in the theory of Partial Differential Equations.

3. Results

We first give some estimates which allow us to determine a first phase space.

3.1 A Priory Estimates

We a priory assume $\|u_0\|_{L^\infty(\Omega)} < 1$ and $\|u\|_{L^\infty((0,T)\times\Omega)} < 1$.
Multiplying (1) by $2\partial_t u$ and (2) by $2\partial_t\alpha$, integrating over Ω and adding the two resulting equations, we have

$$\frac{d}{dt}E_1 + 2\|\partial_t u\|^2 + 2\|\partial_t\alpha\|^2_{H^1} = 0,$$

where

$$E_1 = \epsilon\|\partial_t u\|^2 + \|u\|^2_{H^1} + \|\partial_t\alpha\|^2 + \|\alpha\|^2_{H^1} + 2(F(u),1),$$

which implies, by integrating between 0 and t

$$\epsilon\|\partial_t u(t)\|^2 + \|u(t)\|^2_{H^1} + \|\partial_t\alpha\|^2 + \|\alpha\|^2_{H^1} + \int_0^t \|\partial_t u(s)\|^2 ds + \int_0^t \|\partial_t\alpha(s)\|^2_{H^1} ds \le 4c_0|\Omega| + K, \tag{7}$$

where K is a positive constant.
According to estimate (7), $u,\alpha\in L^\infty(\mathbb{R}_+;H^1_0(\Omega))$, $\partial_t u\in L^\infty(\mathbb{R}_+;L^2(\Omega))\cap L^2(0,T;L^2(\Omega))$ and $\partial_t\alpha\in L^\infty(\mathbb{R}_+;L^2(\Omega))\cap L^2(0,T;H^1_0(\Omega))$, $\forall T>0$.

We multiply (1) by $-2\Delta\partial_t u$ and (2) by $-2\Delta\partial_t\alpha$, integrate over Ω and add two resulting equations. We have

$$\begin{aligned}\frac{d}{dt}\Big(\epsilon\|\partial_t u\|^2_{H^1} + \|u\|^2_{H^2} + \|\partial_t\alpha\|^2_{H^1} + \|\alpha\|^2_{H^2}\Big) + \|\partial_t u\|^2_{H^1} + 2\|\partial_t\alpha\|^2_{H^2} &\le C\|u\|^2_{H^1} &(8)\\ &\le C\|u\|^2_{H^2}. &(9)\end{aligned}$$

Estimate (8) implies $u, \alpha \in L^\infty(0,T; H^2(\Omega) \cap H^1_0(\Omega))$, $\partial_t u \in L^\infty(0,T; H^1_0(\Omega)) \cap L^2(0,T; H^1_0(\Omega))$ and $\partial_t \alpha \in L^\infty(0,T; H^1_0(\Omega)) \cap L^2(0,T; H^2(\Omega) \cap H^1_0(\Omega))$, $\forall T > 0$. Moreover, u is continuous almost for all $t \in [0,T]$.
Multiplying (1) by $2\partial_t^2 u$ and integrate over Ω, we have

$$\frac{d}{dt}\|\partial_t u\|^2 + \epsilon\|\partial_t^2 u\|^2 \quad \leq \quad C_1\|\partial_t \alpha\|^2 + C_2\|u\|^2_{H^2} + C_3\|f(u)\|^2, \tag{10}$$

Which implies $\partial_t u \in L^2(0,T; L^2(\Omega))$.
We multiply (2) by $-2\Delta\partial_t^2\alpha$ and integrate over Ω. We have

$$\frac{d}{dt}\Big(\|\partial_t\alpha\|^2_{H^2} + 2(\Delta\alpha, \Delta\partial_t\alpha)\Big) + \|\partial_t^2\alpha\|^2_{H^1} \quad \leq \quad \|\partial_t u\|^2_{H^1} + 2\|\partial_t\alpha\|^2_{H^2}. \tag{11}$$

Adding (9) and $\gamma_1(11)$ where $\gamma_1 > 0$, we obtain

$$\frac{d}{dt}E_2 + \|\partial_t u\|^2_{H^1} + 2(1-\gamma_1)\|\partial_t\alpha\|^2_{H^2} + \gamma_1\|\partial_t^2\alpha\|^2_{H^1} \quad \leq \quad \gamma_1\|\partial_t u\|^2_{H^1} + C\|u\|^2_{H^1}, \tag{12}$$

where

$$E_2 = \epsilon\|\partial_t u\|^2_{H^1} + \|u\|^2_{H^2} + \|\partial_t\alpha\|^2_{H^1} + \|\alpha\|^2_{H^2} + \gamma_1(\|\partial_t\alpha\|^2_{H^2} + 2(\Delta\alpha, \Delta\partial_t\alpha)).$$

We know that

$$\|\alpha\|^2_{H^2} + \gamma_1(\|\partial_t\alpha\|^2_{H^2} + 2(\Delta\alpha, \Delta\partial_t\alpha)) \quad \geq \quad (1-2\gamma_1)\|\alpha\|^2_{H^2} + \frac{\gamma_1}{2}\|\partial_t\alpha\|^2_{H^2}.$$

Choosing γ_1 such that $1 - 2\gamma_1 > 0$, we have $E_2 \geq 0$ and

$$\|\partial_t\alpha\|^2_{H^1} + (1-2\gamma_1)\|\alpha\|^2_{H^2} + \frac{\gamma_1}{2}\|\partial_t\alpha\|^2_{H^2} \leq E_2. \tag{13}$$

According to (13), Estimate (12) implies $\partial_t\alpha \in L^\infty(0,T; H^2(\Omega) \cap H^1_0(\Omega))$ and $\partial_t^2\alpha \in L^2(0,T; H^1_0(\Omega))$. Since $H^2(\Omega) \subset L^\infty(\Omega)$ with continuous injection , then $\partial_t\alpha \in L^\infty([0,T]\times\Omega)$, there exists $c_2 > 0$ such that $\|\partial_t\alpha(t)\|_{L^\infty(\Omega)} \leq c_2 \quad \forall t \in [0,T]$, where c_2 depends of T and initial conditions.

In our study there are two main results; we prove the existence and uniqueness of solution and the existence of global attractor.

3.2 Existence and Uniqueness of Solution

Theorem 1. (Existence) *We assume* $(u_0, u_1, \alpha_0, \alpha_1) \in (H^2(\Omega) \cap H^1_0(\Omega)) \times H^1_0(\Omega) \times (H^2(\Omega) \cap H^1_0(\Omega)) \times (H^2(\Omega) \cap H^1_0(\Omega))$ *such that* $\| u_0 \|_{L^\infty(\Omega)} < 1$. *Then, the system* $(1)-(2)$ *possesses at least one solution* (u, α) *such that* $u, \alpha \in L^\infty(0,T; H^2(\Omega) \cap H^1_0(\Omega))$, $\partial_t u \in L^\infty(0,T; H^1_0(\Omega)) \cap L^2(0,T; H^1_0(\Omega))$, $\partial_t\alpha \in L^\infty(0,T; H^2(\Omega) \cap H^1_0(\Omega)) \cap L^2(0,T; H^2(\Omega) \cap H^1_0(\Omega))$, $\partial_t^2 u \in L^2(0,T; L^2(\Omega))$ *and* $\partial_t^2\alpha \in L^2(0,T; H^1_0(\Omega)), \forall T > 0$. *Moreover, there exists* $\delta = \delta(T, u_0) \in (0,1)$ *such that* $\| u(t) \|_{L^\infty((0,T)\times\Omega)} \leq \delta, \forall t \in [0,T], \forall T > 0$.

Proof. In order to prove this Theorem, we first show that all solution (u, α) of system $(1)-(2)$ is such that u is separated from the singular points of f, i.e., there exists $\delta \in (0,1)$ depending of T such that $\|u\|_{L^\infty((0,T)\times\Omega)} < \delta$. In the second time, we study the auxiliary problem of the system $(1)-(2)$. Finally, we show that the solution of auxiliary problem is also the solution of system $(1)-(2)$.

From previous section, we have $\partial_t\alpha \in L^\infty([0,T]\times\Omega)$, then there exists $\delta \in (0,1)$ such that

$$\|u_0\|_{L^\infty(\Omega)} \leq \delta, \quad f'(\delta) > 0 \text{ and } \|\partial_t\alpha\|_{L^\infty([0,T]\times\Omega)} < f(\delta) \quad \forall t \in [0,T].$$

We set $U = u - \delta$ and $U^+ = max(U, 0)$, then U satisfies the following equation

$$\epsilon\partial_t^2 U + \partial_t U - \Delta U + f(u) - f(\delta) = \partial_t\alpha - f(\delta). \tag{14}$$

Multiplying (14) par $2\gamma_2 U^+ + 2\gamma_3\partial_t U^+$ and integrating over Ω, we have

$$\frac{d}{dt}E_3 + 2\gamma_2\|U^+\|^2_{H^1} + 2\gamma_3\|\partial_t U^+\|^2 \quad \leq \quad 2(\partial_t\alpha - f(\delta), \gamma_2 U^+ + \gamma_3\partial_t U^+) + C_1(\|U^+\|^2 + \|\partial_t U^+\|^2). \tag{15}$$

where

$$E_3 = \gamma_2(2\epsilon(\partial_t U^+, U^+) + \|U^+\|^2) + \gamma_3(\epsilon\|\partial_t U^+\|^2 + \|U^+\|^2_{H^1}).$$

We have also

$$\begin{aligned}\gamma_2(2\epsilon(\partial_t U^+, U^+) + \|U^+\|^2) + \gamma_3\epsilon\|\partial_t U^+\|^2 &\geq \gamma_3\epsilon\|\partial_t U^+\|^2 + \gamma_2\Big(\|U^+\|^2 - 2(\epsilon^2\|\partial_t U^+\|^2 + \frac{1}{4}\|U^+\|^2)\Big)\\ &\geq \epsilon(\gamma_3 - 2\epsilon\gamma_2)\|\partial_t U^+\|^2 + \frac{\gamma_2}{2}\|U^+\|^2. \quad (16)\end{aligned}$$

Choosing γ_2 and γ_3 such that

$$\begin{aligned}\gamma_3 - 2\epsilon\gamma_2 > 0,\\ \gamma_2 U^+(t) + \gamma_3\partial_t U^+(t) \geq 0 \quad \forall t \in [0, T], \quad (17)\end{aligned}$$

there exists $C > 0$ such that

$$C^{-1}(\|U^+(t)\|^2 + \|\partial_t U^+(t)\|^2 + \|U^+(t)\|^2_{H^1}) \leq E_3(t) \leq C(\|U^+(t)\|^2 + \|\partial_t U^+(t)\|^2 + \|U^+(t)\|^2_{H^1}). \quad (18)$$

Thanks to estimates (17) – (18), there exists $k > 0$ such that estimate (15) implies

$$\frac{d}{dt}E_3(t) \leq kE_3(t),$$

which gives, using Gronwall's lemma

$$E_3(t) \leq E_3(0)e^{kT} \quad \forall t \in [0, T].$$

applying estimate (18) to the above estimate, we have $\forall t \in [0, T]$

$$\|U^+(t)\|^2 + \|\partial_t U^+(t)\|^2 + \|U^+(t)\|^2_{H^1} \leq K(\|U^+(0)\|^2 + \|\partial_t U^+(0)\|^2 + \|U^+(0)\|^2_{H^1})e^{kT} \quad (19)$$

According to the estimate (18), we have $u(0) < \delta$ for almost all $x \in \Omega$, that implies $U^+(0) = 0$. Moreover, since u is continuous for almost all $t \in [0, T]$ and $u(0) < \delta$, there exists $t_0 \in [0, T]$ such that $u(t) < \delta \quad \forall t \in [0, t_0[$, this implies $U^+(t) = 0 \quad \forall t \in [0, t_0[$ and $\partial_t U^+(0) = 0$. Indeed, estimate (19) implies $U^+(t) = 0 \quad \forall t \in [0, T]$, then $u(t) \leq \delta \quad \forall t \in [0, T]$.

In order to show $-\delta \leq u(t) \ \forall t \in [0, T]$, we set $V = u + \delta$ and $V^- = min(V, 0)$, then V satisfies the following equation

$$\epsilon\partial_t^2 V + \partial_t V - \Delta V + f(u) + f(\delta) = \partial_t\alpha + f(\delta). \quad (20)$$

Multiplying (20) by $\gamma_4 V^- + \gamma_5\partial_t V^-$ where γ_4 and $\gamma_5 > 0$ and integrating over Ω, we have

$$\begin{aligned}\frac{d}{dt}E_4 + 2\gamma_5\|\partial_t V^-\|^2 + 2\gamma_4\|V^-\|^2_{H^1} + 2\gamma_4(f(u) + f(\delta), \partial_t V^-) &= 2(\partial_t\alpha + f(\delta), \gamma_4 V^- + \gamma_5\partial_t V^-)\\ &+ 2\gamma_4\epsilon\|\partial_t V^-\|^2, \quad (21)\end{aligned}$$

where

$$E_4 = \gamma_4(2\epsilon(\partial_t V^-, V^-) + \|V^-\|^2) + \gamma_5(\epsilon\|\partial_t V^-\|^2 + \|V^-\|^2_{H^1}).$$

We have

$$\begin{aligned}\gamma_5\epsilon\|\partial_t V^-\|^2 + \gamma_4(2\epsilon(\partial_t V^-, V^-) + \|V^-\|^2) &\geq \gamma_5\epsilon\|\partial_t V^-\|^2 + \gamma_4\Big(-2(\epsilon^2\|\partial_t V^-\|^2 + \frac{1}{4}\|V^-\|^2) + \|V^-\|^2\Big)\\ &\geq \epsilon(\gamma_5 - 2\epsilon\gamma_4)\|\partial_t V^-\|^2 + \frac{1}{2}\|V^-\|^2\end{aligned}$$

Choosing γ_4 and γ_5 such that $\gamma_5 - 2\epsilon\gamma_4 > 0$ and $\gamma_4 V^- + \gamma_5\partial_t V^- \leq 0$ for almost $t \in [0, T]$, there exists $k > 0$ such that (21) implies the following estimate

$$\frac{d}{dt}E_4 \leq kE_4. \quad (22)$$

Moreover, there exists $C > 0$ such that

$$C^{-1}(\epsilon\|\partial_t V^-(t)\|^2 + \|V^-(t)\|^2 + \|V^-(t)\|^2_{H^1}) \leq E_4(t) \leq C(\epsilon\|\partial_t V^-(t)\|^2 + \|V^-(t)\|^2 + \|V^-(t)\|^2_{H^1}). \quad (23)$$

Applying Gronwall's lemma to (22), we have, owing to estimate(23),

$$\epsilon\|\partial_t V^-(t)\|^2 + \|V^-(t)\|^2 + \|V^-(t)\|^2_{H^1} \leq K(\epsilon\|\partial_t V^-(0)\|^2 + \|V^-(0)\|^2 + \|V^-(0)\|^2_{H^1})e^{kT}\forall t \in [0, T]. \tag{24}$$

Thanks to assumptions of Theorem, we have $-\delta < u(0)$ for almost all $x \in \Omega$, which implies $V^-(0) = 0$. Moreover, since u is continuous for almost all $t \in [0, T]$ and $-\delta < u(0)$, there exists $t_1 > 0$ such that $-\delta < u(t)$ for almost all $t \in [0, t_1[$, then $V^-(t) = 0$ for almost all $t \in [0, t_1[$ and $\partial_t V^-(0) = 0$. Indeed, estimate(24) implies $V^-(t) = 0$ for almost all $t \in [0, T]$, then $-\delta \leq u(t)$ for almost all $t \in [0, T]$.

It then proven that there exists $\delta \in (0, 1)$ such that $\|u(t)\|_{L^\infty((0,T)\times\Omega)} \leq \delta$.
We now have to prove the existence of a solution (u^δ, α) of the following auxiliary system

$$\epsilon\frac{\partial^2 u^\delta}{\partial t^2} + \frac{\partial u^\delta}{\partial t} - \Delta u^\delta + f_\delta(u^\delta) = \frac{\partial \alpha}{\partial t} \tag{25}$$

$$\frac{\partial^2 \alpha}{\partial t^2} - \frac{\partial \Delta\alpha}{\partial t} - \Delta\alpha = -\frac{\partial u^\delta}{\partial t} \tag{26}$$

obtained by replacing logarithmic function f by the $C^1(\mathbb{R})$-regular function f_δ in (1), f_δ being defined by

$$f_\delta(s) = \begin{cases} f(-\delta) + f'(-\delta)(s+\delta) & , s < -\delta \\ f(s) & , |s| \leq \delta \\ f(\delta) + f'(\delta)(s-\delta) & , s > \delta \end{cases}$$

where $\delta > 0$ very near of 1 and such that $[-\delta, \delta] \subset (-1, 1)$, with homogenous Dirichlet conditions and initial conditions in $(3) - (4)$.

The existence of a solution of system $(25) - (26)$ is based on estimates $(27) - (30)$ below and a standard Galerkin scheme (see (Moukoko 2015)).

Multiplying (25) by $2\partial_t u^\delta$ and (26) by $2\partial_t\alpha$ integrating over Ω adding the two resulting equations, we have

$$\frac{d}{dt}\Big(\epsilon\|\partial_t u^\delta\|^2 + \| u^\delta \|^2 + \| \partial_t\alpha \|^2 + \| \alpha \|^2 + 2(F_\delta(u^\delta), 1)\Big) + 2 \| \partial_t u^\delta \|^2 + 2 \| \partial_t\alpha \|^2_{H^1} = 0. \tag{27}$$

The above estimate implies $u^\delta, \alpha \in L^\infty(\mathbb{R}; H^1_0(\Omega))$, $\partial_t u^\delta \in L^\infty(\mathbb{R}; L^2(\Omega)) \cap L^2(\mathbb{R}; L^2(\Omega))$ and $\partial_t\alpha \in L^\infty(\mathbb{R}; L^2(\Omega)) \cap L^2(\mathbb{R}; H^1_0(\Omega))$.
Multiplying (25) by $-2\Delta\partial_t u^\delta$ and (26) by $-2\Delta\partial_t\alpha$, integrating over Ω and adding the two resulting equations, we obtain

$$\frac{d}{dt}E_5 + \| \partial_t u^\delta \|^2_{H^1} + 2 \| \partial_t\alpha \|^2_{H^2} \leq C \| u^\delta \|^2_{H^1} .$$

where

$$E_5 = \epsilon\|\partial_t u^\delta\|^2_{H^1} + \| u^\delta \|^2_{H^2} + \| \partial_t\alpha \|^2_{H^1} + \| \alpha \|^2_{H^2} .$$

Therefore $u^\delta, \alpha \in L^\infty(0, T; H^2(\Omega) \cap H^1_0(\Omega))$, $\partial_t u^\delta \in L^\infty(0, T; H^1_0(\Omega)) \cap L^2(0, T; H^1_0(\Omega))$ and $\partial_t\alpha \in L^\infty(0, T; H^1_0(\Omega)) \cap L^2(0, T; H^2(\Omega) \cap H^1_0(\Omega))$.

Multiplying (25) by $2\partial_t^2 u^\delta$ and integrating over Ω , we obtain

$$\frac{d}{dt} \| \partial_t u^\delta \|^2 + \epsilon \| \partial_t^2 u^\delta \|^2 \leq C_1\|u^\delta\|^2_{H^2} + C_2\|f_\delta(u^\delta)\|^2 + C_3\|\partial_t\alpha\|^2,$$

therefore $\partial_t^2 u^\delta \in L^2(0, T; L^2(\Omega))$.

Multiplying (26) by $-2\Delta\partial_t^2\alpha$ and integrate over Ω , we have

$$\frac{d}{dt}\Big(\| \partial_t\alpha \|^2_{H^2} + 2(\Delta\alpha, \Delta\partial_t\alpha)\Big) + \| \partial_t^2\alpha \|^2_{H^1} \leq \| \partial_t u^\delta \|^2_{H^1} + 2 \| \partial_t\alpha \|^2_{H^2} . \tag{28}$$

Adding (29) and γ_6(28) where $\gamma_6 > 0$, we have

$$\frac{d}{dt}\Big(\epsilon \| \partial_t u^\delta \|^2_{H^1} + \| \Delta u^\delta \|^2_{H^2} + \| \partial_t\alpha \|^2_{H^1} + \| \alpha \|^2 + \gamma_6 \| \partial_t\alpha \|^2_{H^2} + 2\gamma_6(\Delta\alpha, \Delta\partial_t\alpha)\Big) + \gamma_6 \| \partial_t^2\alpha \|^2_{H^1}$$
$$+ 2(1-\gamma_6) \| \partial_t\alpha \|^2_{H^2} + (1-\gamma_6) \| \partial_t u^\delta \|^2_{H^1} \leq C \| u^\delta \|^2_{H^2} . \tag{29}$$

We have

$$\| \alpha \|_{H^2}^2 + \gamma_6 \| \partial_t\alpha \|_{H^2}^2 + 2\gamma_6(\Delta\alpha, \Delta\partial_t\alpha \quad \geq \quad (1-2\gamma_6) \| \alpha \|_{H^2}^2 + \frac{\gamma_6}{2} \| \partial_t\alpha \|_{H^2}^2 . \tag{30}$$

We choose $\gamma_6 < \frac{1}{2}$, then (29) implies $\partial_t\alpha \in L^\infty(0,T;H^2(\Omega)\cap H_0^1(\Omega)) \cap L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega))$ et $\partial_t^2\alpha \in L^2(0,T;H_0^1(\Omega))$. Therefore, the auxiliary system possesses a solution (u^δ,α) such that $u^\delta, \alpha \in L^\infty(0,T;H^2(\Omega)\cap H_0^1(\Omega))$, $\partial_t u^\delta \in L^2(0,T;H_0^1(\Omega))$, $\partial_t\alpha \in L^\infty(0,T;H^2(\Omega)\cap H_0^1(\Omega)) \cap L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega))$, $\partial_t^2 u \in L^2(0,T;L^2(\Omega)$ and $\partial_t^2\alpha \in L^2(0,T;H_0^1(\Omega)), \forall T>0$.
Since

$$\|u_0\|_{L^\infty(\Omega)} < \delta, \quad f_\delta'(\delta) > 0 \text{ and } \|\partial_t\alpha\|_{L^\infty((0,T)\times\Omega)} < f_\delta(\delta),$$

then $\|u^\delta\|_{L^\infty((0,T)\times\Omega)} \leq \delta$, then $f_\delta(u^\delta) = f(u^\delta)$ and (u^δ,α) is solution of system (1)-(2).

Theorem 2. (uniqueness) *Let the assumptions of Theorem* 1 *hold. Then, system* (1)-(2) *possesses a unique solution* (u,α) *such that* $u,\alpha \in L^\infty(0,T;H^2(\Omega)\cap H_0^1(\Omega))$, $\partial_t u \in L^2(0,T;H_0^1(\Omega))$, $\partial_t\alpha \in L^\infty(0,T;H^2(\Omega)\cap H_0^1(\Omega)) \cap L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega))$, $\partial_t^2 u \in L^2(0,T;L^2(\Omega), \partial_t^2\alpha \in L^2(0,T;H_0^1(\Omega)), \forall T>0$.

Proof. Let be $T>0$ and $(u^{(i)},\alpha^{(i)})_{i=1,2}$ two solutions of system (1)-(2) with initial conditions $(u_0^{(i)},\alpha_0^{(i)},u_1^{(i)},\alpha_1^{(i)})_{i=1,2}$ respectively, such that $\| u^{(i)}(0) \|_{L^\infty(\Omega)} < \delta^{(i)} < 1$ for $i=1,2$.
We set $u = u^{(1)} - u^{(2)}$ and $\alpha = \alpha^{(1)} - \alpha^{(2)}$. Then, (u,α) satisfies the following system

$$\epsilon\partial_t^2 u + \partial_t u - \Delta u + l(t)u \quad = \quad \partial_t\alpha \tag{31}$$

$$\partial_t^2\alpha - \partial_t\Delta\alpha - \Delta\alpha \quad = \quad -\partial_t u \tag{32}$$

where $l(t) = \int_0^1 f'(su_1(t) + (1-s)u_2(t))ds$, with homogenous Dirichlet conditions and following initial conditions

$$u(0,x) = u_0 = u_0^{(1)} - u_0^{(2)} \qquad \alpha(0,x) = \alpha_0 \quad = \quad \alpha_0^{(1)} - \alpha_0^{(2)}$$

$$\partial_t u|_{t=0} = u_1 = u_1^{(1)} - u_1^{(2)} \qquad \partial_t\alpha|_{t=0} = \alpha_1 \quad = \quad \alpha_1^{(1)} - \alpha_1^{(2)},$$

Since $f \in C^2(-1,1)$ and u_i are separated from ± 1, $i=1,2$, we have

$$\|l(t)\|_{L^\infty} \leq c, \forall t \geq 0,$$

where c depends of $\delta_i, i=1,2$.
Multiplying (31) by $2\partial_t u$ and (32) by $2\partial_t\alpha$, integrating over Ω and adding two resulting equations, we have

$$\frac{d}{dt}\Big(\epsilon \| \partial_t u \|^2 + \|u\|_{H^1}^2 + \| \partial_t\alpha \|^2 + \|\alpha\|_{H^1}^2\Big) + \| \partial_t u \|^2 + 2 \| \partial_t\alpha \|_{H^1}^2 \quad \leq \quad C\|u\|_{H^1}^2, \tag{33}$$

where C depends of δ_i. Applying Gronwall's lemma to (33), we have

$$\epsilon \| \partial_t u(t) \|^2 + \|u(t)\|_{H^1}^2 + \| \partial_t\alpha(t) \|^2 + \|\alpha(t)\|_{H^1}^2 \leq e^{Ct}\Big(\epsilon \| \partial_t u(0) \|^2 + \|u(0)\|_{H^1}^2 + \| \partial_t\alpha(0) \|^2 + \|\alpha(0)\|_{H^1}^2\Big).$$

We have the continuous dependence with respect to initial conditions, hence the uniqueness of solution.

In order to prove the existence of global attractor we seek the solution with more regularity.

Theorem 3. *Assume* $(u_0,u_1,\alpha_0,\alpha_1) \in (H^3(\Omega)\cap H_0^1(\Omega)) \times (H^2(\Omega)\cap H_0^1(\Omega)) \times (H^3(\Omega)\cap H_0^1(\Omega)) \times (H^3(\Omega)\cap H_0^1(\Omega))$ *and* $\| u_0 \|_{L^\infty(\Omega)} < 1$. *Then* (1)-(2) *possesses a unique solution* (u,α) *such that* $u,\alpha \in L^\infty(0,T;H^3(\Omega)\cap H_0^1(\Omega))$, $\partial_t u \in L^\infty(0,T;(H^2(\Omega)\cap H_0^1(\Omega)))\cap L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega))$, $\partial_t\alpha \in L^\infty(0,T;H^3(\Omega)\cap H_0^1(\Omega)) \cap L^2(0,T;H^3(\Omega)\cap H_0^1(\Omega))$ *and* $\partial_t^2\alpha \in L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega)), \forall T>0$. *Moreover, there exists* $\delta = \delta(T,u_0) \in (0,1)$ *such that* $\| u(t) \|_{L^\infty((0,T)\times\Omega)} < \delta, \forall t \in [0,T], \forall T>0$.

Proof. Following Theorem 1, system (1) − (2) has a unique solution (u,α) such that $u,\alpha \in L^\infty(0,T;H^2(\Omega)\cap H_0^1(\Omega))$, $\partial_t u \in L^2(0,T;H_0^1(\Omega))$ and $\partial_t^2\alpha \in L^2(0,T;L^2(\Omega)), \forall T>0$.

Multiplying (1) by $2\Delta^2\partial_t u$ and (2) by $2\Delta^2\partial_t\alpha$, integrating over Ω and adding the two resulting equations, we obtain

$$\frac{d}{dt}\Big(\epsilon\|\partial_t u\|_{H^2}^2 + \|\nabla\Delta u\|^2 + \|\partial_t\alpha\|_{H^2}^2 + \|\nabla\Delta\alpha\|^2\Big) + \|\partial_t u\|_{H^2}^2 + 2\|\nabla\Delta\partial_t\alpha\|^2 \quad \leq \quad C. \tag{34}$$

Therefore $u, \alpha \in L^\infty(0,T;H^3(\Omega)\cap H^1_0(\Omega))$, $\partial_t u \in L^\infty(0,T;H^2(\Omega)\cap H^1_0(\Omega))\cap L^2(0,T;H^2(\Omega)\cap H^1_0(\Omega))$, $\partial_t\alpha \in L^\infty(0,T;H^2(\Omega)\cap H^1_0(\Omega))\cap L^2(0,T;H^3(\Omega)\cap H^1_0(\Omega))$.
Multiply (2) by $-2\Delta^2\partial_t^2\alpha$ and integrate over Ω. We have, owing the continuous injections $H^2(\Omega)\subset L^\infty(\Omega)$ and $H^3(\Omega)\subset L^\infty(\Omega)$,

$$\frac{d}{dt}\|\nabla\Delta\partial_t\alpha\|^2 + \|\partial_t^2\alpha\|^2_{H^2} \le C,$$

which implies $\partial_t\alpha \in L^\infty(0,T;H^3(\Omega)\cap H^1_0(\Omega))\cap L^2(0,T;H^3(\Omega)\cap H^1_0(\Omega))$ and $\partial_t^2\alpha \in L^2(0,T;H^2(\Omega)\cap H^1_0(\Omega))$. This finishes the proof of the Theorem.

3.3 Existence of Global Attractor

The phase-field spaces are $\Phi_\kappa = \{(u,v,\alpha,\beta)\in (H^\kappa(\Omega)\cap H^1_0(\Omega))\times(H^{\kappa-1}(\Omega)\cap H^1_0(\Omega))\times(H^\kappa(\Omega)\cap H^1_0(\Omega))\times(H^\kappa(\Omega)\cap H^1_0(\Omega)) : \|u\|_{L^\infty((0,T)\times\Omega)} < 1\}$ for $\kappa = 2,3$, with energy norm

$$\|(u,\partial_t u,\alpha,\partial_t\alpha)\|^2_{\Phi_\kappa} = \|\zeta_u\|^2_{\varepsilon^\kappa(\epsilon)} + \|\alpha\|^2_{H^\kappa} + \|\partial_t\alpha\|^2_{H^\kappa}.$$

Thanks to theorems 1 and 3, we define the semigroup of operators $S_t(\epsilon)$ resolving the system (1) − (2) by

$$S_t(\epsilon) : \Phi_\kappa \longrightarrow \Phi_\kappa \qquad (u_0,u_1,\alpha_0,\alpha_1)\longmapsto (u(t),\partial_t u(t),\alpha(t),\partial_t\alpha(t)),$$

where $(u(t),\partial_t u(t),\alpha(t),\partial_t\alpha(t))$ is such that (u,α) is uniqueness solution of phase-field system (1) − (2) for initial conditions $(u_0,u_1,\alpha_0,\alpha_1)\in\Phi_\kappa$.

The following lemma gives the uniform estimates of $\| u(t)\|_{H^2}, \|\alpha(t)\|_{H^2}$ and $\|\partial_t\alpha(t)\|_{H^2}$ independent of ϵ, which allow to prove dissipativity of semigroup $S_t(\epsilon)$ in Φ_2.

Lemma 1. *Let the assumptions of Theorem 1 hold, $\epsilon < 1$ and (u,α) the solution of system (1) − (2) such that $(u(0),\partial_t u(0),\alpha(0),\partial_t\alpha(0))\in\Phi_2$. Then, the solution (u,α) verifies the following estimate*

$$\begin{aligned}&\mathbb{D}[u(t)] + \|u(t)\|^2_{H^2} + \epsilon\|\partial_t u(t)\|^2_{H^1} + \|\alpha(t)\|^2_{H^2} + \|\partial_t\alpha(t)\|^2_{H^2}\\ &+\int_0^t e^{-\beta(t-s)}\Big(\|\partial_t u(s)\|^2_{H^1} + \|\partial_t^2\alpha(s)\|^2_{H^1}\Big)ds \le Q(\mathbb{D}[u(0)],\|\zeta_u(0)\|_{\varepsilon^2(\epsilon)},\|\alpha(0)\|_{H^2},\|\partial_t\alpha(0)\|_{H^2})e^{-\beta t} + C, \quad (35)\end{aligned}$$

where the positive constants C,β and the monotonic function Q are independent of ϵ.

Proof. Multiply (1) by $-2\Delta u$ and (2) by $-2\Delta\alpha$ and integrate over Ω. We obtain, thanks to continuous injection $H^2(\Omega)\subset L^\infty(\Omega)$

$$\frac{d}{dt}\Big(2\epsilon(\nabla\partial_t u,\nabla u) + \|u\|^2_{H^2}\Big) + \|u\|^2_{H^2} \le 2\|\partial_t u\|^2_{H^1} + C, \quad (36)$$

$$\frac{d}{dt}\|\alpha\|^2_{H^2} + \|\alpha\|^2_{H^2} \le C_2\|\partial_t u\|^2_{H^1} + C_3\|\partial_t^2\alpha\|^2_{H^1}. \quad (37)$$

Adding $\gamma_7(9)$, $\gamma_8(11)$, $\gamma_9(36)$ et $\gamma_{10}(37)$ where $\gamma_7,\gamma_8,\gamma_9$ and $\gamma_{10} > 0$ are such that

$$\begin{gathered}\gamma_7 - \gamma_8 - 2\gamma_9 - C_2\gamma_{10} > 0,\\ 2\gamma_7 - 2\gamma_8 > 0,\\ \gamma_8 - C_3\gamma_{10} > 0,\\ \gamma_9 - C\gamma_7 > 0,\end{gathered}$$

we have

$$\frac{d}{dt}E_6 + C_1\|\partial_t u\|^2_{H^1} + C_2\|u\|^2_{H^2} + 2\gamma_9\|\alpha\|^2_{H^2} + C_3\|\partial_t\alpha\|^2_{H^2} + C_4\|\partial_t^2\alpha\|^2_{H^1} \le C, \quad (38)$$

where

$$\begin{aligned}E_6 = {}& \gamma_7\Big(\epsilon\|\partial_t u\|^2_{H^1} + \|u\|^2_{H^2} + \|\partial_t\alpha\|^2_{H^1} + \|\alpha\|^2_{H^2}\Big) + \gamma_8\Big(\|\partial_t\alpha\|^2_{H^2} + 2(\Delta\alpha,\Delta\partial_t\alpha)\Big)\\ &+\gamma_9\Big(2\epsilon(\nabla\partial_t u,\nabla u) + \|u\|^2_{H^2}\Big) + \gamma_{10}\|\alpha\|^2_{H^2}.\end{aligned}$$

There exists $\beta > 0$ such that

$$\beta E_6 \leq \frac{C_1}{2}\|\partial_t u\|_{H^1}^2 + C_2\|u\|_{H^2}^2 + 2\gamma_9\|\alpha\|_{H^2}^2 + C_3\|\partial_t\alpha\|_{H^2}^2.$$

Thanks to above estimate, estimate (38) can be written on form

$$\frac{d}{dt}E_6 + \beta E_6 + \frac{C_1}{2}\|\partial_t u\|_{H^1}^2 + C_4\|\partial_t^2\alpha\|_{H^1}^2 \leq C,$$

which implies, thanks to Gronwall's lemma,

$$E_6(t) + \int_0^t e^{\beta(t-s)}\Big(\|\partial_t u\|_{H^1}^2 + \|\partial_t^2\alpha(s)\|_{H^1}^2\Big)ds \leq E_6(0)e^{-\beta t} + C. \tag{39}$$

Moreover, for very small values of γ_8 and γ_9, there exists $C > 0$ independent of ϵ such that

$$C^{-1}\Big(\epsilon\|\partial_t u(t)\|_{H^1}^2 + \|u(t)\|_{H^2}^2 + \|\alpha(t)\|_{H^2}^2 + \|\partial_t\alpha(t))\|_{H^2}^2\Big) \leq E_6(t) \leq C\|(u(t), \partial_t u(t), \alpha(t), \partial_t\alpha(t))\|_{\Phi_2}^2.$$

Thanks to above estimate, (39) implies

$$\begin{aligned}&\mathbb{D}[u(t)] + \epsilon\|\partial_t u(t)\|_{H^1}^2 + \|u(t)\|_{H^2}^2 + \|\alpha(t)\|_{H^2}^2 + \|\partial_t\alpha(t))\|_{H^2}^2 \\ &+ \int_0^t e^{\beta(t-s)}\Big(\|\partial_t u(s)\|_{H^1}^2 + \|\partial_t^2\alpha(s)\|_{H^1}^2\Big)ds \leq Q(\mathbb{D}[u(0)], \|\zeta_u(0)\|_{\varepsilon^2(\epsilon)}, \|\alpha(0)\|_{H^2}, \|\partial_t\alpha(0)\|_{H^2})e^{-\beta t} + C,\end{aligned}$$

this achieves the proof.

Theorem 4. *Let the assumptions of Theorem 1 hold, $\epsilon < 1$ and (u, α) the solution of system* (1) − (2) *such that $(u(0), \partial_t u(0), \alpha(0), \partial_t\alpha(0)) \in \Phi_2$. Then, the following estimate is valid*

$$\begin{aligned}&\| (\zeta_u(t), \alpha(t), \partial_t\alpha(t)) \|_{\Phi_2}^2 + \int_0^t e^{-\beta(t-s)}(\| \partial_t u(s) \|^2 + \| \partial_t\alpha(s) \|^2 + \| \partial_t\alpha(s) \|_{H^1}^2)ds \\ &\leq Q(\mathbb{D}[u(0)], \|\zeta_u(0)\|_{\varepsilon^2(\epsilon)}, \|\alpha(0)\|_{H^2}, \|\partial_t\alpha(0)\|_{H^2})e^{-\beta t} + C,\end{aligned} \tag{40}$$

where positive constants C, β and monotonic function Q are independent of ϵ.

Proof. We first determine standard energy of the initial and boundary value problem for singularly perturbed damped hyperbolic equation (1) . This equation can be written in the following form

$$\epsilon\partial_t^2 u + \partial_t u - \Delta u = -f(u(t)) + \partial_t\alpha(t) = h_{u,\alpha}(t), \quad u(t)|_{\partial\Omega} = h(t)|_{\partial\Omega} = 0. \tag{41}$$

Applying corollary 5.2 of appendix of (Grasselli, Miranville, Pata & Zelik 2007) to equation (41) where $\kappa = 2$, we have

$$\begin{aligned}\| \zeta_u(t) \|_{\varepsilon^2(\epsilon)}^2 + \int_0^t e^{-\beta(t-s)} \| \partial_t u(s) \|_{H^2}^2\, ds \quad &\leq \quad Ce^{-\beta t}(\| \zeta_u(0) \|_{\varepsilon^2(\epsilon)}^2 + \| h_{u,\alpha}(0) \|_{H^1}^2) \\ &+ \quad C\int_0^t e^{-\beta(t-s)}(\| h_{u,\alpha}(s) \|_{H^2}^2 + \| \partial_t h_{u,\alpha}(s) \|^2)ds,\end{aligned} \tag{42}$$

where positive constants C and β are independent of ϵ.
To determine estimate of the last term of second member of (42), we begin by finding an estimate of $\| h_{u,\alpha}(s) \|_{H^2}^2 + \| \partial_t h_{u,\alpha}(s) \|^2$. We have, indeed,

$$\begin{aligned}\| h_{u,\alpha}(s) \|_{H^2}^2 + \| \partial_t h_{u,\alpha}(s) \|^2 \quad &\leq \quad 2 \| f(u(s)) \|_{H^2}^2 + 2 \| \partial_t\alpha(s) \|_{H^2}^2 + 2 \| f'(u(s))\partial_t u(s) \|^2 \\ &+ \quad 2 \| \partial_t^2\alpha(s) \|^2 .\end{aligned} \tag{43}$$

We now determine estimates of $\| f(u) \|_{H^2}$ and $\| f'(u)\partial_t u \|$. We have

$$\| f(u) \|_{H^2} = \|f''(u)(\nabla u)^2 + f'(u)\Delta u\| \leq C(\|u\|_{H^2}^2 + \|u\|_{H^2}) \leq C,$$

where C is independent of ϵ. We have also

$$\| f'(u)\partial_t u \| \quad \leq \quad \|f'(u)\|_{L^\infty}\|\partial_t u\| \leq C\|\partial_t u\|,$$

where C is independent of ϵ. We have, allow for estimate (7),

$$\| h_{u,\alpha}(s) \|_{H^2}^2 + \| \partial_t h_{u,\alpha}(s) \|^2 \leq C(1 + \|\partial_t u(s) \|_{H^1}^2 + \| \partial_t^2 \alpha(s) \|_{H^1}^2). \tag{44}$$

Inserting (44) into (42), we obtain, thanks to estimate (35),

$$\begin{aligned}
&\| \zeta_u(t) \|_{\varepsilon^2(\epsilon)}^2 + \int_0^t e^{-\beta(t-s)} \| \partial_t u(s) \|^2 \, ds \leq \\
&Ce^{-\beta t}(\| \zeta_u(0) \|_{\varepsilon^2(\epsilon)}^2 + \| h_{u,\alpha}(0) \|_{H^1}^2) + C \int_0^t e^{-\beta(t-s)}(1 + \|\partial_t u \|_{H^1}^2 + \| \partial_t^2 \alpha(s) \|_{H^1}^2) ds \\
&\leq Ce^{-\beta t}(\| \zeta_u(0) \|_{\varepsilon^2(\epsilon)}^2 + \| h_{u,\alpha}(0) \|_{H^1}^2) + Q(\mathbb{D}[u](0), \|(\zeta_u(0), \alpha(0), \partial_t \alpha(0))\|_{\Phi_1}) e^{-\beta t} \\
&\leq Q(\mathbb{D}[u(0)], \|\zeta_u(0)\|_{\varepsilon^2(\epsilon)}, \|\alpha(0)\|_{H^2}, \|\partial_t \alpha(0)\|_{H^2}) e^{-\beta t} + C,
\end{aligned}$$

where positives constants C and β are independent of ϵ. Combining above estimate and estimate (35), we obtain the result. This achieves the proof.

Corollary 1. *The semigroup $S_t(\epsilon)$ associated with system $(1)-(2)$ is dissipative in Φ_2, i.e., it possesses a bounded absorbing set in Φ_2.*

This corollary is the straightforward consequence of above Theorem.

In the sequence, we note $\mathcal{B}_{R_0}^1(\epsilon) = \{(u, v, \alpha, v) \in \Phi_2; \|(u, v, \alpha, \omega)\|_{\Phi_2} \leq R_0\}$ where R_0 is enough great, a bounded absorbing set in Φ_2.

Theorem 5. *Let the assumptions of Theorem 3 hold, $\epsilon < 1$ and (u, α) the solution of system $(1) - (2)$ such that $(u(0), \partial_t u(0), \alpha(0), \partial_t \alpha(0)) \in \mathcal{B}_{R_0}^1(\epsilon) \cap \Phi_3$. Then, following estimate is verified*

$$\begin{aligned}
&\mathbb{D}[u](t) + \| (u(t), \partial_t u(t), \alpha(t), \partial_t \alpha(t)) \|_{\Phi_3}^2 + \int_0^t e^{-\beta(t-s)}(\| \partial_t u(s) \|_{H^1}^2 + \epsilon \| \partial_t^2 u(s) \|_{H^1}^2 + \| \partial_t^2 \alpha(s) \|_{H^1}^2) ds \\
&\leq Q(\mathbb{D}[u(0)], \|\zeta_u(0)\|_{\varepsilon^3(\epsilon)}, \|\alpha(0)\|_{H^3}, \|\partial_t \alpha(0)\|_{H^3}) e^{-\beta t} + C,
\end{aligned} \tag{45}$$

where the positive constants C, β and the monotonic function Q are independent of ϵ.

Proof. Multiplying (1) by $2\Delta^2 \partial_t u$ and (2) by $2\Delta^2 \partial_t \alpha$, integrating over Ω and adding the resulting equations, we have

$$\frac{d}{dt}\Big(\epsilon \|\partial_t u\|_{H^2}^2 + \|\nabla \Delta u\|^2 + \|\partial_t \alpha\|_{H^2}^2 + \|\nabla \Delta \alpha\|^2\Big) + \|\partial_t u\|_{H^2}^2 + 2\|\nabla \Delta \partial_t \alpha\|^2 \quad \leq \quad C. \tag{46}$$

Multiply (1) by $2\Delta^2 u$ and (2) by $2\Delta^2 \alpha$, and integrate over Ω. We obtain

$$\frac{d}{dt}\Big(2\epsilon(\Delta \partial_t u, \Delta u) + \|u\|_{H^2}^2\Big) + \|\nabla \Delta u\|^2 \quad \leq \quad C \tag{47}$$

$$\frac{d}{dt}\Big(2(\Delta \partial_t \alpha, \Delta \alpha) + \|\alpha\|_{H^3}^2\Big) + \|\nabla \Delta \alpha\|^2 \quad \leq \quad \|\partial_t u\|_{H^1}^2 + C. \tag{48}$$

Multiply (2) by $2\Delta^2 \partial_t^2 \alpha$ and integrate over Ω. We find

$$\frac{d}{dt}\Big(\|\partial_t \alpha\|_{H^3}^2 + 2(\nabla \Delta \alpha, \nabla \Delta \partial_t \alpha)\Big) + \|\partial_t^2 \alpha\|_{H^2}^2 \quad \leq \quad 2\|\nabla \Delta \partial_t \alpha\|^2 + C. \tag{49}$$

Multiplying (1) by $-2\Delta \partial_t^2 u$, integrating over Ω, we have

$$\frac{d}{dt} E_7 + 2\epsilon \|\partial_t^2 u\|_{H^1}^2 \quad \leq \quad C_6 \|\partial_t u\|_{H^1}^2 + \|\partial_t^2 \alpha\|_{H^1}^2 + C, \tag{50}$$

where

$$E_7 = \|\partial_t u\|_{H^1}^2 + 2(\Delta u, \Delta \partial_t u) - 2(f(u), \Delta \partial_t u) + 2(\partial_t \alpha, \Delta \partial_t u).$$

Add (38), (46), (47), γ_{11}(48), γ_{12}(49) and γ_{13}(50) where γ_{11}, γ_{12} and $\gamma_{13} > 0$ are such that

$$C_1 - \gamma_{11} - C_6\gamma_{13} > 0;$$
$$C_4 - \gamma_{13} > 0;$$
$$1 - \gamma_{12} > 0;$$

we obtain

$$\frac{d}{dt}E_8 + C_1\|\partial_t u\|^2_{H^1} + C_2\|\partial_t u\|^2_{H^2} + C_3\|u\|^2_{H^2} + C_4\|\nabla\Delta u\|^2 + C_5\|\alpha\|^2_{H^2} + C_6\|\nabla\Delta\alpha\|^2$$
$$+C_7\|\partial_t\alpha\|^2_{H^2} + C_8\|\nabla\Delta\partial_t\alpha\|^2 + C_9\|\partial_t^2\alpha\|^2_{H^1} + C_{10}\|\partial_t^2\alpha\|^2_{H^2} + C_{10}\epsilon\|\partial_t^2 u\|^2_{H^1} \le C, \tag{51}$$

where

$$\begin{aligned} E_8 &= E_6 + \left(\epsilon\|\partial_t u\|^2_{H^2} + \|\nabla\Delta u\|^2 + \|\partial_t\alpha\|^2_{H^2} + \|\nabla\Delta\alpha\|^2\right) + \left(2\epsilon(\Delta\partial_t u, \Delta u) + \|u\|^2_{H^2}\right) \\ &+ \gamma_{11}\left(2(\Delta\partial_t\alpha, \Delta\alpha) + \|\nabla\Delta\alpha\|^2\right) + \gamma_{12}\left(\|\nabla\Delta\partial_t\alpha\|^2 + 2(\nabla\Delta\alpha, \nabla\Delta\partial_t\alpha)\right) \\ &+ \gamma_{13}\left(\|\partial_t u\|^2_{H^1} + 2(\Delta u, \Delta\partial_t u) + 2(f(u), \Delta\partial_t u) - 2(\partial_t\alpha, \Delta\partial_t u)\right). \end{aligned}$$

There exists $\beta > 0$ independent of ϵ such that

$$\beta E_8 \le C_1\|\partial_t u\|^2_{H^1} + C_2\|\partial_t u\|^2_{H^2} + C_4\|u\|^2_{H^3} + C_6\|\alpha\|^2_{H^3} + C_8\|\partial_t\alpha\|^2_{H^3}. \tag{52}$$

Choosing $\gamma_{11}, \gamma_{12}, \gamma_{13}$ and γ_{14} very small, there exists $C > 0$ independent of ϵ such that

$$C^{-1}(\|\zeta_u(t)\|^2_{\mathcal{E}^3(\epsilon)} + \|\partial_t\alpha(t)\|^2_{H^3} + \|\alpha(t)\|_{H^3}) \le E_8(t) \le C(\|\zeta_u(t)\|^2_{\mathcal{E}^3(\epsilon)} + \|\partial_t\alpha(t)\|^2_{H^3} + \|\alpha(t)\|_{H^3}). \tag{53}$$

Inserting estimate (52) into (51), we obtain

$$\frac{d}{dt}E_8 + \beta E_8 + \|\partial_t u\|^2_{H^1} + \epsilon\|\partial_t^2 u\|^2_{H^1} + \|\partial_t^2\alpha\|^2_{H^1} \le C.$$

Applying Gronwall's lemma to above estimate, thanks to estimate (53), we obtain the result.

Corollary 2. *Assume assumptions of Theorem 3 hold and* $0 < \epsilon < 1$. *Then, the semigroup* $S_t(\epsilon)$ *associated with system* $(1)-(2)$ *is dissipative in* Φ_3, *i.e., it possesses a bounded absorbing set in* Φ_3.

This corollary is the straightforward consequence of above Theorem.

Theorem 6. *Assume assumptions of Theorem 3 hold and* $0 < \epsilon < 1$. *Then, the semigroup* $S_t(\epsilon)$ *associated with system* $(1)-(2)$ *possesses a global attractor* $\mathcal{A}_\epsilon$ *which is compact in* Φ_2, *bounded in* Φ_3 *and connected.*

Proof. we already proved existence of bounded absorbed set $\mathcal{B}^1_{R_0}(\epsilon)$ in Φ_2; it remains to split the solution $(u, \alpha) \in B^1_{R_0}(\epsilon)$ as following

$$(u, \alpha) = (v, \eta) + (\omega, \xi)$$

such that semigroup $S_t(\epsilon)$ can be written as $S_t(\epsilon) = S_t^1(\epsilon) + S_t^2(\epsilon)$ with

$$S_t^1(\epsilon)(u(0), \partial_t u(0), \alpha(0), \partial_t\alpha(0)) = (v(t), \partial_t v(t), \eta(t), \partial_t\eta(t)),$$
$$S_t^2(\epsilon)(0,0,0,0) = (\omega(t), \partial_t\omega(t), \xi(t), \partial_t\xi(t)),$$

where $S_t^1(\epsilon)$ is resolving operator of following linear hyperbolic system

$$\epsilon\partial_t^2 v + \partial_t v - \Delta v = \partial_t\eta \tag{54}$$
$$\partial_t^2\eta - \partial_t\Delta\eta - \Delta\eta = -\partial_t v \tag{55}$$
$$v = \eta = 0 \quad \text{over} \quad \partial\Omega,$$

$S_t^2(\epsilon)$ is resolving operator of following nonlinear hyperbolic system

$$\epsilon\partial_t^2\omega + \partial_t\omega - \Delta\omega + f(u) = \partial_t\xi \tag{56}$$
$$\partial_t^2\xi - \partial_t\Delta\xi - \Delta\xi = -\partial_t\omega \tag{57}$$
$$\omega = \xi = 0 \quad \text{over} \quad \partial\Omega$$

and to show that $S_t^1(\epsilon)$ uniformly converges to 0 over all bounded set of Φ_2 and $S_t^2(\epsilon)$ is regularizing in Φ_3, when t tends to $+\infty$.
We first prove that the semigroup of operator $S_t^1(\epsilon)$ uniformly converges to 0 in Φ_2, when t tends $+\infty$.
Multiplying (54) by $-2\Delta\partial_t\nu$ and (55) by $-2\Delta\partial_t\eta$, integrating over Ω and adding resulting equations, we have

$$\frac{d}{dt}\Big(\epsilon \parallel \partial_t\nu \parallel_{H^1}^2 + \parallel \nu \parallel_{H^2}^2 + \parallel \partial_t\eta \parallel_{H^1}^2 + \parallel \eta \parallel_{H^2}^2 \Big) + 2 \parallel \partial_t\nu \parallel_{H^1}^2 + 2 \parallel \partial_t\eta \parallel_{H^2}^2 = 0. \tag{58}$$

Multiply (54) by $-2\Delta\nu$ and (55) by $-2\Delta\eta$ and integrate over Ω. We obtain

$$\frac{d}{dt}\Big(\parallel \nu \parallel_{H^1}^2 + 2\epsilon(\nabla\nu, \nabla\partial_t\nu)\Big) + \parallel \nu \parallel_{H^2}^2 \leq C_1 \parallel \partial_t\eta \parallel_{H^2}^2 + 2 \parallel \partial_t\nu \parallel_{H^1}^2 \tag{59}$$

$$\frac{d}{dt}\Big(\parallel \eta \parallel_{H^2}^2 + 2(\nabla\eta, \nabla\partial_t\eta)\Big) + \parallel \eta \parallel_{H^2}^2 \leq C_2 \parallel \partial_t\nu \parallel_{H^1}^2 + C_3 \parallel \partial_t\eta \parallel_{H^2}^2 . \tag{60}$$

Multiply (54) by $2\partial_t^2\nu$ and (55) by $-2\Delta\partial_t^2\eta$, and integrate over Ω. We have

$$\frac{d}{dt}\Big(\|\partial_t\nu\|^2 + 2(\nabla\nu, \nabla\partial_t\nu)\Big) + \epsilon\|\partial_t^2\nu\|^2 \leq C_4\|\partial_t\eta\|_{H^2}^2 + 2\|\partial_t\nu\|_{H^1}^2 \tag{61}$$

$$\frac{d}{dt}\Big(\|\partial_t\eta\|_{H^2}^2 + 2(\Delta\eta, \Delta\partial_t\eta)\Big) + \|\partial_t^2\eta\|_{H^1}^2 \leq \|\partial_t\nu\|_{H^1}^2 + 2\|\partial_t\eta\|_{H^2}^2. \tag{62}$$

Adding (58), γ_{14}(59), γ_{15}(60), γ_{16}(61) and γ_{17}(62) where $\gamma_{14}, \gamma_{15}, \gamma_{16}$ and $\gamma_{17} > 0$ such that

$$1 - 2\epsilon\gamma_{14} - C_2\gamma_{15} - 2\gamma_{16} - \gamma_{17} > 0;$$
$$1 - C_1\gamma_{14} - C_3\gamma_{15} - C_4\gamma_{16} - 2\gamma_{17} > 0,$$

we have

$$\frac{d}{dt}E_9 + \parallel \nu \parallel_{H^2}^2 + C_1 \parallel \partial_t\nu \parallel_{H^1}^2 + \parallel \eta \parallel_{H^2}^2 + \parallel \partial_t\eta \parallel_{H^2}^2 + \parallel \partial_t^2\nu \parallel_{H^1}^2 + \parallel \partial_t^2\eta \parallel_{H^1}^2 \leq 0, \tag{63}$$

where $C_1 > 0$ and

$$\begin{aligned} E_9 &= \epsilon \parallel \partial_t\nu \parallel_{H^1}^2 + \parallel \nu \parallel_{H^2}^2 + \parallel \partial_t\eta \parallel_{H^1}^2 + \parallel \eta \parallel_{H^2}^2 + \gamma_{14}\Big(\parallel \nu \parallel_{H^1}^2 + 2\epsilon(\nabla\nu, \nabla\partial_t\nu)\Big) \\ &+ \gamma_{15}\Big(\parallel \eta \parallel_{H^2}^2 + 2(\nabla\eta, \nabla\partial_t\eta)\Big) + \gamma_{16}\Big(\|\partial_t\nu\|^2 + 2(\nabla\nu, \nabla\partial_t\nu)\Big) \\ &+ \gamma_{17}\Big(\|\partial_t\eta\|_{H^2}^2 + 2(\Delta\eta, \Delta\partial_t\eta)\Big). \end{aligned}$$

Moreover, for γ_{15}, γ_{16} and γ_{17} sufficiently small, there exists β and $C > 0$ such that

$$\beta E_9(t) \leq \parallel \nu(t) \parallel_{H^2}^2 + C_1 \parallel \partial_t\nu(t) \parallel_{H^1}^2 + \parallel \eta(t) \parallel_{H^2}^2 + \parallel \partial_t\eta(t) \parallel_{H^2}^2 \tag{64}$$

$$C^{-1}\|(\nu(t), \partial_t\nu(t), \eta(t), \partial_t\eta(t))\|_{\Phi_2}^2 \leq E_9(t) \leq C\|(\nu(t), \partial_t\nu(t), \eta(t), \partial_t\eta(t))\|_{\Phi_2}^2. \tag{65}$$

Thanks to (64), estimate (63) can be written of the following form

$$\frac{d}{dt}E_9 + \beta E_9 + \parallel \partial_t^2\nu \parallel^2 + \parallel \partial_t^2\eta \parallel_{H^1}^2 \leq 0.$$

Applying Gronwall's lemma to above estimate, thanks to estimate (65), we find

$$\begin{aligned} &\mathbb{D}[u](t) + \parallel (\nu(t), \partial_t\nu(t), \eta(t), \partial_t\eta(t)) \parallel_{\Phi_2}^2 + \int_0^t \Big(\parallel \partial_t^2\nu(s) \parallel^2 + \parallel \partial_t^2\eta(s) \parallel_{H^1}^2 \Big)e^{-\beta(t-s)}ds \\ &\leq Q(\mathbb{D}[u](0), \|(u(0)\|_{H^3}, \|\partial_t u(0)\|_{H^2}, \|\alpha(0)\|_{H^3}, \|\partial_t\alpha(0)\|_{H^3})e^{-\beta t}. \end{aligned}$$

where positive constants C and β are independent of ϵ. Therefore the semigroup of operators $S_t^1(\epsilon)$ uniformly converges to 0 when t tends to $+\infty$. To end, we prove that the semigroup $S_t^2(\epsilon)$ is regularizing in Φ_3.
Thanks to assumptions of function f and the fact that u is continuous on $[0, T]$ such that $-1 < u(t, x) < 1 \quad \forall(t, x) \in [0, T] \times \Omega$, we have $f'(u)\nabla u, f''(u)(\nabla u)^2$ and $f'(u)\Delta u \in L^2(0, T; L^2(\Omega)), \forall T > 0$.

Multiplying (56) by $2(-\Delta\partial_t\omega + \Delta^2\partial_t\omega)$ and (57) by $2(-\Delta\partial_t\xi + \Delta^2\partial_t\xi)$, integrating over Ω and adding two resulting equations, we have

$$\frac{d}{dt}E_{10}+ \| \partial_t\omega \|^2_{H^1} +2 \| \partial_t\xi \|^2_{H^2} + \| \partial_t\omega \|^2_{H^2} +2 \| \nabla\Delta\partial_t\xi \|^2 \leq \|f'(u)\nabla u\|^2 + 2 \| f''(u)(\nabla u)^2 \|^2 + 2\|f'(u)\Delta u\|^2, \quad (66)$$

where

$$E_{10} = \epsilon \| \partial_t\omega \|^2_{H^1} +\epsilon \| \partial_t\omega \|^2_{H^2} + \| \omega \|^2_{H^2} + \| \nabla\Delta\omega \|^2 + \| \partial_t\xi \|^2_{H^1} + \| \partial_t\xi \|^2_{H^2} + \| \xi \|^2_{H^2} + \| \nabla\Delta\xi \|^2_{H^3} .$$

Multiplying (57) by $2\Delta^2\partial_t^2\xi$ and integrating over Ω, we have, owing to continuous injections $H^2(\Omega) \subset L^\infty(\Omega)$ and $H^3(\Omega) \subset L^\infty(\Omega)$

$$\frac{d}{dt}\Big(\|\nabla\Delta\partial_t\xi\|^2 + 2(\nabla\Delta\xi, \nabla\Delta\partial_t\xi)\Big) + \|\partial_t^2\xi\|^2_{H^2} \leq C. \quad (67)$$

Summing (66) and γ_{18}(67) where $\gamma_{18} > 0$, we obtain

$$\frac{d}{dt}E_{11} \leq C(\|f'(u)\nabla u\|^2+ \| f'(u)(\nabla u)^2 \|^2 +2\|f''(u)\Delta u\|^2 + 1), \quad (68)$$

where

$$E_{11} = E_{10} + \gamma_{18}\Big(\|\nabla\Delta\partial_t\xi\|^2 + 2(\nabla\Delta\xi, \nabla\Delta\partial_t\xi)\Big),$$

which implies, by integration between 0 and t

$$E_{12}(t) \leq C(T^2 + 1)Q(\|u_0\|_{H^2}, \|\partial_t u_0\|, \|\alpha_0\|_{H^2}, \|\alpha_1\|_{H^2}).$$

For sufficiently small $\gamma_{18} > 0$, there exists $C > 0$ such that

$$C\|(\omega(t), \partial_t\omega(t), \xi(t), \partial_t\xi(t))\|^2_{\Phi_3} \leq E_{12}(t).$$

Thanks to above estimate, estimate (69) implies

$$\|(\omega(t), \partial_t\omega(t), \xi(t), \partial_t\xi(t))\|^2_{\Phi_3} \leq C(T^2 + 1)Q(\mathbb{D}[u](0), \|u_0\|_{H^2}, \|u_1\|, \|\alpha_0\|_{H^2}, \|\alpha_1\|_{H^2}),$$

then, the semigroup $S_t^2(\epsilon)$ is regularizing on Φ_3. This finishes the proof of the Theorem.

4. Discussion

In this article we have proven as in (Moukoko, 2014, 2015) that the Caginalp hyperbolic phase-field system with singular potential, has a unique solution and the global attractor. It remains to study the existence of exponential attractors and the robust family of exponential attractors for Caginalp hyperbolic phase-field system with singular potential.
We can also complete this work by studying Caginalp hyperbolic phase-field system with other types of boundary conditions and regular or singular potential.

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