Some New Families of Mean Graphs

S K Vaidya (Corresponding author)

Department of Mathematics, Saurashtra University

Rajkot 360 005, Gujarat, India

E-mail: samirkvaidya@yahoo.co.in

Lekha Bijukumar

Shanker Sinh Vaghela Bapu Institute of Technology

Gandhinagar, GUJARAT, India

E-mail: dbijuin@yahoo.co.in

Abstract

We contribute some new results for mean labeling of graphs. It has been proved that the graphs obtained by the composition of paths P_m and P_2 denoted by $P_m[P_2]$, the square of path P_n and the middle graph of path P_n admit mean labeling. We also investigate mean labeling for some cycle related graphs.

Keywords: Mean labeling, Mean graphs, Middle graphs

AMS Subject classification number(2010): 05C78.

1. Introduction

We begin with simple,finite,connected and undirected graph $G = (V(G), E(G))$ with p vertices and q edges. For standard terminology and notations we follow (Harary, F., 1972). We will provide brief summary of definitions and other information which are prerequisites for the present investigations.

Definition 1.1 The *composition* of two graphs G_1 and G_2 denoted by $G = G_1[G_2]$ has vertex set $V(G_1[G_2])=V(G_1)\times V(G_2)$ and edge set $E(G_1[G_2]) = \{(u_1, v_1)(u_2, v_2) \,/\, u_1u_2 \in E(G_1)$ or $u_1 = u_2$ and $v_1v_2 \in E(G_2)\}$

Definition 1.2 *Duplication* of a vertex v_k by a new edge $e = v_k^{'}v_k^{''}$ in a graph G produces a new graph G_1 such that $N(v_k^{'}) \cap N(v_k^{''}) = v_k$.

Definition 1.3 *Duplication* of an edge e=v_iv_{i+1} by a vertex v_k in a graph G produces a new graph G_1 such that $N(v_k)$=$\{v_i, v_{i+1}\}$.

Definition 1.4 *Square of a graph G* denoted by G^2 has the same vertex set as of G and two vertices are adjacent in G^2 if they are at a distance of 1 or 2 apart in G.

Definition 1.5 *The middle graph $M(G)$* of a graph G is the graph whose vertex set is $V(G) \cup E(G)$ and in which two vertices are adjacent if and only if either they are adjacent edges of G or one is a vertex of G and the other is an edge incident on it.

Definition 1.6 If the vertices are assigned values subject to certain conditions then it is known as *graph labeling*.

For comprehensive survey on graph labeling we refer to (Gallian. J., 2009).

Definition 1.7 A function f is called a *mean labeling* of graph G if $f : V(G) \rightarrow \{0, 1, 2, \ldots, q\}$ is injective and the induced function f^*:$E(G) \rightarrow \{1, 2, \ldots, q\}$ defined as

$$f^*(e = uv) = \frac{f(u)+f(v)}{2}; \text{ if } f(u) + f(v) \text{ is even}$$

$$= \frac{f(u)+f(v)+1}{2}; \text{ if } f(u) + f(v) \text{ is odd}$$

is bijective. The graph which admits mean labeling is called a *mean graph.*

The mean labeling was introduced by (Somasundaram, S., 2003, p.29-35) and they proved the graphs P_n, C_n, $P_n \times P_m$, $P_m \times C_n$ etc. admit mean labeling. The same authors (Somasundaram, S., 2004, p.47-58) have discussed the mean labeling of subdivision of $K_{1,n}$ for $n < 4$ while in (Somasundaram, S., 2003, p.103-111) they proved that the wheel W_n does not admit mean labeling for $n > 3$. In the present work seven new results corresponding to mean labeling and some new families of mean graphs are investigated.

Observations 1.8 In a mean graph G

(1) $v \in V(G)$ and $d(v) \geq 2$, having label 0, then edge label 1 can be produced only if v is adjacent to the vertex having label either 1 or 2 and edge label 2 can be produced only if v is adjacent to the vertex having label either 3 or 4 or the vertex with label 1 is adjacent to a vertex having label 3.

(2) The edge label q can be produced only when the vertices having labels q and $q-1$ are adjacent in G.

2. Main Results

Theorem-2.1: The composition of paths P_m and P_2 denoted as $P_m[P_2]$ admits mean labeling except for $m = 2$.

Proof: Let $u_1, u_2, \ldots, u_m$ be the vertices of the path P_m and v_1, v_2 be the vertices of path P_2. The composition $P_m[P_2]$ consists of $2m$ vertices, can be partitioned into two sets $T_1 = \{(u_i, v_1), i = 1, 2, \ldots, m\}$ and $T_2 = \{(u_i, v_2), i = 1, 2, \ldots, m\}$. To define $f: V(G) \to \{0, 1, 2, \ldots, q\}$ following two cases are to be considered.

Case 1: m is even, $m \neq 2$

We define the labeling as follows.

$f(u_1, v_1) = 0$

$f(u_1, v_2) = 2$

$f(u_2, v_1) = 4$

$f(u_2, v_2) = 7$

<u>For $3 \leq i \leq (m-1)$</u>

$f(u_i, v_1) = f(u_{i-1}, v_1) + 5; \forall i.$

$f(u_i, v_2) = f(u_{i-1}, v_2) + 6$; when i is odd.

$\quad = f(u_{i-1}, v_2) + 4$; when i is even

<u>For $i = m$</u>;

$f(u_i, v_1) = f(u_{i-1}, v_1) + 6$

$f(u_i, v_2) = f(u_{i-1}, v_2) + 3$

Case 2: m is odd.

We define the labeling as follows.

$f(u_1, v_1) = 0$

$f(u_1, v_2) = 2$

<u>For $2 \leq i \leq m$</u>

$f(u_i, v_1) = f(u_{i-1}, v_1) + 6$; when i is odd.

$\quad = f(u_{i-1}, v_1) + 4$; when i is even.

<u>For $2 \leq i \leq (m-1)$</u>

$f(u_i, v_2) = f(u_{i-1}, v_2) + 6$; when i is even.

$\quad = f(u_{i-1}, v_2) + 4$; when i is odd.

<u>For $i = m$</u>;

$f(u_i, v_1) = f(u_{i-1}, v_1) + 6$

$f(u_i, v_2) = f(u_{i-1}, v_2) + 3$

In view of the above defined labeling pattern the graph under consideration admits mean labeling.

Case 3: m=2.

When $m = 2$ the resultant graph is K_4 and as proved by (Somasundaram S., 2004, p.47-58) the complete graph K_n does not admit mean labeling for $n > 3$.

Thus we conclude that the graph $P_m[P_2]$ admits mean labeling except for $m = 2$.

Remark:2.2

It is obvious from the observations 1.8 that the composition $P_2[P_m]$ does not admit mean labeling.

Illustration 2.3:

Consider the composition of P_7 and P_2. This is the case when n is odd. The mean labeling is as shown in *Figure 1*.

Theorem-2.4: The graph obtained by duplication of an arbitrary vertex by a new edge in cycle C_n admits mean labeling.

Proof: Let $v_1, v_2, \ldots, v_n$ be the vertices of cycle C_n and let G be the graph obtained by duplicating an arbitrary vertex of C_n by a new edge. Without loss of generality let this vertex be v_1 and the edge be $e = v'_1v''_1$. To define $f : V(G) \rightarrow \{0, 1, 2, \ldots, q\}$ the following two cases are to be considered.

Case 1: n is odd.

We define the labeling as follows.

$f(v_1) = 3$

$f(v'_1) = 0$

$f(v''_1) = 2$

$f(v_i) = 2(i + 1)$; For $2 \leq i \leq (\frac{n+1}{2})$

$f(v_i) = 2(n + 3 - i) - 1$; For $\frac{n+3}{2} \leq i \leq n$

Case 2: n is even.

We define the labeling as follows.

$f(v_1) = 3$

$f(v'_1) = 0$

$f(v''_1) = 2$

$f(v_i) = 2(i + 1)$; For $2 \leq i \leq (\frac{n}{2})$

$f(v_i) = 2(n + 3 - i) - 1$; For $\frac{n+2}{2} \leq i \leq n$

Above defined labeling pattern exhaust all the possibilities and in each case the graph under consideration admits mean labeling.

Illustration 2.5:

Consider the graph obtained by duplicating the vertex v_1 by an edge $v'_1v''_1$ in C_8. This is the case when n is even.The corresponding mean labeling is as shown in *Figure 2*.

Theorem-2.6: Duplication of an arbitrary edge by a new vertex in cycle C_n produces a mean graph.

Proof: Let v_1,v_2,..., v_n be the vertices of cycle C_n. Let G be the graph obtained by duplication of an arbitrary edge in C_n by a new vertex. Without loss of generality let this edge be $e = v_1v_2$ and the vertex be v'. To define $f : V(G) \rightarrow \{0, 1, 2, \ldots, q\}$ the following two cases are to be considered.

Case 1: n is odd.

We define the labeling as follows.

$f(v') = 0$

$f(v_1)$=4

$f(v_2)$=2

$f(v_i) = 2i$; For $3 \leq i \leq (\frac{n+1}{2})$

$f(v_i) = 2(n + 2 - i) + 1$; For $\frac{n+3}{2} \leq i \leq n$

Case 2: n is odd.

We define the labeling as follows.

$f(v') = 0$

$f(v_1) = 4$

$f(v_2) = 2$

$f(v_i) = 2i$; For $3 \leq i \leq (\frac{n+2}{2})$

$f(v_i) = 2(n + 2 - i) + 1$; For $\frac{n+4}{2} \leq i \leq n$

In view of the above defined labeling pattern the graph under consideration admits mean labeling.

Illustration 2.7: Consider the graph obtained by duplicating an arbitrary edge v_1v_2 by a vertex v' in cycle C_9. This is the case when n is odd. Labeling pattern is shown in *Figure 3*.

Theorem-2.8: P_n^2 is a mean graph.

Proof: Let v_1,v_2,....v_n be the vertices of path P_n. Define f:$V(G) \rightarrow \{0, 1, 2, \ldots, q\}$ as follows.

$f(v_i) = 2(i-1)$; For $i \leq (n-1)$

$f(v_i) = 2i-3$; For $i = n$

Above defined labeling pattern provides mean labeling for P_n^2. i.e. P_n^2 is a mean graph.

Illustration 2.9:

The graph P_9^2 and the corresponding mean labeling is shown in *Figure 4*.

Theorem-2.10: Crown (cycle with pendant edge attached at each vertex) is a mean graph.

Proof: Let v_1,v_2,..., v_n be the vertices of cycle C_n. Let G_1 be the crown with $2n$ vertices v_1,v_2,..., v_n,u_1,u_2,..., u_n and $2n$ edges (n edges of cycle C_n together with n pendant edges).

Case 1: n is odd.

we can label the graph as follows.

$f(v_1) = 0$

$f(u_1) = 1$

$f(v_i) = 2i-1$; For $2 \leq i \leq \frac{n+1}{2}$

$f(v_i) = 2i$; For $\frac{n+3}{2} \leq i \leq n$

$f(u_i) = 2i-2$; For $2 \leq i < \frac{n+1}{2}$

$f(u_i) = 2i$; For $i = \frac{n+1}{2}$

$f(u_i) = 2i-1$; For $\frac{n+3}{2} \leq i \leq n$

Case 2: n is even.

we can label the graph as follows.

$f(v_1) = 0$

$f(u_1) = 1$

<u>For $2 \leq i \leq \frac{n}{2}$</u>

$f(v_i) = 2i-1$

$f(u_i) = 2i-2$

<u>For $\frac{n+2}{2} \leq i \leq n$;</u>

$f(v_i) = 2i$

$f(u_i) = 2i-1$

The above defined function f provides mean labeling for crown. i.e. crown is a mean graph.

Illustration 2.11 Consider the cycle C_{12} with 12 pendant vertices attached. This is the case when n is even. The mean labeling pattern is as shown in *Figure 5*.

Theorem-2.12: Tadpoles $T(n, k)$ (The graph obtained by identifying a vertex of cycle C_n to an end vertex of path P_k) admits mean labeling.

Proof: Let v_1,v_2,..., v_n be the vertices of cycle C_n and u_1,u_2,..., u_k be the vertices of the path P_k. Let G_1 be the resultant graph obtained by identifying a vertex of cycle C_n to an end vertex of the path P_k. To define f:$V(G) \rightarrow \{0, 1, 2, \ldots, q\}$ the following two cases are to be considered.

Case 1: n is odd.

Identify one of the end vertices of P_k to a vertex of C_n in such a way that $v_{\frac{n+3}{2}}$=u_1. We can label the graph as follows.

$f(v_i) = 2(i-1)$; For $1 \leq i \leq \frac{n+1}{2}$

$f(v_i) = 2(n-i)+3$; For $\frac{n+3}{2} \leq i \leq n$

$f(u_1) = f(v_{\frac{n+3}{2}})$

$f(u_i) = n+i-1, i = 2, 3, \ldots, k.$

Case 2: n is even.

Identify one of the end vertices of P_k to a vertex of C_n in such a way that $v_{\frac{n+2}{2}}=u_1$. We label the graph as follows.

$f(v_i) = 2(i-1)$;For $1 \leq i \leq \frac{n+2}{2}$

$f(v_i) = 2(n-i)+3$;For $\frac{n+4}{2} \leq i \leq n$

$f(u_1) = f(v_{\frac{n+2}{2}})$

$f(u_i) = n+i-1,\ i = 2, 3 \ldots, k$

In both the cases the above described function f provides mean labeling for the graph under consideration.

Illustration 2.13 In *Figure 6* the graph T(9,4) and its mean labeling is shown.

Theorem-2.14: The middle graph of a path P_n denoted by $M(P_n)$ admits mean labeling.

Proof: Let v_1,v_2,$\ldots v_n$ be the vertices and e_1,e_2,$\ldots$, e_{n-1} be the edges of path P_n.The middle graph of a path P_n denoted by $M(P_n)$ is a graph with $V[M(P_n)] = V(P_n) \bigcup E(P_n)$ and two vertices are adjacent in $M(P_n)$ if and only if they are adjacent in P_n or one is a vertex and other is an incident edge to that vertex in P_n. $M(P_n)$ consists of two types of vertices $\{v_1,v_2,\ldots, v_n\}$ and $\{e_1,e_2,\ldots, e_{n-1}\}$. To define f:$V(M(P_n)) \rightarrow \{0, 1, 2, \ldots, q\}$ as follows.

$f(e_1) = 1$

$f(e_2) = 4$

$f(e_3) = 8$

$f(e_i) = 3i-1$; $4 \leq i \leq (n-1)$

$f(v_1) = 0$

$f(v_i) = 2i-1$; For $2 \leq i \leq 4$

$f(v_i) = 3i-5$; For $5 \leq i \leq n$

The above defined function f provides mean labeling for $M(P_n)$. i.e. $M(P_n)$ is a mean graph.

Illustration 2.15 The mean labeling for $M(P_6)$ is as shown in *Figure 7* .

3. Concluding Remarks

We contribute here seven new results to the theory of mean graphs. This work is an effort to relate some graph operations with mean labeling. We also introduce two concepts namely duplication of a vertex by an edge as well as duplication of an edge by a vertex. We apply these operations to cycle C_n and prove that the resultant graphs are mean graphs. In addition to this we prove that the middle graph of a path P_n admits mean labeling.

Future Scope:

It is possible to investigate similar results for other graph families and in the context of different labeling techniques.

References

Gallian, J. A. (2009). A dynamic survey of graph labeling, *The Electronics Journal of Combinatorics*, 16, $\sharp DS$ 6.

Harary, F. (1972). *Graph Theory*, Addison Wesley, Massachusetts.

Somasundaram, S. and Ponraj, R. (2003). Some results on mean graphs. *Pure and Applied Mathematical Sciences*, 58, 29-35.

Somasundaram, S. and Ponraj, R, (2004). On mean graphs of order < 5. *J.Decision and Mathematical Sciences*, 9, 47-58.

Somasundaram, S. and Ponraj, R. (2003). Non existence of mean labeling for wheel. *Bulletin of Pure and Applied Sciences*, 22E, 103-111.

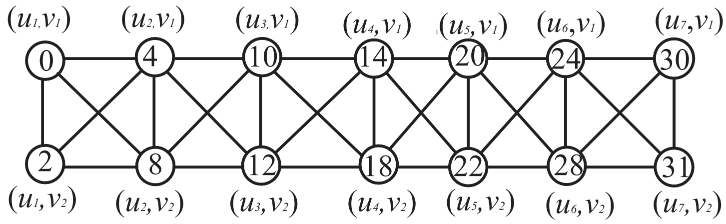

Figure 1. The mean labeling of graph $P_7[P_2]$

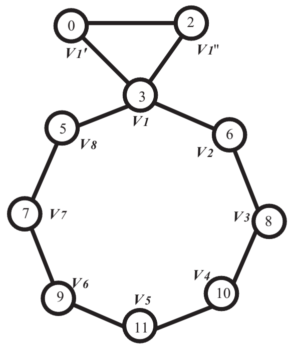

Figure 2. Duplication of a vertex by an edge in C_8 and its mean labeling

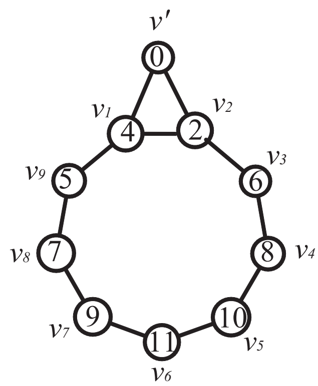

Figure 3. Duplication of an edge by a vertex in C_9 and its mean labeling

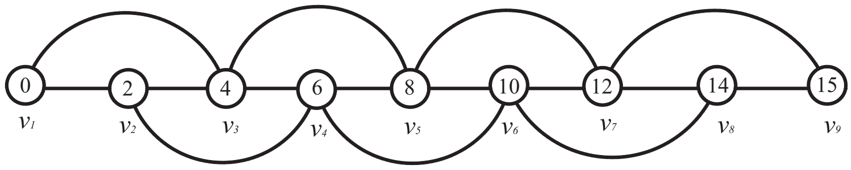

Figure 4. The mean labeling of P_9^2

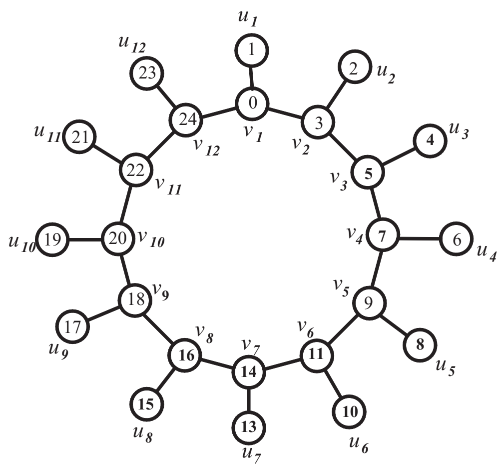

Figure 5. The mean labeling of crown

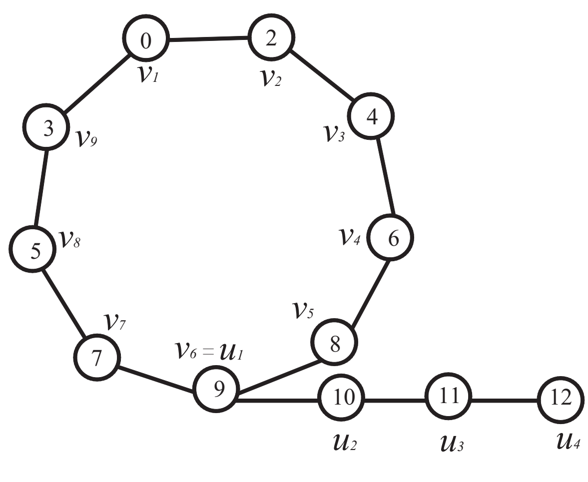

Figure 6. The mean labeling of $T(9,4)$

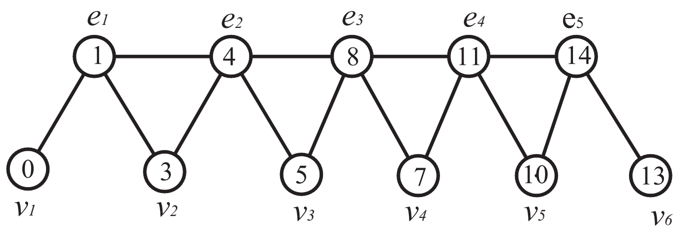

Figure 7. The mean labeling of $M(P_6)$