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The Differential Properties of Functions from Sobolev-Morrey Type Spaces of Fractional Order

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Abstract

The main goal of this paper is study a fractional order Sobolev-Morrey type spaces and obtained integral estimates for the generalized derivatives of fractional order of functions in this spaces. Also, we study a smoothness of solution of one class of high order fractional quasielliptic equations.

Keywords: Sobolev-Morrey type space of fractional order, embedding theorems, Hölder condition, smoothness, fractional order derivatives

1. Introduction and preliminary notes

In this paper in connection with the investigation of differential equation of higher fractional order of type

$$\sum_{(\alpha,\lambda)\leq 1,(\beta,\lambda)\leq 1} D^{\alpha}\left(a_{\alpha\beta}(x)\,D^{\beta}u\right) = \sum_{(\alpha,\lambda)\leq 1} D^{\alpha} f_{\alpha}, \tag{1.1}$$

where $x = (x_1, ..., x_n)$, $\alpha = (\alpha_1, ..., \alpha_n)$, $\beta = (\beta_1, ..., \beta_n)$, and $\alpha_j, \beta_j \geq 0$ $(j = 1, ..., n)$ we introduce the new form of description of norm of the spaces $W^l_p(G)$ and $W^l_{p,a,\varkappa,\tau}(G)$, when $l = (l_1, ..., l_n)$, $l_j > 0$, $j = 1, ..., n$. Also, such approach were studied in (A. M. Najafov, 2010) and (A. M. Najafov, 2013). In other words the norms of the Sobolev and Sobolev-Morrey spaces of fractional order's the generalized derivatives of fractional order $D^{l_i}_i f = D^{[l_i]}_i D^{\{l_i\}}_{+i} f$ ($[l_i]$ is the integer part, $\{l_i\}$ is the non-integer part of the number l_i) expression by the ordinary Riemann-Liouville fractional derivatives of functions. But in the papers (T. I. Amanov, 1976; N. Aronszajn and K. Smith, 1961; A. Calderon and A. Zygmund, 1961; A. D. Jabrailov, 1972; P. I. Lizorkin, 1963; P. I. Lizorkin, 1972; A. M. Najafov, 2005a,b; A. M. Najafov and A. T. Orujova, 2012; Yu. V. Netrusov, 1984; L. N. Slobodetskiy, 1958a,b; H. Triebel, 1986) and etc the Sobolev and Sobolev-Morrey type spaces the generalized derivatives of fractional order expression by the differences of derivatives of functions. Also, we study the differential properties of functions from spaces $W^l_{p,a,\varkappa,\tau}(G)$ $(G \in R^n, l \in (0,\infty)^n,\ p \in [1,\infty),\ a \in [0,1]^n, \tau \in [1,\infty])$ with parameters in terms of embedding theory and some properties of fractional order Sobolev-Morrey type spaces is proved. As application of obtained results we study a smoothness of solution of one class of higher order fractional quasielliptic equations (1.1).The fundamental difference of this work from earlier work is to obtain estimates for generalized derivatives of fractional order.

The Hölder continuity of solutions of integer order quasielliptic equations with continuous or Hölder continuous coefficients of the leading derivatives was considered in (E. Guisti, 1967). In (L. Arkeryd, 1969), L_p- estimates for solutions were studied, under the condition that the coefficients of leading derivatives are infinitely differentiable, and in (L. A. Bagirov, 1979; S. V. Uspenskii, G. V. Demidenko and V. G. Perepelkin, 1984) some other problems of the theory of quasielliptic equations were considered. In (R. V. Guseinov, 1992) and (A. M. Nadzhafov, 2005) the theorems were proved claiming that the solution belongs to the Hölder class inside the domain, and in (P. S. Filatov, 1997) local "interior" Hölder estimates were obtained for solutions to a quasielliptic type equation in the case when the right-hand side satisfies the anistropic Hölder condition. In this paper, as in (R. V. Guseinov, 1992) and (A. M. Nadzhafov, 2005), we study the Hölder continuity of a solution without any smoothness conditions on $a_{\alpha\beta}(x)$.

In recent years, different problems of partial fractional differential equation were studied in (A. M. Nakhushev, 2001; M. Kh. Shkhanukov, 1996; A. V. Pskhu, 2010; A. A. Kilbas, H. M. Strivastava and J. J. Trujillo, 2006; J. Öztürk, 2010; F. M. Nakhusheva, 2005) and others.

Let G be a domain of $\mathbb{R}^n, t > 0$. Given $x \in \mathbb{R}^n$, we put

$$I_{t^{\varkappa}}(x) = \left\{y : \left|y_j - x_j\right| < (1/2)\, t^{\varkappa_j}, j = 1, 2, ..., n\right\}, \quad G_{t^{\varkappa}}(x) = G \cap I_{t^{\varkappa}}(x).$$

Definition 1 *Denote by* $W^l_{p,a,\varkappa,\tau}(G)$ *the space of locally summable functions* f *on* G *having the weak derivatives* $D^{l_i}_i f$ *on* G $(i = 1, 2, ..., n)$ *with the finite norm*

$$\|f\|_{W^l_{p,a,\varkappa,\tau}(G)} = \|f\|_{L_{p,a,\varkappa,\tau}(G)} + \sum_{i=1}^{n} \left\|D^{l_i}_i f\right\|_{L_{p,a,\varkappa,\tau}(G)}, \tag{1.2}$$

where $(1 \le \tau \le \infty)$,

$$\|f\|_{L_{p,a,\varkappa,\tau}(G)} = \|f\|_{p,a,\varkappa,\tau;G} = \sup_{x\in G} \left\{ \int_0^{\infty} \left[[t]_1^{-\sum_{j=1}^{n} \frac{\varkappa_j a_j}{p}} \|f\|_{p,G_{t^{\varkappa}}(x)} \right]^{\tau} \frac{dt}{t} \right\}^{1/\tau}, \tag{1.3}$$

$[t]_1 = \min\{1, t\}$, $D^{l_i}_i f = D^{[l_i]}_i D^{\{l_i\}}_{+i} f$, $[l_i]$ *is the integer part,* $\{l_i\}$ *is the non-integer part of the number* l_i. *The partial generalized fractional derivatives* $D^{\{l_i\}}_{+i}$ *in S. L. Sobolev's sense are understood in the following sense:*

$$\int_G f(x) \left(D^{[l_i]}_i D^{\{l_i\}}_{-i} \varphi\right)(x)\, dx = (-1)^{[l_i]} \int_G \varphi(x) \left(D^{[l_i]}_i D^{\{l_i\}}_{+i} f\right)(x)\, dx$$

for $\varphi \in C^{\infty}_0(G)$. *The symbol* $D^{\{l_i\}}_{+i}$ *and* $D^{\{l_i\}}_{-i}$ *are the ordinary Riemann-Liouville fractional derivatives of order* $\{l_i\}\,(0 < \{l_i\} < 1)$ *in the domain are understood as (A. M. Najafov, 2010) and (A. M. Najafov, 2013)*

$$\left(D^{\{l_i\}}_{+i} f\right)(x) = \frac{1}{\Gamma(1 - \{l_i\})} \frac{\partial}{\partial x_i} \int_{G^{(i)}} \frac{f(x_1, x_2, ..., x_{i-1}, s_i, x_{i+1}, ..., x_n)}{(x_i - s_i)^{\{l_i\}}} ds_i,$$

$$\left(D^{\{l_i\}}_{-i} f\right)(x) = -\frac{1}{\Gamma(1 - \{l_i\})} \frac{\partial}{\partial x_i} \int_{\overline{G}^{(i)}} \frac{f(x_1, x_2, ..., x_{i-1}, s_i, x_{i+1}, ..., x_n)}{(s_i - x_i)^{\{l_i\}}} ds_i,$$

where x *is the inner point of the domain* G. $\Gamma(\alpha)$ *is a gamma function, the sets* $G^{(i)}$ *and* $\overline{G}^{(i)}$ *are determined as*

$$G^{(i)} = \left\{(x_1, x_2, ..., x_{i-1}, s_i, x_{i+1}, ..., x_n) \in G : x_j = const\,(j \neq i)\,; s_i < x_i\right\},$$

$$\overline{G}^{(i)} = \left\{(x_1, x_2, ..., x_{i-1}, s_i, x_{i+1}, ..., x_n) \in G : x_j = const\,(j \neq i)\,; s_i > x_i\right\}.$$

It should be noted that ordinary Riemann-Liouville fractional derivative on the segments and the real line are reminded in the monograph (S. G. Samko, A. A. Kilbas and O. N. Marichev, 1987).

Note that, the fractional order Sobolev space $W^l_{p,0,\varkappa,\infty}(G) \equiv W^l_p(G)$ was introduced in (A. M. Najafov, 2010) and (A. M. Najafov, 2013). In the case $l \in N^n, \tau = \infty, a = (a, ..., a)$ the Sobolev-Morrey spaces $W^l_{p,a,\varkappa,\infty}(G) \equiv W^l_{p,a,\varkappa}(G)$ were defined and studied by (V.P.Ilyin, 1971).

Observe some properties of $L_{p,a,\varkappa,\tau}(G)$ and $W^l_{p,a,\varkappa,\tau}(G)$.

1. The following embeddings hold for arbitrary $\varkappa_j > 0$ and $0 \le a_j \le 1$ $(j = 1, 2, ..., n)$:

$$L_{p,a,\varkappa,\tau}(G) \hookrightarrow L_{p,a,\varkappa}(G), \quad W^l_{p,a,\varkappa,\tau}(G) \hookrightarrow W^l_{p,a,\varkappa}(G);$$

i.e.,

$$\|f\|_{p,a,\varkappa;G} \le C\, \|f\|_{p,a,\varkappa,\tau;G} \tag{1.4}$$

and

$$\|f\|_{W^l_{p,a,\varkappa}(G)} \le C\, \|f\|_{W^l_{p,a,\varkappa,\tau}(G)}. \tag{1.5}$$

2. The spaces $L_{p,a,\varkappa,\tau}(G)$ and $W^l_{p,a,\varkappa,\tau}(G)$ are complete.

3. For every real $c > 0$,

$$\|f\|_{p,a,c\varkappa,\tau;G} = c^{-\frac{1}{\tau}} \|f\|_{p,a,\varkappa,\tau;G}, \quad \|f\|_{W^l_{p,a,c\varkappa,\tau}(G)} = c^{-\frac{1}{\tau}} \|f\|_{W^l_{p,a,\varkappa,\tau}(G)}.$$

4. The following relations are valid for every $\varkappa_j > 0\,(j = 1, 2, ..., n)$:

$$(a)\ \|f\|_{p,0,\varkappa,\infty;G} = \|f\|_{p;G}, \quad \|f\|_{W^l_{p,0,\varkappa,\infty}(G)} = \|f\|_{W^l_p(G)};$$

$$(b)\ \|f\|_{p,1,\varkappa,\tau;G} \geq \|f\|_{\infty;G}, \quad \|f\|_{W^l_{p,1,\varkappa,\tau}(G)} \geq \|f\|_{W^l_\infty(G)}.$$

5. If G is a bounded domain, $p\leq q$, $\frac{1-b_j}{q} \leq \frac{1-a_j}{p}$, $j = 1, ..., n$, and $1 \leq \tau_1 < \tau_2 \leq \infty$ then

$$L_{q,b,\varkappa,\tau_1}(G) \hookrightarrow L_{p,a,\varkappa,\tau_2}(G).$$

To prove the main theorems, we need some auxiliary inequalities in the lemmas below. Assume that $M(\cdot, y, z) \in C_0^\infty$ is such that

$$S(M) = \sup pM \subset I_1 = \{y : |y_i| < 1/2, j = 1, 2, ..., n\},$$

$0 < T \leq 1$, $\lambda = (\lambda_1, ..., \lambda_n)$ and $\lambda_j > 0$, $j = 1, 2, ..., n$. Put

$$V = \bigcup_{0<t\leq T} \left\{y : \left(y/t^\lambda\right) \in S(M)\right\}.$$

Clearly,$V \subset I_{T^\lambda}$. Let U be an open subset of G. Henceforth we always assume that $U + V \subset G$. Let

$$G_{T^\varkappa}(U) = \bigcup_{x\in U} G_{T^\varkappa}(x) = (U + I_{T^\varkappa}(x)) \cap G.$$

Note that if $0 < \varkappa_j \leq \lambda_j\,(j = 1, 2, ..., n)$ and $0 < T \leq 1$ then $I_{T^\lambda} \subset I_{T^\varkappa}$ and so

$$U + V \subset G_{T^\varkappa}(U) = Q.$$

Lemma 1 *Let* $1 \leq p \leq q \leq r \leq \infty, 0 < \varkappa_j \leq \lambda_j\,(j = 1, 2, ..., n), 0 < t \leq T \leq 1, 0 < \rho < \infty, 1 \leq \tau \leq \infty, 0 < \eta \leq T, \nu = (\nu_1, ..., \nu_n).\nu_j \geq 0\ j = 1, 2, ..., n, \Phi \in L_{p,a,\varkappa,\tau}(G)$ and

$$\mu_i = \lambda_i l_i - \sum_{j=1}^{n} \left[\nu_j \lambda_j + \left(\lambda_j - \varkappa_j a_j\right)(1/p - 1/q)\right], \tag{1.6}$$

$$A^{(i)}_\eta(x) = \int_0^\eta t^{-1-|\lambda|-|\nu,\lambda|+\lambda_i l_i} \int_{\mathbb{R}^n} \Phi(x + y)\, M\left(\frac{y}{t^\lambda}, \frac{\rho\left(t^\lambda, x\right)}{t^\lambda}, \rho\prime\left(t^\lambda, x\right)\right) dydt, \tag{1.7}$$

$$A^{(i)}_{\eta T}(x) = \int_\eta^T t^{-1-|\lambda|-|\nu,\lambda|+\lambda_i l_i} \int_{\mathbb{R}^n} \Phi(x + y)\, M\left(\frac{y}{t^\lambda}, \frac{\rho\left(t^\lambda, x\right)}{t^\lambda}, \rho\prime\left(t^\lambda, x\right)\right) dydt, \tag{1.8}$$

where

$$\rho\prime(u, x) = \frac{\partial}{\partial u}\rho(u, x), \qquad |\lambda| = \sum_{j=1}^{n} \lambda_j, \qquad |\nu, \lambda| = \sum_{j=1}^{n} \nu_j \lambda_j$$

Then

$$\sup_{\bar{x}\in U} \left\|A^{(i)}_\eta\right\|_{q,U_{\rho^\varkappa}(\bar{x})} \leq C_1 \|\Phi\|_{p,a,\varkappa,\tau;Q} [\rho]_1^{\sum_{j=1}^{n} \frac{\varkappa_j a_j}{q}} \eta^{\mu_i} \qquad (\mu_i > 0) \tag{1.9}$$

$$\sup_{\bar{x}\in U} \left\|A^{(i)}_{\eta T}\right\|_{q,U_{\rho^\varkappa}(\bar{x})} \leq C_2 \|\Phi\|_{p,a,\varkappa,\tau;Q} [\rho]_1^{\sum_{j=1}^{n} \frac{\varkappa_j a_j}{q}} \tag{1.10}$$

Here $U_{\rho^\varkappa}\left(\bar{x}\right) = \left\{x : \left|x_j - \bar{x}_j\right| < \frac{1}{2}\rho^{\varkappa_j},\ j = 1, 2, ..., n\right\}$ and C_1 and C_2 are constants independent of Φ, ρ, η and T.

Proof. Given $\bar{x} \in U$ and applying the generalized Minkowski inequality, we obtain

$$\left\|A_\eta^{(i)}\right\|_{q,U_{\rho^{\varkappa}}(x)} \le C \int^{\eta} t^{-1-|\lambda|-|\nu,\lambda|+\lambda_i l_i} \|F(\cdot .t)\|_{q,U_{\rho^{\varkappa}}(x)}\, dt. \tag{1.11}$$

where

$$F(x,t) = \int_{R^n} \Phi(x+y)\, M\left(\frac{y}{t^\lambda}, \frac{\rho\left(t^\lambda, x\right)}{t^\lambda}, \acute{\rho}\left(t^\lambda, x\right)\right) dy$$

Estimate the norm $\|F(\cdot,t)\|_{q,U_{\rho^{\varkappa}}(x)}$. By Holder's inequality $(q \le r)$,

$$\|F(\cdot .t)\|_{q,U_{\rho^{\varkappa}}(x)} \le \|F(\cdot .t)\|_{r,U_{\rho^{\varkappa}}(x)}\, \rho^{\left(\frac{1}{q}-\frac{1}{r}\right)\sum\limits_{j=1}^{n}}. \tag{1.12}$$

Let χ be the characteristic function of $S(M)$. Observing that $1 \le p \le r \le \infty$, $s \le r\left(\frac{1}{s} = 1 - \frac{1}{p} + \frac{1}{r}\right)$, and

$$|\Phi M| = \left(|\Phi|^p |M|^s\right)^{\frac{1}{r}} \left(|\Phi|^p \chi\right)^{\frac{1}{p}-\frac{1}{r}} \left(|M|^s\right)^{\frac{1}{s}-\frac{1}{r}}$$

and applying Holder's inequality $\left(\frac{1}{r} + \left(\frac{1}{p} - \frac{1}{r}\right) + \left(\frac{1}{s} - \frac{1}{r}\right) = 1\right)$, we obtain

$$\|F(\cdot .t)\|_{r,U_{\rho^{\varkappa}}(x)} \le \sup_{x \in U_{\rho^{\varkappa}}(x)} \left(\int_{R^n} |\Phi(x+y)|^p \chi\left(y : t^\lambda\right) dy\right)^{\frac{1}{p}-\frac{1}{r}} \times$$

$$\sup_{x \in V} \left(\int_{U_{\rho^{\varkappa}}(x)} |\Phi(x+y)|^p\, dx\right)^{\frac{1}{r}} \left(\int_{R^n} \left| M\left(\frac{y}{t^\lambda}, \frac{\rho\left(t^\lambda, x\right)}{t^\lambda}, \acute{\rho}\left(t^\lambda, x\right)\right)\right|^s dy\right)^{\frac{1}{s}}, \tag{1.13}$$

Since $Q_{t^\lambda}(x) \subset Q_{t^\varkappa}(x)$ for arbitrary $0 \le t \le 1$and $\varkappa_j \le \lambda_j\ (j = 1, 2, ..., n)$, given $x \in U$ we have

$$\int_{R^n} |\Phi(x+y)|^p \chi\left(y : t^\lambda\right) dy \le \int_{Q_{t^\varkappa}(x)} |\Phi(y)|^p\, dy \le \|\Phi\|_{p,a,\varkappa;Q}^p\, t^{\sum\limits_{j=1}^{n}\varkappa_j a_j} \tag{1.14}$$

For $y \in V$

$$\int_{U_{\rho^{\varkappa}}(x)} |\Phi(x+y)|^p\, dx \le \int_{Q_{\rho^{\varkappa}}(x+y)} |\Phi(x)|^p\, dx \le \|\Phi\|_{p,a,\varkappa;Q}^p\, [\rho]_1^{\sum\limits_{j=1}^{n}\varkappa_j a_j} \tag{1.15}$$

$$\int_{R^n} \left| M\left(\frac{y}{t^\lambda}, \frac{\rho\left(t^\lambda, x\right)}{t^\lambda}, \acute{\rho}\left(t^\lambda, x\right)\right)\right|^s dy = t^{|\lambda|} \|M\|_s^s \tag{1.16}$$

It follows from $(1.12) - (1.16)$ that

$$\|F(\cdot,t)\|_{q,U_{\rho^{\varkappa}}(x)} \le$$

$$\le C \|\Phi\|_{p,a,\varkappa;Q}\, t^{\sum\limits_{j=1}^{n}\lambda_j - \left(\frac{1}{p}-\frac{1}{r}\right)\sum\limits_{j=1}^{n}\left(\lambda_j - \varkappa_j a_j\right)}\, [\rho]_1^{\frac{1}{r}\sum\limits_{j=1}^{n}\varkappa_j a_j}\, \rho^{\left(\frac{1}{q}-\frac{1}{r}\right)\sum\limits_{j=1}^{n}\varkappa_j} \tag{1.17}$$

Using (1.4) $(1 \le r \le \infty)$, inserting (1.17) in (1.11), we arrive at (1.9). Similarly, we prove (1.10).

Corollary 1 *Putting* $r = \infty$ *for* $0 < p \le 1$*or* $r = q$ *for* $p > 1$ *in* (1.17), *we infer*

$$\sup_{x \in U} \|F(\cdot,t)\|_{q,U_{\rho^{\varkappa}}(x)} \le C \|\Phi\|_{p,a,\varkappa,;Q}\, [\rho]_1^{\sum\limits_{j=1}^{n}\varkappa_j \frac{1}{q}}$$

or

$$\|F(\cdot,t)\|_{q,b,\varkappa;U} \leq C\,\|\Phi\|_{p,a,\varkappa,;Q}$$

hence, using (1.4) *for* $1 \leq \tau_1 \leq \tau_2 \leq \infty$ *we obtain*

$$\|F(\cdot,t)\|_{q,b,\varkappa,\tau_2;U} \leq \acute{C}\,\|\Phi\|_{p,a,\varkappa,\tau_1;Q} \tag{1.18}$$

Lemma 2 *Let* $1 \leq p \leq q < \infty, 0 < \varkappa_j \leq \lambda_j\,(j = 1,2,...,n), 0 < T \leq 1, \nu = (\nu_1,...,\nu_n), \nu_j \geq 0,\ j = 1,2,...,n, 1 \leq \tau_1 \leq \tau_2 \leq \infty, \mu_i > 0$ *and*

$$\mu_{i,0} = \lambda_i l_i - \sum_{j=1}^{n}\left[\nu_j\lambda_j + \left(\lambda_j - \varkappa_j a_j\right)\frac{1}{p}\right]$$

Then the function $A_T^{(i)}(x)$ *defined by (1.7) satisfies the estimate*

$$\left\|A_T^{(i)}(x)\right\|_{q,b,\varkappa,\tau_2;U} \leq \|\Phi\|_{p,a,\varkappa,\tau_1;Q} \tag{1.19}$$

where $b = (b_1,...,b_n)$ *and* b_j *is an arbitrary number satisfying the inequalities*

$$0 \leq b_j \leq 1 \text{ if } \mu_{i,0} > 0,$$

$$0 \leq b_j < 1 \text{ if } \mu_{i,0} = 0,$$

$$0 \leq b_j < 1 + \frac{\mu_{i,0}q\left(1-a_j\right)}{n\left(\lambda_j - \varkappa_j a_j\right)} = a_j + \frac{\mu_i q\left(1-a_j\right)}{n\left(\lambda_j - \varkappa_j a_j\right)} \text{ if } \mu_{i,0} < 0. \tag{1.20}$$

2. Main Results

Now we reduce main result of this paper.

Theorem 1 *Assume that an open set* $G \subset R^n$ *satisfies the flexible* λ*−horn condition (O. V. Besov, V. P. Ilyin and S. M. Nikolskii, 1996),* $\lambda \in (0,\infty)^n$, $1 \leq p \leq q \leq \infty, \widetilde{\varkappa} = c\varkappa$, *where* $\frac{1}{c} = \max\limits_{1\leq j\leq n}\frac{\varkappa_j}{\lambda_j}$, $\nu = (\nu_1,...,\nu_n), \nu_j \geq 0$, $j = 1,...,n, 1 \leq \tau_1 \leq \tau_2 \leq \infty, \mu_i > 0\,(i = 1,...,n), \mu_i$ *and* $\mu_{i,0}$ *are defined in Lemmas1 and 2, and* $f \in W^l_{p,a,\varkappa,\tau_1}(G)$.

Then $D^\nu : W^l_{p,a,\varkappa,\tau_1}(G) \hookrightarrow L_{q,b,\varkappa,\tau_2}(G)$ *,i.e. there exists generalized mixed derivatives of fractional order* $D^\nu f$ *and the following inequalities are valid:*

$$\|D^\nu f\|_{q,G} \leq C_1\left(T^{\mu_0}\,\|f\|_{p,a,\varkappa.\tau_1;G} + \sum_{j=1}^{n}T^{\mu_i}\left\|D_i^{l_i}f\right\|_{p,a,\varkappa.\tau_1;G}\right), \tag{2.1}$$

$$\|D^\nu f\|_{q,b,\varkappa.\tau_2;G} \leq C_2\,\|f\|_{W^l_{p,a,\varkappa.\tau_1}(G)} \qquad (\ p \leq q < \infty\ ), \tag{2.2}$$

where

$$\mu_0 = \mu_i - \lambda_i l_i = -\sum_{j=1}^{n}\left[\nu_j\lambda_j + \left(\lambda_j - \varkappa_j a_j\right)\left(\frac{1}{p} - \frac{1}{q}\right)\right];$$

here $T \leq \min(1,T_0)$, $C_1 - C_4$ *are constants independent of* f *and* C_1 *and* C_3 *are also independent of* T.

In particular, if $\mu_{i,0} > 0$, $i = 1,...,n$ then D^ν is continuous on G and

$$\sup_{x\in G}|D^\nu f| \leq C\left(T^{\mu_{i,0}-l_i\lambda_i}\,\|f\|_{p,a,\varkappa.\tau_1;G} + \sum_{j=1}^{n}T^{\mu_{i,0}}\,\|D^\nu f\|_{p,a,\varkappa.\tau_1;G}\right) \tag{2.3}$$

where $D^\nu f = D^{[\nu]}D_+^{\{\nu\}}f$, and

$$\left(D_+^\nu f\right)(x) = \frac{1}{\Gamma(1-\{\nu\})}\frac{\partial^n}{\partial x_1\partial x_2...\partial x_n}\int\limits_{G^{(n)}}\frac{f(s)\,ds}{(x-s)^{\{\nu\}}},$$

$$(D_{-}^{\nu} f)(x) = (-1)^n \frac{1}{\Gamma(1-\{\nu\})} \frac{\partial^n}{\partial x_1 \partial x_2 ... \partial x_n} \int_{\overline{G}^{(n)}} \frac{f(s)\, ds}{(s-x)^{\{\nu\}}}.$$

where $ds = ds_1 ds_2 \cdots ds_n$; $\Gamma(1-\{\nu\}) = \Gamma(1-\{\nu_1\}) ... \Gamma(1-\{\nu_n\})$; $(x-s)^{\{\nu\}} = (x_1-s_1)^{\{\nu_1\}} ... (x_n-s_n)^{\{\nu_n\}}$,

Proof. First of all,observe that,since $\widetilde{\varkappa} = c\varkappa$ using the property 4, we can assume that $f \in W^l_{p,a,\widetilde{\varkappa},\tau_1}(G)$ and substitute $\widetilde{\varkappa}$ for $\varkappa$ everywhere in (2.1)-(2.3) and for μ_i in (1.6). We will prove these very inequalities (the greater $\varkappa$, the greater μ_i). Existence of the generalized mixed derivatives of fractional order $D^\nu f$ under the conditions of the theorem follows from (A. M. Najafov, 2010) and (A. M. Najafov, 2013).

Indeed, if $\mu_i > 0$ then $\lambda_i l_i - |\nu : l| > 0$, for $p \le q$,$0 \le a \le 1$ and $\varkappa \le \lambda$. Since $f \in W^l_{p,a,\varkappa,\tau}(G) \hookrightarrow W^l_{p,a,\varkappa}(G) \hookrightarrow W^l_p(G)$, by Theorem 1 (A. M. Najafov, 2010) and (A. M. Najafov, 2013) the generalized mixed derivatives of fractional order exists on G and$D^\nu f \in L_p(G)$. Then it is obtained integral representation for generalized mixed derivatives of fractional order of functions from Sobolev spaces of fractional order defined on the n-dimensional domains in R^n and satisfying flexible horn conditions (The domains satisfying flexible horn condition introduced in (O. V. Besov, V. P. Ilyin and S. M. Nikolskii, 1996):

$$D^\nu f(x) = f^{(\nu)}_{T^\lambda} + \int_0^T \int_{\mathbb{R}^n} \sum_{i=1}^n t^{-1-|\lambda|-|\nu,\lambda|+\lambda_i l_i} \times$$

$$\times L_i^{(\nu)}\left(\frac{y}{t^\lambda}, \frac{\rho\left(t^\lambda, x\right)}{t^\lambda}, \rho\prime\left(t^\lambda, x\right)\right) D_i^{l_i} f(x+y)\, dy dt, \qquad (2.4).$$

$$f^{(\nu)}_{T^\lambda} = T^{-|\lambda|-|\nu,\lambda|} \int_{\mathbb{R}^n} f(x+y)\, \Omega^{(\nu)}\left(\frac{y}{T^\lambda}, \frac{\rho\left(t^\lambda, x\right)}{T^\lambda}\right) dy \qquad (2.5).$$

where $0 < T \le \min(1, T_0)$ the functions $\Omega^{(\nu)}(\cdot, y)$ and $L_i^{(\nu)}(\cdot, y, z)$ are of the class $C_0^\infty(\mathbb{R}^n)$ with support in I_1 and the support of (2.4), (2.5) is contained in the flexible horn $x + V(\lambda, x, 0) \subset G$. Using Minkowski's inequality hence, we obtain that

$$\|D^\nu f\|_{q;G} \le \left\|f^{(\nu)}_{T^\lambda}\right\|_{q;G} + \sum_{i=1}^n \left\|A_T^{(i)}\right\|_{q;G} \qquad (2.6)$$

From (1.17) for $U = G, t = T$ as $\rho \to \infty$ and $r = q$ we derive

$$\left\|f^{(\nu)}_{T^\lambda}\right\|_{q;G} \le C_1 T^{\mu_0} \|f\|_{p,a,\widetilde{\varkappa}.\tau_1;G} \qquad (2.7).$$

and from inequality (1.9) for $U = G$ as $\rho \to \infty$, which we may apply, since $\mu_i > 0$, $i = 1, ..., n$, we find that

$$\left\|A_T^{(i)}\right\|_{q;G} \le C_1 T^{\mu_i} \left\|D_i^{l_i} f\right\|_{p,a,\widetilde{\varkappa}.\tau_1;G} \qquad (2.8)$$

Consequently,

$$\|D^\nu f\|_{q;G} \le C_1 \left(T^{\mu_0} \|f\|_{p,a,\widetilde{\varkappa}.\tau_1;G} + \sum_{j=1}^n T^{\mu_i} \left\|D_i^{l_i} f\right\|_{p,a,\widetilde{\varkappa}.\tau_1;G}\right)$$

Similarly, using (1.18) and (1.19), we establish (2.2).

Assume now that $\mu_{i,0} > 0$,$i = 1, ..., n$. Show that then $D^\nu f$ is continuous on G. From (2.4),(2.5), and (2.8) for $q = \infty$ and $\mu_i = \mu_{i,0} > 0$ we derive

$$\|D^\nu f - D^\nu f_{T^\lambda}\|_{q;G} \le \sum_{i=1}^n T^{\mu_i} \left\|D_i^{l_i} f\right\|_{p,a,\widetilde{\varkappa}.\tau_1;G}$$

Hence, the left-hand side of the inequality tends to zero as $T \to 0$. Since $D^\nu f_{T^\lambda}$ is continuous on G, in this case the convergence of $L_\infty(G)$ coincides with uniform convergence; consequently, the limit function $D^\nu f$ is continuous on G. The theorem is proved.

Theorem 2 *Suppose that the conditions of Theorem 1 are satisfied. Then for $\mu_i > 0, i = 1, ..., n$, the generalized mixed derivatives of fractional order $D^\nu f$ satisfies the Holder condition on G in the metric of L_q with exponent β^1; more exactly,*

$$\|\Delta(\gamma, G) D^\nu f\|_{q;G} \leq C \|f\|_{W^l_{p,a,\varkappa,\tau}(G)} |\gamma|^\varepsilon,$$

where ε is an arbitrary number satisfying the inequalities

$$0 \leq \varepsilon \leq 1 \quad \text{if } \frac{\mu^0}{\lambda_0} > 1,$$

$$0 \leq \varepsilon < 1 \quad \text{if } \frac{\mu^0}{\lambda_0} = 1,$$

$$0 \leq \varepsilon < \frac{\mu^0}{\lambda_0} \quad \text{if } \frac{\mu^0}{\lambda_0} < 1,$$

with $\mu^0 = \min \mu_i, i = 1, ..., n$, and $\lambda_0 = \max \lambda_j, j = 1, ..., n$.

If $\mu_{i,0} > 0$ then

$$\sup_{x \in G} |\Delta(\gamma, G) D^\nu f|_{q;G} \leq C \|f\|_{W^l_{p,a,\varkappa,\tau}(G)} |\gamma|^{\varepsilon_0},$$

where ε_0 satisfies the same conditions as ε, but with $\mu_{i,0}$ instead of μ_i.

Now, consider the problem of smoothness of solutions of higher order fractional quasielliptic equations (1.1). Suppose that $p = 2, \lambda = (\lambda_1, ..., \lambda_n), \lambda_j^{-1} = l_j > 0, j = 1, ..., n$, and the coefficients $a_{\alpha\beta}(x) \equiv a_{\beta\alpha}(x), a_{\alpha\beta}(x)$ are bounded, measurable in G, and such that

$$\sum_{(\alpha,\lambda)=(\beta,\lambda)=1} (-1)^{|\alpha|} a_{\alpha\beta}(x) \xi_\alpha \xi_\beta \geq C_0 \sum_{(\alpha,\lambda)=1} |\xi_\alpha|^2, \; C_0 = const.$$

We suppose that $f_\alpha \in L_2(G)$ for $(\alpha, \lambda) < 1$ and $f_\alpha \in L_{2,a,\varkappa}(G)$ for $(\alpha, \lambda) = 1$.

A work solution to (1.1) in G is a function $u(x) \in W_2^l(G)$ such that

$$\sum_{(\alpha,\lambda)\leq 1, (\beta,\lambda)\leq 1} \int_G (-1)^{|\alpha|} a_{\alpha\beta}(x) D^\beta u D^\alpha \vartheta = \sum_{(\alpha,\lambda)\leq 1} \int_G (-1)^{|\alpha|} f_\alpha D^\alpha \vartheta dx$$

for every function $\vartheta(x) \in \overset{\circ}{W}{}_2^l(G)$.

Theorem 3 *If $\frac{|\lambda|}{2} + |\nu, \lambda| \leq 1, \nu = (\nu_1, ..., \nu_n), \nu_j \geq 0, \; j = 1, ..., n$, then every weak solution to (1.1) in $W_2^l(G)$ belongs to the space $C_{\nu+\varepsilon_0}\left(G^d\right), \overline{G}^d \subset G$.*

Remark 1 Note that Theorem 1, Theorem 2 and Theorem 3 in the case when $l = (l_1, l_2, ..., l_n), \left(l_j \in N, \;\; j = 1, 2, ..., n\right)$ were proved by the author (A. M. Nadzhafov, 2005).

Theorem 4 *Let the domain $G \subset R^n$ such that there exists $\rho = const > 0$, for any point $x_0 \in \partial G$ and the number $r < 1$, there exists a parallelepiped $\Pi_{\rho r}\left(x^1\right) \subset \Pi_r(x_0) \cap (R^n \backslash G)$ and $u(x)$ is solution of equation (1.1) from the space $\overset{\circ}{W}{}_2^l(G)$. If $\frac{|\lambda|}{2} + |\nu, \lambda| \leq 1$, then $u(x)$ belongs to the space $C_{\nu+\varepsilon_0}\left(\overline{G}\right)$.*

Proof. It is sufficiently in this case, to let all $a_{\alpha\beta}(x) \equiv 0$,except for ones for which $|\alpha, \lambda| = |\beta, \lambda| = 1$. Let $x_0 \in \partial G$, and all $f_a \equiv 0$ in $\Pi_b(x_0)$, $u(x) \equiv 0$ outside of G.

From the variational principle it follows that

$$A(u(x), \Pi_b(x_0)) \leq A(\theta(x) u(x), \Pi_b(x_0)).$$

As $\theta(x) \equiv 0$ in $\Pi_{\frac{b}{2}}(x_0)$, then

$$A(u(x), \Pi_b(x_0)) \leq A\left(u(x), \Pi_b(x_0) \backslash \Pi_{\frac{b}{2}}(x_0)\right) +$$

$$+C\sum_{|\alpha,\lambda|<1}\int\limits_{\Pi_b(x_0)\setminus\Pi_{\frac{b}{2}}(x_0)} b^{-2+2|\alpha,\lambda|}\left(D^{\alpha}u(x)\right)^2dx.$$

As $u|_{\Pi_{\rho b}\left(x^1\right)}=0$, where $\Pi_{\rho b}\left(x^1\right)\subset\Pi_b(x_0)\setminus\Pi_{\frac{b}{2}}(x_0)$, then we have

$$A\left(u(x),\Pi_b(x_0)\right)\leq qA\left(u(x),\Pi_b(x_0)\setminus\Pi_{\frac{b}{2}}(x_0)\right),$$

and hence it follows

$$A\left(u(x),\Pi_{\frac{b}{2^k}}(x_0)\right)\leq\left(1-\frac{1}{q}\right)^kA\left(u(x),\Pi_b(x_0)\right).$$

Therefore

$$A\left(u(x),\Pi_{\delta}(x_0)\right)\leq\left(\frac{\delta}{b}\right)^{\xi-\sigma}A\left(u(x),G\right), \tag{2.9}$$

if $\delta<b,\forall x_0\in\partial G, f_b(x_0)$. Let $0<\delta<1, x\in G, f_{\alpha}\neq0$ and us consider two cases:

a) $x_0\in G^{\sqrt{\delta}}$;

b) $x_0\notin G^{\sqrt{\delta}}$.

a) In this case for all $\delta<b$, assuming that $b=\sqrt{\delta}$.We have

$$A\left(u(x),\Pi_{\delta}(x_0)\right)\leq C_1\left(\frac{\delta}{b}\right)^{\xi-\sigma}A\left(u(x),G\right)+C_2b^{\triangle}\leq$$

$$\leq C_3\left(\frac{\delta}{b}\right)^{\xi-\sigma}A\left(u(x),G\right)+1.$$

b) In this case there is a point $x_1\in\partial G$, such that $\Pi_{2\sqrt{\delta}}\left(x^1\right)\supset\Pi_{\sqrt{\delta}}(x_0)$. Let $b>2\sqrt{\delta}$, u_{b,x^1} - solution of equation (1.1) in $\Pi_b\left(x^1\right)\cap G$ form the space $\overset{\circ}{W}{}_2^l\left(\Pi_b\left(x^1\right)\cap G\right)$, for which inequality

$$A\left(u_{b,x^1},\Pi_{\delta}\left(x^1\right)\right)\leq C_4b^{\triangle}. \tag{2.10}$$

The function $u(x)-u_{b,x^1}$ is a solution of equation (1.1) in $\Pi_b\left(x^1\right)$, where $f_{\alpha}\equiv0$. From inequalities (2.9) and (2.10) we have

$$A\left(u(x),\Pi_{2\sqrt{\delta}}\left(x^1\right)\right)\leq C_5A\left(u-u_{b,x^1},\Pi_{2\sqrt{\delta}}\left(x^1\right)\right)+C_5A\left(u_{b,x^1},\Pi_{2\sqrt{\delta}}\left(x^1\right)\right)\leq$$

$$\leq 2C_6\left(\frac{\sqrt{\delta}}{b}\right)^{\xi-\sigma}A\left(u-u_{b,x^1},\Pi_{2\sqrt{\delta}}\left(x^1\right)\right)+2C_7b^{\triangle}\leq C_8\left(\frac{\delta}{b}\right)^{\xi-\sigma}A\left(u(x),G\right),$$

$$A\left(u(x),\Pi_{\sqrt{\delta}}(x_0)\right)\leq A\left(u(x),\Pi_{2\sqrt{\delta}}\left(x^1\right)\right)\leq C\left(\frac{\delta}{b}\right)^{\xi-\sigma}A\left(u(x),G\right).$$

Consequently

$$A\left(u(x),\Pi_{\delta}(x_0)\right)\leq C\left(\frac{\delta}{b}\right)^{\xi-\sigma}A\left(u(x),G\right),$$

$$\int\limits_0^1\left[\eta^{-\xi}\int\limits_{\Pi_{\eta}(x_0)}u^2(x)\,dx\right]^{\frac{1}{2}}\frac{d\eta}{\eta}\leq C\int\limits_0^1\frac{db}{b^{1-\frac{1}{2}\sigma}}<\infty.$$

This implies that $u\in L_{2,a,\varkappa,1}\left(\overline{G}\right)\subset L_{2,a,\varkappa,\tau}\left(\overline{G}\right)$ and also $D_i^{l_i}u\in L_{2,a,\varkappa,\tau}\left(\overline{G}\right)$, $i=1,2,...,n$, then it follows that $u\in W_{2,a,\varkappa,\tau}^l\left(\overline{G}\right)$. Then in this case the conditions in theorems 1 and 2 are satisfied. Thus by theorems1 and 2 it follows that $u\in C_{\nu+\varepsilon_0}\left(\overline{G}\right)$.

The theorem is proved.

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