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Hamiltonian Vector Fields on Weil Bundles

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Abstract

Let M be a paracompact smooth manifold, A a Weil algebra and M^A the associated Weil bundle. In this paper, we give a characterization of hamiltonian field on M^A in the case of Poisson manifold and of Symplectic manifold.

Keywords: Weil algebra, Weil bundle, Poisson manifold, hamiltonian vector fields

1. Introduction

In what follows, we denote by M, a paracompact smooth manifold of dimension n, $C^\infty(M)$ the algebra of smooth functions on M and A a Weil algebra i.e a real commutative algebra of finite dimension, with unit, and with an unique maximal ideal $\mathfrak{m}$ of codimension 1 over $\mathbb{R}$ (Weil, 1953). In this case, there exists an integer h such that $\mathfrak{m}^{h+1} = (0)$ and $\mathfrak{m}^h \neq (0)$. The integer h is the height of A. Also we have $A = \mathbb{R} \oplus \mathfrak{m}$.

We recall that a near point of $x \in M$ of kind A (Weil, 1953) is a morphism of algebras

$$\xi : C^\infty(M) \longrightarrow A$$

such that

$$\xi(f) - f(x) \in \mathfrak{m}$$

for any $f \in C^\infty(M)$. We denote M_x^A the set of near points of $x \in M$ of kind A and $M^A = \bigcup_{x\in M} M_x^A$ the manifold of infinitely near points of M of kind A and

$$\pi_M : M^A \longrightarrow M$$

the projection which assigns every infinitely near point of $x \in M$ to its origin x. The triplet (M^A, π_M, M) defines a bundle called bundle of infinitely near points or simply Weil bundle (Kolár, Michor, Slovak, 1993).

When M and N are smooth manifolds and when $h : M \longrightarrow N$ is a differentiable map of class C^∞, then the map

$$h^A : M^A \longrightarrow N^A, \xi \longmapsto h^A(\xi)$$

such that for all g in $C^\infty(N)$,

$$[h^A(\xi)](g) = \xi(g \circ h)$$

is differentiable (Morimoto, 1976). Thus, for $f \in C^\infty(M)$, the map

$$f^A : M^A \longrightarrow \mathbb{R}^A = A, \xi \longmapsto [f^A(\xi)](id_{\mathbb{R}}) = \xi(id_{\mathbb{R}} \circ f) = \xi(f)$$

is differentiable of class C^∞. The set, $C^\infty(M^A, A)$ of smooth functions on M^A with values in A, is a commutative algebra over A with unit and the map

$$C^\infty(M) \longrightarrow C^\infty(M^A, A), f \longmapsto f^A$$

is an injective morphism of algebras. Then, we have (Bossoto & Okassa, 2008):

$$(f+g)^A = f^A + g^A; (\lambda \cdot f)^A = \lambda \cdot f^A; (f \cdot g)^A = f^A \cdot g^A$$

for $f, g \in C^\infty(M)$ and $\lambda \in \mathbb{R}$.

1.1 Vector Fields on Weil Bundles

In (Bossoto & Okassa, 2008) and (Nkou, Bossoto & Okassa, 2015), we showed that the following assertions are equivalent:

1) A vector field on M^A is a differentiable section of the tangent bundle (TM^A, π_{M^A}, M^A).

2) A vector field on M^A is a derivation of $C^\infty(M^A)$.

3) A vector field on M^A is a derivation of $C^\infty(M^A, A)$ which is A-linear.

4) A vector field on M^A is a linear map $X : C^\infty(M) \longrightarrow C^\infty(M^A, A)$ such that

$$X(f \cdot g) = X(f) \cdot g^A + f^A \cdot X(g), \quad \text{for any } f, g \in C^\infty(M).$$

In all that follows, we denote by $\mathfrak{X}(M^A)$ the set of vector fields on M^A and $Der_A[C^\infty(M^A, A)]$ the set of A-linear maps

$$X : C^\infty(M^A, A) \longrightarrow C^\infty(M^A, A)$$

such that

$$X(\varphi \cdot \psi) = X(\varphi) \cdot \psi + \varphi \cdot X(\psi), \quad \text{for any } \varphi, \psi \in C^\infty(M^A, A).$$

Then (Nkou, Bossoto & Okassa, 2015),

$$\mathfrak{X}(M^A) = Der_A[C^\infty(M^A, A)].$$

The map

$$\mathfrak{X}(M^A) \times \mathfrak{X}(M^A) \longrightarrow \mathfrak{X}(M^A), (X, Y) \longmapsto [X, Y] = X \circ Y - Y \circ X$$

is skew-symmetric A-bilinear and defines a structure of an A-Lie algebra over $\mathfrak{X}(M^A)$.

If

$$\theta : C^\infty(M) \longrightarrow C^\infty(M),$$

is a vector field on M, then there exists one and only one A-linear derivation,

$$\theta^A : C^\infty(M^A, A) \longrightarrow C^\infty(M^A, A)$$

called prolongation of the vector field θ, such that

$$\theta^A(f^A) = [\theta(f)]^A, \quad \text{for any } f \in C^\infty(M).$$

If θ, θ_1 and θ_2 are vector fields on M and if $f \in C^\infty(M)$, then we have:

$$(\theta_1 + \theta_2)^A = \theta_1^A + \theta_2^A; (f \cdot \theta)^A = f^A \cdot \theta^A ; [\theta_1, \theta_2]^A = [\theta_1^A, \theta_2^A].$$

The map

$$\mathfrak{X}(M) \longrightarrow Der_A[C^\infty(M^A, A)], \theta \longmapsto \theta^A$$

is an injective morphism of $\mathbb{R}$-Lie algebras.

1.2 Structure of A-Poisson Manifold on M^A When M is a Poisson Manifold

We recall that a Poisson structure on a smooth manifold M is due to the existence of a bracket $\{,\}$ on $C^\infty(M)$ such that the pair $(C^\infty(M), \{,\})$ is a real Lie algebra such that, for any $f \in C^\infty(M)$ the map

$$ad(f) : C^\infty(M) \longrightarrow C^\infty(M), g \longmapsto \{f, g\}$$

is a derivation of commutative algebra i.e

$$\{f, g \cdot h\} = \{f, g\} \cdot h + g \cdot \{f, h\}$$

for $f, g, h \in C^\infty(M)$. In this case we say that M is a Poisson manifold and $C^\infty(M)$ is a Poisson algebra (Vaisman, 1994, 1995).

We denote by

$$C^\infty(M) \longrightarrow Der_{\mathbb{R}}[C^\infty(M)], f \longmapsto ad(f),$$

the adjoint representation and d_{ad} the operator of cohomology associated to this representation. For any $p \in \mathbb{N}$,

$$\Lambda^p_{Pois}(M) = \mathcal{L}^p_{sks}[C^\infty(M), C^\infty(M)]$$

denotes the $C^\infty(M)$-module of skew-symmetric multilinear forms of degree p from $C^\infty(M)$ into $C^\infty(M)$. We have

$$\Lambda^0_{Pois}(M) = C^\infty(M).$$

When M is a smooth manifold, A a weil algebra and M^A the associated Weil bundle, the A-algebra $C^\infty(M^A, A)$ is a Poisson algebra over A if there exists a bracket $\{,\}$ on $C^\infty(M^A, A)$ such that the pair $(C^\infty(M^A, A), \{,\})$ is a Lie algebra over A satisfying

$$\{\varphi_1 \cdot \varphi_2, \varphi_3\} = \{\varphi_1, \varphi_3\} \cdot \varphi_2 + \varphi_1 \cdot \{\varphi_2, \varphi_3\}$$

for any $\varphi_1, \varphi_2, \varphi_3 \in C^\infty(M^A, A)$ (Bossoto & Okassa, 2012).
When M is a Poisson manifold with bracket $\{,\}$, for any $f \in C^\infty(M)$, let

$$[ad(f)]^A : C^\infty(M) \longrightarrow C^\infty(M^A, A), g \longmapsto \{f, g\}^A,$$

be the prolongation of the vector field $ad(f)$ and let

$$\widetilde{[ad(f)]^A} : C^\infty(M^A, A) \longrightarrow C^\infty(M^A, A)$$

be the unique A-linear derivation such that

$$\widetilde{[ad(f)]^A}(g^A) = [ad(f)]^A(g) = \{f, g\}^A$$

for any $g \in C^\infty(M)$.

For $\varphi \in C^\infty(M^A, A)$, the application

$$\tau_\varphi : C^\infty(M) \longrightarrow C^\infty(M^A, A), f \longmapsto -\widetilde{[ad(f)]^A}(\varphi)$$

is a vector field on M^A considered as derivation of $C^\infty(M)$ into $C^\infty(M^A, A)$ and

$$\widetilde{\tau_\varphi} : C^\infty(M^A, A) \longrightarrow C^\infty(M^A, A)$$

the unique A-linear derivation (vector field) such that

$$\widetilde{\tau_\varphi}(f^A) = \tau_\varphi(f) = -\widetilde{[ad(f)]^A}(\varphi)$$

for any $f \in C^\infty(M)$. We have for $f \in C^\infty(M)$,

$$\widetilde{\tau_{f^A}} = \widetilde{[ad(f)]^A},$$

and for $\varphi, \psi \in C^\infty(M^A, A)$ and for $a \in A$,

$$\widetilde{\tau_{\varphi+\psi}} = \widetilde{\tau_\varphi} + \widetilde{\tau_\psi}; \widetilde{\tau_{a\cdot\varphi}} = a \cdot \widetilde{\tau_\varphi}; \widetilde{\tau_{\varphi\cdot\psi}} = \varphi \cdot \widetilde{\tau_\psi} + \psi \cdot \widetilde{\tau_\varphi}.$$

For any $\varphi, \psi \in C^\infty(M^A, A)$, we let

$$\{\varphi, \psi\}_A = \widetilde{\tau_\varphi}(\psi).$$

In (Bossoto & Okassa, 2012), we showed that this bracket defines a structure of A-Poisson algebra on $C^\infty(M^A, A)$.

Thus when M is a Poisson manifold with bracket $\{,\}$, then $\{,\}_A$ is the prolongation on M^A of the structure of Poisson on M defined by $\{,\}$.

The map

$$C^\infty(M^A, A) \longrightarrow Der_A[C^\infty(M^A, A)], \varphi \longmapsto \widetilde{\tau_\varphi},$$

is a representation from $C^\infty(M^A, A)$ into $C^\infty(M^A, A)$. We denote $\widetilde{d_A}$ the cohomology operator associated to this adjoint representation (Nkou & Bossoto, 2014).

For any $p \in \mathbb{N}$, $\Lambda^p_{Pois}(M^A, \sim_A) = \mathcal{L}^p_{sks}[C^\infty(M^A, A), C^\infty(M^A, A)]$ denotes the $C^\infty(M^A, A)$-module of skew-symmetric multilinear forms of degree p from $C^\infty(M^A, A)$ into $C^\infty(M^A, A)$. We have

$$\Lambda^0_{Pois}(M^A, \sim_A) = C^\infty(M^A, A).$$

We denote

$$\Lambda_{Pois}(M^A, \sim_A) = \bigoplus_{p=0}^{n} \Lambda^p_{Pois}(M^A, \sim_A).$$

For $\Omega \in \Lambda^p_{Pois}(M^A, \sim_A)$ and $\varphi_1, \varphi_2, ..., \varphi_{p+1} \in C^\infty(M^A, A)$, we have

$$\begin{aligned}\widetilde{d_A}\Omega(\varphi_1, ..., \varphi_{p+1}) &= \sum_{i=1}^{p+1}(-1)^{i-1}\widetilde{\tau_{\varphi_i}}[\Omega(\varphi_1, ..., \widehat{\varphi_i}, ..., \varphi_{p+1})] \\ &+ \sum_{1\le i<j\le p+1}(-1)^{i+j}\Omega(\{\varphi_i, \varphi_j\}_A, \varphi_1, ..., \widehat{\varphi_i}, ..., \widehat{\varphi_j}, ..., \varphi_{p+1})\end{aligned}$$

where $\widehat{\varphi_i}$ means that the term φ_i is omitted.

Proposition 1.1 (Nkou & Bossoto, 2014) *For any $\eta \in \Lambda^p_{Pois}(M)$, we have*

$$\widetilde{d_A}(\eta^A) = (d_{ad}\eta)^A.$$

2. Hamiltonian Vector Fields on Weil Bundles

When M is a Poisson manifold with bracket $\{,\}$, we recall that a vector field

$$\theta : C^\infty(M) \longrightarrow C^\infty(M)$$

is locally hamiltonian if θ is closed for the cohomology associated with the adjoint representation

$$ad : C^\infty(M) \longrightarrow Der_{\mathbb{R}}\left[C^\infty(M)\right]$$

i.e. $d_{ad}\theta = 0$ and θ is globally hamiltonian if θ is exact for the cohomology associated with the adjoint representation

$$ad : C^\infty(M) \longrightarrow Der_{\mathbb{R}}\left[C^\infty(M)\right]$$

i.e. there exists $f \in C^\infty(M)$ such that $\theta = d_{ad}(f)$.

Thus a vector field on M^A

$$X : C^\infty(M^A, A) \longrightarrow C^\infty(M^A, A)$$

is locally hamiltonian if X is closed for the cohomology associated with the adjoint representation

$$C^\infty(M^A, A) \longrightarrow Der_A\left[C^\infty(M^A, A)\right], \varphi \longmapsto \widetilde{\tau_\varphi}$$

i.e $\widetilde{d_A}X = 0$ and X is globally hamiltonian if X is exact for the cohomology associated with the adjoint representation

$$C^\infty(M^A, A) \longrightarrow Der_A\left[C^\infty(M^A, A)\right], \varphi \longmapsto \widetilde{\tau_\varphi}$$

i.e. there exists $\varphi \in C^\infty(M^A, A)$ such that $X = \widetilde{d_A}(\varphi)$.

Proposition 2.1 *When M is a Poisson manifold with bracket $\{,\}$, then a vector field*

$$\theta : C^\infty(M) \longrightarrow C^\infty(M)$$

is locally hamiltonian if and only if the vector field

$$\theta^A : C^\infty(M^A, A) \longrightarrow C^\infty(M^A, A).$$

is locally hamiltonian.

Proof Indeed, for any $\eta \in \Lambda^p_{Pois}(M)$, we have

$$\widetilde{d_A}(\eta^A) = (d_{ad}\eta)^A.$$

In particular, for $p = 1$, we have

$$\widetilde{d_A}(\theta^A) = (d_{ad}\theta)^A.$$

Thus, $d_{ad}\theta = 0$ if and only if $\widetilde{d_A}(\theta^A) = 0$.

Proposition 2.2 *When M^A is a A-Poisson manifold with bracket $\{,\}_A$, then, a vector field*

$$X : C^\infty(M^A, A) \longrightarrow C^\infty(M^A, A)$$

locally hamiltonian is a derivation of the Poisson A-algebra $C^\infty(M^A, A)$.

Proof We have

$$\begin{array}{rccc} \widetilde{d_A}X : & C^\infty(M^A, A) \times C^\infty(M^A, A) & \longrightarrow & C^\infty(M^A, A) \\ & (\varphi, \psi) & \longmapsto & (\widetilde{d_A}X)(\varphi, \psi) \end{array}$$

and if $\widetilde{d_A}X = 0$, then for any $\varphi, \psi \in C^\infty(M^A, A)$,

$$\begin{aligned} 0 &= (\widetilde{d_A}X)(\varphi, \psi) \\ &= \widetilde{\tau_\varphi}[X(\psi)] - \widetilde{\tau_\psi}[X(\varphi)] - X(\{\varphi, \psi\}_A) \\ &= \{\varphi, X(\psi)\}_A - \{\psi, X(\varphi)\}_A - X(\{\varphi, \psi\}_A) \end{aligned}$$

i.e

$$X(\{\varphi, \psi\}_A) = \{X(\varphi), \psi\}_A + \{\varphi, X(\psi)\}_A.$$

That ends the proof.

Proposition 2.3 *Let M be a Poisson manifold with bracket $\{,\}$. If a vector field*

$$\theta : C^\infty(M) \longrightarrow C^\infty(M)$$

is globally hamiltonian then the vector field

$$\theta^A : C^\infty(M^A, A) \longrightarrow C^\infty(M^A, A)$$

is globally hamiltonian.

Proof Based on the assumptions, there exists $f \in C^\infty(M)$ such that $\theta = d_{ad}(f)$. Thus,

$$\begin{aligned} \theta^A &= [ad(f)]^A \\ &= \widetilde{d_A}(f^A). \end{aligned}$$

Thus, $\theta = d_{ad}(f)$ then $\theta^A = \widetilde{d_A}(f^A)$ is globally hamiltonian.

Proposition 2.4 *When M^A is a A-Poisson manifold with bracket $\{,\}_A$, then a vector field*

$$X : C^\infty(M^A, A) \longrightarrow C^\infty(M^A, A)$$

globally hamiltonian is the derivation interior of the Poisson A-algebra $C^\infty(M^A, A)$.

Proof If the vector field

$$X : C^\infty(M^A, A) \longrightarrow C^\infty(M^A, A)$$

is globally hamiltonian, there exists $\varphi \in C^\infty(M^A, A)$ suth that $X = \widetilde{d_A}\varphi$. For any $\psi \in C^\infty(M^A, A)$, we have

$$\begin{aligned} X(\psi) &= (\widetilde{d_A}\varphi)(\psi) \\ &= \widetilde{\tau_\varphi}(\psi) \\ &= \{\varphi, \psi\}_A \end{aligned}$$

i.e. $X = ad(\varphi)$. where

$$ad(\varphi) : C^\infty(M^A, A) \longrightarrow C^\infty(M^A, A), \psi \longmapsto \{\varphi, \psi\}_A$$

Thus, X is globally hamiltonian if there exists $\varphi \in C^\infty(M^A, A)$ such that $X = \widetilde{\tau_\varphi} = ad(\varphi)$ i.e. X is the interior derivation of the Poisson A-algebra $C^\infty(M^A, A)$.

3. Hamiltonian Vector Fields on M^A When M is a Symplectic Manifold

When (M, Ω) is a symplectic manifold, then (M^A, Ω^A) is a symplectic A-manifold (Bossoto & Okassa, 2012).

For any $f \in C^\infty(M)$, we denote X_f the unique vector field on M such that

$$i_{X_f}\Omega = df$$

where

$$d : \Lambda(M) \longrightarrow \Lambda(M)$$

is the operator of de Rham cohomology. We denote

$$d^A : \Lambda(M^A, A) \longrightarrow \Lambda(M^A, A)$$

the operator of cohomology associated with the representation

$$\mathfrak{X}(M^A) \longrightarrow Der_A\left[C^\infty(M^A, A)\right], X \longmapsto X.$$

For $\varphi \in C^\infty(M^A, A)$, we denote by X_φ the unique vector field on M^A, considered as a derivation from $C^\infty(M^A, A)$ into $C^\infty(M^A, A)$, such that

$$i_{X_\varphi}\Omega^A = d^A(\varphi).$$

The bracket

$$\begin{aligned} \{\varphi, \psi\}_{\Omega^A} &= -\Omega^A(X_\varphi, X_\psi) \\ &= X_\varphi(\psi) \end{aligned}$$

defines a structure of A-Poisson manifold on M^A and for any $f \in C^\infty(M)$, $X_{f^A} = (X_f)^A$ and

$$i_{(X_f)^A}\Omega^A = i_{X_{f^A}}\Omega^A.$$

We deduce that (Bossoto & Okassa, 2012):

Theorem 3.1 *If (M, Ω) is a symplectic manifold, the structure of A-Poisson manifold on M^A defined by Ω^A coincide with the prolongation on M^A of the Poisson structure on M defined by the symplectic form Ω i.e for any $\varphi \in C^\infty(M^A, A)$, $\widetilde{\tau_\varphi} = X_\varphi$.*

Therefore, for any $\varphi, \psi \in C^\infty(M^A, A)$, we have

$$\{\varphi, \psi\}_{\Omega^A} = \{\varphi, \psi\}_A .$$

Proposition 3.2 *If ω is a differential form on M and if θ is a vector field on M, then*

$$(i_\theta\omega)^A = i_{\theta^A}(\omega^A).$$

Proof If the degree of ω is p, according (Bossoto & Okassa, 2012, Proposition 9), $(i_\theta\omega)^A$ is the unique differential A-form of degree $p - 1$ such that

$$\begin{aligned} (i_\theta\omega)^A(\theta_1^A, ..., \theta_{p-1}^A) &= \left[(i_\theta\omega)(\theta_1, ..., \theta_{p-1})\right]^A \\ &= \left[\omega(\theta, \theta_1, ..., \theta_{p-1})\right]^A \end{aligned}$$

for any $\theta_1, \theta_2, ..., \theta_{p-1} \in \mathfrak{X}(M)$. As $i_{\theta^A}(\omega^A)$ is of degree $p-1$ and is such that

$$\begin{aligned} i_{\theta^A}(\omega^A)\left[\theta_1^A, ..., \theta_{p-1}^A\right] &= \omega^A(\theta^A, \theta_1^A, ..., \theta_{p-1}^A) \\ &= \left[\omega(\theta, \theta_1, ..., \theta_{p-1})\right]^A \end{aligned}$$

for any $\theta_1, \theta_2, ..., \theta_{p-1} \in \mathfrak{X}(M)$, we conclude that $(i_\theta\omega)^A = i_{\theta^A}(\omega^A)$.

When (M, Ω) is a symplectic manifold, we recall that a vector field θ on M is locally hamiltonian if the form $i_\theta\Omega$ is closed for the de Rham cohomology and θ is globally hamiltonian if there exists $f \in C^\infty(M)$ such that $i_\theta\Omega = d(f)$, i.e. the form $i_\theta\Omega$ is d-exact.

Thus a vector field X on M^A is locally hamiltonian if the form $i_X\Omega^A$ is d^A-closed and X is globally hamiltonian if there exists $\varphi \in C^\infty(M^A, A)$ such that $i_X\Omega^A = d^A(\varphi)$, i.e. the form $i_X\Omega^A$ is d^A-exact.

Proposition 3.3 *A vector field $\theta : C^\infty(M) \longrightarrow C^\infty(M)$ on a symplectic manifold M is locally hamiltonian, if and only if $\theta^A : C^\infty(M^A, A) \longrightarrow C^\infty(M^A, A)$ is a locally hamiltonian vector field.*

Proof For any $\theta \in \mathfrak{X}(M)$, we have

$$\begin{aligned} d^A(i_{\theta^A}\Omega^A) &= d^A[(i_\theta\Omega)^A] \\ &= [d(i_\theta\Omega)]^A. \end{aligned}$$

Thus θ is locally hamiltonian, i.e $d(i_\theta\Omega) = 0$ if and only if, $d^A(i_{\theta^A}\Omega^A) = 0$ i.e $\theta^A : C^\infty(M^A, A) \longrightarrow C^\infty(M^A, A)$ is a locally hamiltonian vector field.

Theorerm 3.4 *A vector field $X : C^\infty(M^A, A) \longrightarrow C^\infty(M^A, A)$ on M^A locally hamiltonian is a derivation of the A-Lie algebra induced by the A-structure of Poisson defined by the symplectic A-manifold (M^A, Ω^A).*

Proof Let (M^A, Ω^A) be a symplectic manifold. For any $\varphi, \psi \in C^\infty(M^A, A)$,

$$\begin{aligned} \{\varphi, \psi\}_{\Omega^A} &= -\Omega^A(X_\varphi, X_\psi) \\ &= X_\varphi(\psi) \end{aligned}$$

If X is locally hamiltonian vector field, we have $d^A(i_X\Omega^A) = 0$ i.e. for any Y and $Z \in \mathfrak{X}(M^A)$,

$$d^A(i_X\Omega^A)(Y, Z) = 0.$$

In particular, for any $\varphi, \psi \in C^\infty(M^A, A)$, we have

$$\begin{aligned} 0 &= (d^A(i_X\Omega^A))(X_\varphi, X_\psi) \\ &= X_\varphi[i_X\Omega^A(X_\psi)] - X_\psi[i_X\Omega^A(X_\varphi)] - i_X\Omega^A([X_\varphi, X_\psi]) \end{aligned}$$

Therefore

$$i_X\Omega^A([X_\varphi, X_\psi]) = X_\varphi[i_X\Omega^A(X_\psi)] - X_\psi[i_X\Omega^A(X_\varphi)]$$

i.e

$$\Omega^A(X, [X_\varphi, X_\psi]) = X_\varphi[\Omega^A(X, X_\psi)] - X_\psi[\Omega^A(X, X_\varphi)]$$

Hence

$$X(\{\varphi, \psi\}_{\Omega^A}) = \{X(\varphi), \psi\}_{\Omega^A} + \{\varphi, X(\psi)\}_{\Omega^A}.$$

That ends the proof.

Proposition 3.5 *Let (M, Ω) be a symplectic manifold. If a vector field*

$$\theta : C^\infty(M) \longrightarrow C^\infty(M)$$

is globally hamiltonian then the vector field

$$\theta^A : C^\infty(M^A, A) \longrightarrow C^\infty(M^A, A)$$

is globally hamiltonian.

Proof If θ is globally hamiltonian, then there exists $f \in C^{\infty}(M)$ such that $i_\theta\Omega = d(f)$. Then,

$$\begin{aligned}(i_\theta\Omega)^A &= [d(f)]^A \\ &= d^A\left(f^A\right)\end{aligned}$$

Thus

$$i_{\theta^A}\Omega^A = d^A\left(f^A\right).$$

i.e θ^A is globally hamiltonian.

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