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A Counterexample to the Generalized Ho-Zhao Problem

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Abstract

In this paper we find the answer to the open question in (Ho & Zhao, 2009), which states that we do not know whether the isomorphism of complete lattices $C(P)$ and $C(Q)$ implies that of the dcpo's P and Q, where $C(P)$ and $C(Q)$ are the lattices of all Scott closed subsets of P and Q respectively. We proved that is not necessarily satisfied in general case.

Keywords: directed set, directed complete poset (dcpo), Scott closed sets, lattice of Scott closed set

1. Introduction

1.1 Introducing the Problem

This paper depends on the work of (Ho & Zhao, 2009) about the nature of the order relation in the lattice of Scott-closed sets over semi-lattice. They mentioned at end of their paper that we still do not know whether the isomorphism of complete lattices $C(P)$ and $C(Q)$ implies that of the dcpo's P and Q, so further work must be done to achieve a better understanding of the lattices of Scott-closed sets.

1.2 What is The Question?

The remained question is: Can one prove or deny the statement: $C(P) \cong C(Q)$ implays $P \cong Q$ for two arbitrary directed complete partly ordered sets P and Q.

1.3 What We Are Proving in This Paper?

In this paper, we prove that it's not necessarily satisfied in general case, through defining two dcpo Υ and Ψ such that $\Upsilon \ncong \Psi$ and $C(\Upsilon) \cong C(\Psi)$.

2. Method

At first, we give some preliminaries on directed complete partly ordered sets.

2.1 Definition

A nonempty subset D of a poset is said to be directed if any two elements in D have an upper bound in D. See (Kelley, 1975, p 81)

A poset L in which every directed subset D has a supremum (donate by $\bigvee D$) is called a directed-complete partial order, or dcpo for short. See (Abramsky & Jung, 1995, p 14)

2.2 Definition

For any subset A of a poset L, the subset $\uparrow A$ is defined by:

$$\uparrow A = \{b \in L: \exists a \in A, a \leq b\}$$

And the subset $\downarrow A$ defined dually by:

$$\downarrow A = \{b \in L: \exists a \in A, b \leq a\}$$

If $A = \{x\}$ then $\uparrow\{x\} = \uparrow x$ and $\downarrow\{x\} = \downarrow x$.

A subset A of a poset L is said to be upper if $A = \uparrow A$ and said to be lower if $A = \downarrow A$. See (Gierz, et al., 2003)

2.3 Definition

Let L be a dcpo and $U \subseteq L$. Then U said to be Scott-open if and only if the following two conditions are satisfied:

i. U is upper set

ii. $\sup D \in U$ implies $D \cap U \neq \phi$ for all directed sets $D \subseteq L$.

The collection of all Scott-open subsets of L is called the Scott topology of L and will be denoted by $\sigma(L)$.

The complement of a Scott-open set is called Scott-closed, The collection of all Scott-closed subsets of L will be denoted by $C(L)$.

One can prove that a subset $F \subseteq L$ is Scott-closed if and only if the following two conditions are satisfied:

i. F is lower set

ii. For any directed set $D \subseteq F$, If D has a supremum $\bigvee D$ then $\bigvee D \in F$.

Both $\sigma(P)$ and $C(P)$ are complete, distributive lattices with respect to the inclusion relation. See (Gierz, et al., 2003; Gierz, et al., 1980)

3. Results

Now we define two dcpo Υ and Ψ such that $\Upsilon \ncong \Psi$ and $C(\Upsilon) \cong C(\Psi)$.

First, let $\Upsilon = [0,1]$ the real interval ordered by real order relation $\leq$.

The upper subsets in Υ are the closed intervals $[a, 1]$ and the half opened intervals $]a, 1]$, and every subset of Υ is directed because $\leq$ is a total order relation (for every two elements, one must be upper bound of the other).

Since $\leq$ is a total order relation, ever (directed) subset of Υ has an upper bound, so Υ is directed complete poset (dcpo).

Now let us characterize the Scott-open sets of Υ, the first condition is to be upper set.

For every upper set of the form $[a, 1]$ with $0 < a < 1$, we have the directed set $D = [0, a[$ that does not have any intersection with it.

But the supremum of D is $\bigvee D = a \in [a, 1]$, so those upper sets of the form $[a, 1]$ are not Scott-open, because they don't satisfy the second condition.

On other hand, for every upper set of the form $U =]a, 1]$, every subset D satisfies that: D doesn't have any intersection with U, will have a supremum $\bigvee D \leq a$, that means $\bigvee D \notin U$, so U is Scott-open.

As a result the lattice of Scott-open sets of $\Upsilon = [0,1]$ is:

$$\sigma(\Upsilon) = \{\Phi, [0,1]\} \cup \{]a, 1]: 0 < a < 1\}$$

Since the complement of a Scott-open set is Scott-closed, the lattice of Scott-closed sets of $\Upsilon = [0,1]$ is:

$$C(\Upsilon) = \{\Phi, [0,1]\} \cup \{[0, a]: 0 < a < 1\}$$

Second, let $\Psi = \{[0, a]: 0 < a \leq 1\}$ then Ψ is dcpo with inclusion relation $\subseteq$.

We want to prove that Ψ is isomorphic to $\Upsilon\backslash\{0\}$:

Let us define $f: \Upsilon\backslash\{0\} \to \Psi$ by:

$$f(x) = [0, x]: 0 < x \leq 1$$

f Order preserving:

For every two elements $x, y \in \Upsilon\backslash\{0\}$ where $x \leq y$:

$$x \leq y \Longrightarrow [0, x] \subseteq [0, y] \Longrightarrow f(x) \subseteq f(y)$$

f Injective function:

For every two elements $x, y \in \Upsilon\backslash\{0\}$ where $f(x) = f(y)$:

$$f(x) = f(y) \Longrightarrow [0, x] = [0, y] \Longrightarrow x = y$$

f Surjective function:

For every $[0, a] \in \Psi$ where $0 < a \leq 1$, there is $a \in \Upsilon\backslash\{0\}$ satisfies $f(a) = [0, a]$

So f is isomorphism.

Now, $\Upsilon\backslash\{0\} \subset \Upsilon$, $0 \notin \Upsilon\backslash\{0\}$, $0 \in \Upsilon$ this means $\Upsilon\backslash\{0\} \ncong \Upsilon$. So $\Upsilon \ncong \Psi$, because we proved $\Upsilon\backslash\{0\} \cong \Psi$.

Now let us characterize the Scott-open sets of Ψ.

Every subset of Ψ is directed, since $\subseteq$ is a total order relation, because for every two intervals $[0, x]$, $[0, y]$

in Ψ one of the following is satisfied:

$$0 < x < y < 1 \Leftrightarrow [0,x] \subseteq [0,y] \text{ or } 0 < y < x < 1 \Leftrightarrow [0,y] \subseteq [0,x]$$

The upper subsets in Ψ are of the form $U_x = \{[0,a]: x \leq a\}$ or of the form $U'_x = \{[0,a]: x < a\}$ for every $0 < x \leq 1$.

For every upper set of the form $U_x = \{[0,a]: x \leq a\}$ where $0 < x \leq 1$, there is a directed set:

$$D = \{[0,a]: a < x\}$$

which has no intersection with U_x, but the supremum of D is $\bigvee D = [0,x] \in U_x$. So the sets of the form $U_x = \{[0,a]: x \leq a\}$ are not Scott-open.

On the other hand, for the upper sets of the form $U'_x = \{[0,a]: x < a\}$, the supremum of any set D where D has no intersection with U'_x, is $\bigvee D \subseteq [0,x]$, this means $\bigvee D \notin U'_x$.

Therefore, the sets of the form U'_x are Scott-open.

As a result, the lattice of Scott-open sets of Ψ is:

$$\sigma(\Psi) = \{\Phi, \Psi\} \cup \{\{[0,a]: x < a\}: 0 < x < 1\}$$

Since the complement of a Scott-open set is Scott-closed, the lattice of Scott-closed sets of Ψ is:

$$C(\Psi) = \{\Phi, \Psi\} \cup \{\{[0,a]: a \leq x\}: 0 < x < 1\}$$

In the following we will prove that the lattice $C(\Upsilon)$ is isomorphic to the lattice $C(\Psi)$:

Let us define $f: C(\Upsilon) \to C(\Psi)$ by:

$$f([0,x]) = \{[0,a]: a \leq x\}: 0 < x \leq 1$$

f Order preserving:

For every two intervals $[0,x],[0,y] \in C(\Upsilon)$, If $[0,x] \subseteq [0,y]$ then:

$$[0,x] \subseteq [0,y] \Longrightarrow x \leq y \Longrightarrow \{[0,a]: a \leq x\} \subseteq \{[0,a]: a \leq y\} \Longrightarrow f([0,x]) \subseteq f([0,y])$$

f Injective function:

For every two intervals $[0,x],[0,y] \in C(\Upsilon)$, If $f([0,x]) = f([0,y])$ then:

$$\begin{aligned} f([0,x]) = f([0,y]) &\Longrightarrow \{[0,a]: a \leq x\} = \{[0,a]: a \leq y\} \\ &\Longrightarrow \sup\{[0,a]: a \leq x\} = \sup\{[0,a]: a \leq y\} \\ &\Longrightarrow [0,x] = [0,y] \end{aligned}$$

f Surjective function:

For every $\{[0,a]: a \leq x\} \in \Psi$ where $0 < x \leq 1$, there is $[0,x] \in C(\Upsilon)$ satisfies $f([0,x]) = \{[0,a]: a \leq x\}$

Therefore, f is isomorphism.

As a result, for the two dcpo Υ and Ψ defined above, $C(\Psi) \cong C(\Upsilon)$ but $\Psi \ncong \Upsilon$.

4. Conclusions

The counterexample we provided in this paper gives the answer to the open question in (Ho & Zhao, 2009). Thus in general case, we know now that the isomorphism of complete lattices $C(P)$ and $C(Q)$ doesn't imply that of the dcpo's P and Q, where $C(P)$ and $C(Q)$ are the lattices of all Scott closed subsets of P and Q respectively.

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