Journal of Mathematics Research
http://ccsenet.org/jmr
Vol. 2, No. 1
February 2010

Evaluation of Certain Elliptic Type Single, Double Integrals of Ramanujan and Erdélyi

M. I. Qureshi
Department of Applied Sciences and Humanities
Faculty of Engineering and Technology, Jamia Millia Islamia (A Central University)
New Delhi 110025, India
E-mail: miqureshi_delhi@yahoo.co.in

Salahuddin
Department of Applied Sciences and Humanities
Faculty of Engineering and Technology, Jamia Millia Islamia (A Central University)
New Delhi 110025, India
E-mail: sludn@yahoo.com

M. P. Chaudhary (Corresponding author)
Department of Applied Sciences and Humanities
Faculty of Engineering and Technology, Jamia Millia Islamia (A Central University)
New Delhi 110025, India
E-mail: mpchaudhary_2000@yahoo.com

K. A. Quraishi
Department of Applied Sciences and Humanities
Faculty of Engineering and Technology, Jamia Millia Islamia (A Central University)
New Delhi 110025, India
E-mail: kaleemspn@yahoo.co.in

Abstract

This paper is in continuation of earlier work of Denis *et. al.* associated with Ramanujan's Seventh Entry of Chapter XVII of Second Notebook.

Keywords: Pochhammer symbol, Gaussian hypergeometric function, Complete elliptic integrals, Kampé de fériet double hypergeometric function and srivastava's triple hypergeometric function

1. Introduction and preliminaries

The Pochhammer's symbol or Appell's symbol or shifted factorial or rising factorial or generalized factorial function is defined by

$$(b,\ k) = (b)_k = \frac{\Gamma(b+k)}{\Gamma(b)} = \begin{cases} b(b+1)(b+2)\cdots(b+k-1); & \text{if } k = 1,2,3,\cdots \\ 1 & ; \text{ if } k = 0 \\ k! & ; \text{ if } b = 1,\ k = 1,2,3,\cdots \end{cases}$$

where b is neither zero nor negative integer and the notation Γ stands for Gamma function.

1.1 Generalized gaussian hypergeometric function

Generalized ordinary hypergeometric function of one variable is defined by

$$ {}_AF_B\left[\begin{array}{ll} a_1, a_2, \cdots, a_A & ; \\ & z \\ b_1, b_2, \cdots, b_B & ; \end{array}\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_A)_k z^k}{(b_1)_k (b_2)_k \cdots (b_B)_k k!} $$

or

$$ {}_AF_B\left[\begin{array}{ll} (a_A) & ; \\ & z \\ (b_B) & ; \end{array}\right] \equiv {}_AF_B\left[\begin{array}{ll} (a_j)_{j=1}^{A} & ; \\ & z \\ (b_j)_{j=1}^{B} & ; \end{array}\right] = \sum_{k=0}^{\infty} \frac{((a_A))_k z^k}{((b_B))_k k!} \tag{1.1} $$

where denominator parameters $b_1, b_2, \cdots, b_B$ are neither zero nor negative integers and $A,\ B$ are non-negative integers.

1.2 Kampé de fériet's general double hypergeometric function

In 1921, Appell's four double hypergeometric functions F_1, F_2, F_3, F_4 and their confluent forms $\Phi_1, \Phi_2, \Phi_3, \Psi_1, \Psi_2, \Xi_1, \Xi_2$ were unified and generalized by Kampé de Fériet.
We recall the definition of general double hypergeometric function of Kampé de Fériet in slightly modified notation of H.M.Srivastava and R.Panda:

$$ \mathrm{F}^{A:B;D}_{E:G;H}\left[\begin{array}{ll} (a_A) : (b_B) ; (d_D) & ; \\ & x,\ y \\ (e_E) : (g_G) ; (h_H) & ; \end{array}\right] = \sum_{m,n=0}^{\infty} \frac{((a_A))_{m+n}\, ((b_B))_m\, ((d_D))_n\, x^m\, y^n}{((e_E))_{m+n}\, ((g_G))_m\, ((h_H))_n\, m!\, n!} \tag{1.2} $$

where for convergence

(i) $A + B < E + G + 1,\ A + D < E + H + 1\ \ ; |x| < \infty,\ \ |y| < \infty,$ or

(ii) $A + B = E + G + 1,\ A + D = E + H + 1,$ and

$$ \begin{cases} |x|^{\frac{1}{(A-E)}} + |y|^{\frac{1}{(A-E)}} < 1 & ,\text{if } E < A \\ \max\ \{|x|, |y|\} < 1 & ,\text{if } E \geq A \end{cases} $$

1.3 Srivastava's general triple hypergeometric function

In 1967, H. M. Srivastava defined a general triple hypergeometric function $F^{(3)}$ in the following form

$$ F^{(3)}\left[\begin{array}{ll} (a_A) :: (b_B); (d_D); (e_E) : (g_G); (h_H); (l_L); & \\ & x,\ y,\ z \\ (m_M) :: (n_N); (p_P); (q_Q) : (r_R); (s_S); (t_T); & \end{array}\right] $$

$$ = \sum_{i,j,k=0}^{\infty} \frac{((a_A))_{i+j+k}\ ((b_B))_{i+j}\ ((d_D))_{j+k}\ ((e_E))_{k+i}\ ((g_G))_i\ ((h_H))_j\ ((l_L))_k\ x^i\ y^j\ z^k}{((m_M))_{i+j+k}\ ((n_N))_{i+j}\ ((p_P))_{j+k}\ ((q_Q))_{k+i}\ ((r_R))_i\ ((s_S))_j\ ((t_T))_k\ i!\ j!\ k!} \tag{1.3} $$

1.4 Wright's generalized hypergeometric function

$$ {}_p\Psi_q\left[\begin{array}{ll} (\alpha_1, A_1), \cdots, (\alpha_p, A_p) & ; \\ & x \\ (\lambda_1, B_1), \cdots, (\lambda_q, B_q) & ; \end{array}\right] = \sum_{m=0}^{\infty} \frac{\Gamma(\alpha_1 + mA_1)\Gamma(\alpha_2 + mA_2) \cdots \Gamma(\alpha_p + mA_p) x^m}{\Gamma(\lambda_1 + mB_1)\Gamma(\lambda_2 + mB_2) \cdots \Gamma(\lambda_q + mA_q) m!} \tag{1.4} $$

$$ {}_p\Psi^*_q\left[\begin{array}{ll} (\alpha_1, A_1), \cdots, (\alpha_p, A_p) & ; \\ & x \\ (\lambda_1, B_1), \cdots, (\lambda_q, B_q) & ; \end{array}\right] = \sum_{m=0}^{\infty} \frac{(\alpha_1)_{mA_1} (\alpha_2)_{mA_2} \cdots (\alpha_p)_{mA_p} x^m}{(\lambda_1)_{mB_1} (\lambda_2)_{mB_2} \cdots (\lambda_q)_{mB_q} m!} \tag{1.5} $$

2. Some integrals of ramanujan and erdélyi

Entry 7 (ix). If $|x| < 1$, then

$$ \frac{\pi}{2} \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\phi}{\sqrt{(1 + x \sin\phi)}} = \int_0^{\frac{\pi}{2}} \frac{\cos^{-1}(x \sin^2\phi)\mathrm{d}\phi}{\sqrt{(1 - x^2 \sin^4\phi)}} = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\theta \mathrm{d}\phi}{(1 + x \sin\theta \sin^2\phi)} \tag{2.1} $$

Entry 7 (x). If $|x| < 1$, then

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\theta \mathrm{d}\phi}{\sqrt{(1-x\sin^2\theta)(1-x\sin^2\theta\sin^2\phi)}} = \left(\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\phi}{\sqrt{(1-x\sin^4\phi)}}\right)^2 \tag{2.2}$$

Entry 7 (xi). If $|x| < 1$, then

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{x\sin\phi \mathrm{d}\theta \mathrm{d}\phi}{\sqrt{(1-x^2\sin^2\phi)(1-x^2\sin^2\theta\sin^2\phi)}} = \int_0^{\frac{\pi}{2}}\int_0^{\sin^{-1}x} \frac{\mathrm{d}\theta \mathrm{d}\phi}{\sqrt{(1-x^2\sin^2\phi-\sin^2\theta\cos^2\phi)}}$$

$$= \frac{1}{2}\left(\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\phi}{\sqrt{(1-\frac{(1+x)}{2}\sin^2\phi)}}\right)^2 - \frac{1}{2}\left(\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\phi}{\sqrt{(1-\frac{(1-x)}{2}\sin^2\phi)}}\right)^2 \tag{2.3}$$

Erdélyi *et. al.*[p.315(7)]

$$\Pi^*(\phi,\psi,k) = \left(\int_0^{\phi} \frac{k^2\cos\psi\sin\psi\sqrt{(1-k^2\sin^2\psi)}\,\sin^2 t}{(1-k^2\sin^2\psi\sin^2 t)\sqrt{(1-k^2\sin^2 t)}}\,\mathrm{dt}\right) \tag{2.4}$$

Kyrala [p.287(Q.27)]

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{[k^2\cos^2\theta + (1-k^2)\cos^2\phi]\,\mathrm{d}\theta \mathrm{d}\phi}{\sqrt{(1-k^2\sin^2\theta)}\sqrt{(1-(1-k^2)\sin^2\phi)}} \tag{2.5}$$

Above integral was considered to prove Legendre relation $E(k)K'(k) + E'(k)K(k) - K'(k)K(k) = \frac{\pi}{2}$

3. Evaluation of integrals

$$\frac{\pi}{2}\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\phi}{\sqrt{(1+x\sin\phi)}} = \frac{\pi^2}{4}\,{}_2F_1\left[\begin{array}{ccc} \frac{1}{4}, \frac{3}{4} & ; & \\ & & x^2 \\ 1 & ; & \end{array}\right] - \frac{\pi\, x}{4}\,{}_3F_2\left[\begin{array}{ccc} \frac{3}{4}, \frac{5}{4}, 1 & ; & \\ & & x^2 \\ \frac{3}{2}, \frac{3}{2} & ; & \end{array}\right] \tag{3.1}$$

$$\int_0^{\frac{\pi}{2}} \frac{\cos^{-1}(x\sin^2\phi)\mathrm{d}\phi}{\sqrt{(1-x^2\sin^4\phi)}} = \frac{\pi^2}{4}{}_2F_1\left[\begin{array}{ccc} \frac{1}{4}, \frac{3}{4} & ; & \\ & & x^2 \\ 1 & ; & \end{array}\right] - \frac{x\pi}{4}{}_3F_2\left[\begin{array}{ccc} 1, \frac{3}{4}, \frac{5}{4} & ; & \\ & & x^2 \\ \frac{3}{2}, \frac{3}{2} & ; & \end{array}\right] \tag{3.2}$$

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\theta \mathrm{d}\phi}{(1+x\sin\theta\sin^2\phi)} = \frac{\pi^2}{4}{}_2F_1\left[\begin{array}{ccc} \frac{1}{4}, \frac{3}{4} & ; & \\ & & x^2 \\ 1 & ; & \end{array}\right] - \frac{x\pi}{4}{}_3F_2\left[\begin{array}{ccc} 1, \frac{3}{4}, \frac{5}{4} & ; & \\ & & x^2 \\ \frac{3}{2}, \frac{3}{2} & ; & \end{array}\right] \tag{3.3}$$

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\theta \mathrm{d}\phi}{\sqrt{(1-x\sin^2\theta)(1-x\sin^2\theta\sin^2\phi)}} = \frac{\pi^2}{4}\,F^{1:1;2}_{1:0;1}\left[\begin{array}{ccc} \frac{1}{2}:\frac{1}{2}\;;\frac{1}{2}, \frac{1}{2} & ; & \\ & & x,\ x \\ 1:\text{—};\,1 & ; & \end{array}\right] \tag{3.4}$$

$$\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\phi}{\sqrt{(1-x\sin^4\phi)}} = \frac{\pi}{2}{}_2F_1\left[\begin{array}{ccc} \frac{1}{4}, \frac{3}{4} & ; & \\ & & x \\ 1 & ; & \end{array}\right] \tag{3.5}$$

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{x\sin\phi \mathrm{d}\theta \mathrm{d}\phi}{\sqrt{(1-x^2\sin^2\phi)(1-x^2\sin^2\theta\sin^2\phi)}} = \frac{\pi x}{2}\,F^{1:1;2}_{1:0;1}\left[\begin{array}{ccc} 1:\frac{1}{2}\;;\frac{1}{2}, \frac{1}{2} & ; & \\ & & x^2,\ x^2 \\ \frac{3}{2}:\text{—};\,1 & ; & \end{array}\right] \tag{3.6}$$

$$\int_0^{\frac{\pi}{2}}\int_0^{\sin^{-1}x} \frac{\mathrm{d}\theta \mathrm{d}\phi}{\sqrt{(1-x^2\sin^2\phi-\sin^2\theta\cos^2\phi)}} = \frac{\pi\sin^{-1}x}{2}\,F^{1:1;2}_{1:0;1}\left[\begin{array}{ccc} \frac{1}{2}:\frac{1}{2}\;;\frac{1}{2}, \frac{1}{2} & ; & \\ & & x^2,\ 1 \\ 1:\text{—};\,1 & ; & \end{array}\right] -$$

$$- \frac{\pi x\sqrt{(1-x^2)}}{16}\,F^{(3)}\left[\begin{array}{ccc} \frac{3}{2}::\text{—};\frac{3}{2}, \frac{3}{2};\text{—}:\frac{1}{2};\,1;1, 1 & ; & \\ & & x^2,\ 1,\ x^2 \\ 2::\text{—};\,2,\ 2\,;\text{—}:\text{—};\text{—};\frac{3}{2} & ; & \end{array}\right] \tag{3.7}$$

$$\frac{1}{2}\left(\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\phi}{\sqrt{(1-\frac{(1+x)}{2}\sin^2\phi)}}\right)^2 - \frac{1}{2}\left(\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\phi}{\sqrt{(1-\frac{(1-x)}{2}\sin^2\phi)}}\right)^2$$

$$= \frac{\pi^2}{8}\left\{ {}_2F_1\left[\begin{array}{ll} \frac{1}{2},\frac{1}{2} \;; & \\ & \frac{(1+x)}{2} \\ 1 \;; & \end{array}\right]\right\}^2 - \frac{\pi^2}{8}\left\{ {}_2F_1\left[\begin{array}{ll} \frac{1}{2},\frac{1}{2} \;; & \\ & \frac{(1-x)}{2} \\ 1 \;; & \end{array}\right]\right\}^2 \tag{3.8}$$

$$\int_0^{\phi} \frac{k^2\cos\psi\sin\psi\sqrt{(1-k^2\sin^2\psi)}\ \sin^2 t}{(1-k^2\sin^2\psi\sin^2 t)\sqrt{(1-k^2\sin^2 t)}}\,\mathrm{d}t = -\frac{k^2\sin(2\psi)\sin(2\phi)\sqrt{(1-k^2\sin^2\psi)}}{8}\times$$

$$\times F^{(3)}\left[\begin{array}{ll} \frac{3}{2} :: -;-;-: 1\;;\frac{1}{2}\;;1,1; & \\ & k^2\sin^2\psi, k^2,\ k^2\sin^2\psi\sin^2\phi \\ 2 :: -;-\;;-:-\;;-;\frac{3}{2}; & \end{array}\right] -$$

$$-\frac{k^4\sin(2\psi)\sin(2\phi)\sqrt{(1-k^2\sin^2\psi)}}{32}\times$$

$$\times F^{(3)}\left[\begin{array}{ll} \frac{5}{2} ::-;\frac{3}{2}\;;2:1\;;1\;;1; & \\ & k^2\sin^2\psi\sin^2\phi, k^2,\ k^2\sin^2\phi \\ 3 ::-;2\;;\frac{5}{2}:-;-;-; & \end{array}\right] +$$

$$+\frac{\phi\, k^2\sin(2\psi)\sqrt{(1-k^2\sin^2\psi)}}{4} F^{1:1;1}_{1:0;0}\left[\begin{array}{ll} \frac{3}{2} : 1\;;\frac{1}{2}; & \\ & k^2\sin^2\psi,\ k^2 \\ 2 : -;-; & \end{array}\right] \tag{3.9}$$

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{[k^2\cos^2\theta + (1-k^2)\cos^2\phi]\,\mathrm{d}\theta\mathrm{d}\phi}{\sqrt{(1-k^2\sin^2\theta)}\sqrt{(1-(1-k^2)\sin^2\phi)}}$$

$$= \frac{k^2\pi^2}{8}\,{}_2F_1\left[\begin{array}{ll} \frac{1}{2},\frac{1}{2} \;; & \\ & k^2 \\ 2 \;; & \end{array}\right]{}_2F_1\left[\begin{array}{ll} \frac{1}{2},\frac{1}{2} \;; & \\ & (1-k^2) \\ 1 \;; & \end{array}\right] +$$

$$+\frac{(1-k^2)\pi^2}{8}\,{}_2F_1\left[\begin{array}{ll} \frac{1}{2},\frac{1}{2} \;; & \\ & k^2 \\ 1 \;; & \end{array}\right]{}_2F_1\left[\begin{array}{ll} \frac{1}{2},\frac{1}{2} \;; & \\ & (1-k^2) \\ 2 \;; & \end{array}\right] \tag{3.10}$$

4. Derivation

4.1 Evaluation of integrals involved in entry 7(ix)

$$\text{Let}\quad \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\phi}{\sqrt{(1+x\sin\phi)}} = \frac{\pi}{2}\sum_{m=0}^{\infty}\frac{(\frac{1}{2})_m(-x)^m}{m!}\ \int_0^{\frac{\pi}{2}}\sin^m\phi\ \mathrm{d}\phi = \frac{\pi^2}{4}\sum_{m=0}^{\infty}\frac{(\frac{1}{2})_m(\frac{1}{2})_{m/2}(-x)^m}{m!(1)_{m/2}}$$

$$= \frac{\pi^2}{4}\,{}_2\Psi_1^*\left[\begin{array}{ll} (\frac{1}{2},1),(\frac{1}{2},\frac{1}{2}) \;; & \\ & -x \\ (1,\frac{1}{2}) \;; & \end{array}\right] = \frac{\pi}{4}\,{}_2\Psi_1\left[\begin{array}{ll} (\frac{1}{2},1),(\frac{1}{2},\frac{1}{2}) \;; & \\ & -x \\ (1,\frac{1}{2}) \;; & \end{array}\right]$$

$$= \frac{\pi^2}{4}\,{}_2F_1\left[\begin{array}{ll} \frac{1}{4},\frac{3}{4} \;; & \\ & x^2 \\ 1 \;; & \end{array}\right] - \frac{\pi\, x}{4}\,{}_3F_2\left[\begin{array}{ll} \frac{3}{4},\frac{5}{4},1 \;; & \\ & x^2 \\ \frac{3}{2},\frac{3}{2} \;; & \end{array}\right] \tag{4.1}$$

$$\text{Again}\qquad \int_0^{\frac{\pi}{2}} \frac{\cos^{-1}(x\sin^2\phi)\mathrm{d}\phi}{\sqrt{(1-x^2\sin^4\phi)}} = \int_0^{\frac{\pi}{2}} \frac{\left(\frac{\pi}{2} - \ \sin^{-1}(x\sin^2\phi)\right)\mathrm{d}\phi}{\sqrt{(1-x^2\sin^4\phi)}}$$

$$= \int_0^{\frac{\pi}{2}} \frac{(\frac{\pi}{2})\mathrm{d}\phi}{\sqrt{(1-x^2\sin^4\phi)}} - \int_0^{\frac{\pi}{2}} \frac{\sin^{-1}(x\sin^2\phi)\mathrm{d}\phi}{\sqrt{(1-x^2\sin^4\phi)}}$$

$$= \frac{\pi}{2}\int_0^{\frac{\pi}{2}}(1-x^2\sin^4\phi)^{-\frac{1}{2}}\mathrm{d}\phi\ -\ \int_0^{\frac{\pi}{2}}(x\sin^2\phi)\ {}_2F_1\left[\begin{array}{ll} 1,1 \;; & \\ & x^2\sin^4\phi \\ \frac{3}{2} \;; & \end{array}\right]\mathrm{d}\phi$$

$$= \frac{\pi}{2}\int_0^{\frac{\pi}{2}} {}_1F_0\left[\begin{matrix} \frac{1}{2} & ; \\ & & x^2\sin^4\phi \\ - & ; \end{matrix}\right]\mathrm{d}\phi - x\int_0^{\frac{\pi}{2}} \sin^2\phi\ {}_2F_1\left[\begin{matrix} 1,\,1 & ; \\ & & x^2\sin^4\phi \\ \frac{3}{2} & ; \end{matrix}\right]\mathrm{d}\phi$$

$$= \frac{\pi}{2}\sum_{m=0}^{\infty}\frac{(\frac{1}{2})_m x^{2m}}{m!}\int_0^{\frac{\pi}{2}}\sin^{4m}\phi\mathrm{d}\phi \ - \ x\sum_{m=0}^{\infty}\frac{(1)_m(1)_m x^{2m}}{(\frac{3}{2})_m m!}\int_0^{\frac{\pi}{2}}\sin^{4m+2}\phi\mathrm{d}\phi$$

$$= \frac{\pi}{2}\sum_{m=0}^{\infty}\frac{(\frac{1}{2})_m x^{2m}\Gamma(\frac{4m+1}{2})\Gamma(\frac{0+1}{2})}{m!\ 2\ \Gamma(\frac{4m+0+2}{2})} \ - \ x\sum_{m=0}^{\infty}\frac{(1)_m(1)_m x^{2m}\Gamma(\frac{4m+3}{2})\Gamma(\frac{0+1}{2})}{(\frac{3}{2})_m m!\ 2\ \Gamma(\frac{4m+4}{2})}\mathrm{d}\phi$$

$$= \frac{\pi^2}{4}\sum_{m=0}^{\infty}\frac{x^{2m}(\frac{1}{4})_m(\frac{3}{4})_m}{m!m!} \ - \ \frac{x\pi}{4}\sum_{m=0}^{\infty}\frac{(1)_m x^{2m}(\frac{3}{4})_m(\frac{5}{4})_m}{(\frac{3}{2})_m m!(\frac{3}{2})_m}$$

$$= \frac{\pi^2}{4}\ {}_2F_1\left[\begin{matrix} \frac{1}{4},\frac{3}{4} & ; \\ & & x^2 \\ 1 & ; \end{matrix}\right] - \frac{x\pi}{4}\ {}_3F_2\left[\begin{matrix} 1,\,\frac{3}{4},\,\frac{5}{4} & ; \\ & & x^2 \\ \frac{3}{2},\,\frac{3}{2} & ; \end{matrix}\right] \tag{4.2}$$

Also $\displaystyle\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\theta\mathrm{d}\phi}{(1+x\sin\theta\sin^2\phi)} = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}(1+x\sin\theta\sin^2\phi)^{-1}\mathrm{d}\theta\mathrm{d}\phi$

$$= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\sum_{m=0}^{\infty}(-1)^m x^m\sin^m\theta\sin^{2m}\phi\mathrm{d}\theta\mathrm{d}\phi = \sum_{m=0}^{\infty}(-1)^m x^m\Big(\int_0^{\frac{\pi}{2}}\sin^m\theta\mathrm{d}\theta\Big)\Big(\int_0^{\frac{\pi}{2}}\sin^{2m}\phi\mathrm{d}\phi\Big)$$

$$= \sum_{m=0}^{\infty}x^{2m}\Big(\int_0^{\frac{\pi}{2}}\sin^{2m}\theta\mathrm{d}\theta\Big)\Big(\int_0^{\frac{\pi}{2}}\sin^{4m}\phi\mathrm{d}\phi\Big) - \sum_{m=0}^{\infty}x^{2m+1}\Big(\int_0^{\frac{\pi}{2}}\sin^{2m+1}\theta\mathrm{d}\theta\Big)\Big(\int_0^{\frac{\pi}{2}}\sin^{4m+2}\phi\mathrm{d}\phi\Big)$$

$$= \sum_{m=0}^{\infty}x^{2m}\Big(\frac{\Gamma(\frac{2m+1}{2})\Gamma(\frac{0+1}{2})}{2\Gamma(\frac{2m+2}{2})}\Big)\Big(\frac{\Gamma(\frac{4m+1}{2})\Gamma(\frac{0+1}{2})}{2\Gamma(\frac{4m+2}{2})}\Big) - \sum_{m=0}^{\infty}x^{2m+1}\Big(\frac{\Gamma(\frac{2m+2}{2})\Gamma(\frac{0+1}{2})}{2\Gamma(\frac{2m+3}{2})}\Big)\Big(\frac{\Gamma(\frac{4m+3}{2})\Gamma(\frac{0+1}{2})}{2\Gamma(\frac{4m+4}{2})}\Big)$$

$$= \sum_{m=0}^{\infty}x^{2m}\Big(\frac{\Gamma(\frac{1}{2}+m)\sqrt{\pi}}{2\ m!}\Big)\Big(\frac{\Gamma(\frac{1}{2}+2m)\sqrt{\pi}}{4\ m!}\Big) - \sum_{m=0}^{\infty}x^{2m+1}\Big(\frac{(1)_m\sqrt{\pi}}{2\ \Gamma(\frac{3}{2}+m)}\Big)\Big(\frac{\Gamma(\frac{3}{2}+2m)\sqrt{\pi}}{2\ (1)_{2m+1}}\Big)$$

$$= \frac{\pi^2}{4}\sum_{m=0}^{\infty}\frac{(\frac{1}{4})_m(\frac{3}{4})_m(x^2)^m}{m!(1)_m} - \frac{x\pi}{4}\sum_{m=0}^{\infty}\frac{(1)_m(\frac{3}{4})_m(\frac{5}{4})_m(x^2)^m}{m!(\frac{3}{2})_m(\frac{3}{2})_m}$$

$$= \frac{\pi^2}{4}\ {}_2F_1\left[\begin{matrix} \frac{1}{4},\frac{3}{4} & ; \\ & & x^2 \\ 1 & ; \end{matrix}\right] - \frac{\pi\ x}{4}\ {}_3F_2\left[\begin{matrix} 1,\,\frac{3}{4},\,\frac{5}{4} & ; \\ & & x^2 \\ \frac{3}{2},\,\frac{3}{2} & ; \end{matrix}\right] \tag{4.3}$$

4.2 Evaluation of integrals involved in entry 7(x)

$$\text{L.H.S.} = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\theta\mathrm{d}\phi}{\sqrt{(1-x\sin^2\theta)(1-x\sin^2\theta\sin^2\phi)}}$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}(1-x\sin^2\theta)^{-\frac{1}{2}}(1-x\sin^2\theta\sin^2\phi)^{-\frac{1}{2}}\mathrm{d}\theta\mathrm{d}\phi$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\sum_{m=0}^{\infty}\frac{(\frac{1}{2})_m x^{2m}\sin^{2m}\theta}{m!}\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n x^n\sin^2\theta\sin^{2n}\phi}{n!}\mathrm{d}\theta\mathrm{d}\phi$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_m x^m(\frac{1}{2})_n x^n}{m!n!}\Big(\int_0^{\frac{\pi}{2}}\sin^{2m+2n}\theta\mathrm{d}\theta\Big)\Big(\int_0^{\frac{\pi}{2}}\sin^{2n}\phi\mathrm{d}\phi\Big)$$

$$= \pi\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_m x^m(\frac{1}{2})_n x^n}{m!n!}\Big(\frac{\Gamma(\frac{2m+2n+1}{2})\Gamma(\frac{0+1}{2})}{2\Gamma(\frac{2m+2n+0+2}{2})}\Big)\Big(\frac{\Gamma(\frac{2n+1}{2})\Gamma(\frac{0+1}{2})}{2\Gamma(\frac{2n+0+2}{2})}\Big)$$

$$= \frac{\pi}{4}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_m x^m(\frac{1}{2})_n x^n}{m!n!}\Big(\frac{\Gamma(\frac{1}{2}+m+n)}{(1)_{m+n}}\Big)\Big(\frac{\Gamma(\frac{1}{2}+n)}{(1)_n}\Big)$$

$$= \frac{\pi^2}{4}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_m x^m(\frac{1}{2})_n x^n(\frac{1}{2})_{m+n}\frac{1}{2})_n}{m!n!(1)_{m+n}(1)_n} = \frac{\pi^2}{4}\ F^{1:1;2}_{1:0;1}\left[\begin{matrix} \frac{1}{2}:\frac{1}{2}\ ;\frac{1}{2},\frac{1}{2} & ; \\ & & x,\ x \\ 1:-;\ 1 & ; \end{matrix}\right] \tag{4.4}$$

$$\text{R.H.S.} = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{(1-x\sin^4\phi)}} = \int_0^{\frac{\pi}{2}} \left(1-x\sin^4\phi\right)^{-\frac{1}{2}} d\phi = \int_0^{\frac{\pi}{2}} \sum_{m=0}^{\infty} \frac{(\frac{1}{2})_m x^m \sin^{4m}\phi}{m!} d\theta$$

$$= \sum_{m=0}^{\infty} \frac{(\frac{1}{2})_m x^m}{m!} \Big(\frac{\Gamma(\frac{4m+1}{2})\Gamma(\frac{0+1}{2})}{2\Gamma(\frac{4m+2}{2})} \Big) = \frac{\pi}{2} \sum_{m=0}^{\infty} \frac{(\frac{1}{4})_m (\frac{3}{4})_m x^m}{m! m!} = \frac{\pi}{2} {}_2F_1 \left[\begin{array}{ccc} \frac{1}{4}, \frac{3}{4} & ; & \\ & & x \\ 1 & ; & \end{array} \right] \tag{4.5}$$

4.3 Evaluation of integrals involved in entry 7(xi)

Let
$$\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{x\sin\phi \, d\theta d\phi}{\sqrt{(1-x^2\sin^2\phi)(1-x^2\sin^2\theta\sin^2\phi)}}$$

$$= x \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \sin\phi (1-x^2\sin^2\phi)^{-\frac{1}{2}} (1-x^2\sin^2\phi\sin^2\theta)^{-\frac{1}{2}} d\theta d\phi$$

$$= x \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_m (\frac{1}{2})_n \, x^{2m+2n}}{m! \, n!} \Big(\int_0^{\frac{\pi}{2}} \sin^{2m+2n+1}\phi d\phi \Big) \Big(\int_0^{\frac{\pi}{2}} \sin^{2n}\theta d\theta \Big)$$

$$= \frac{\pi x}{2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(1)_{m+n} (\frac{1}{2})_m (\frac{1}{2})_n (\frac{1}{2})_n \, x^{2m+2n}}{(\frac{3}{2})_{m+n} (1)_n m! \, n!} = \frac{\pi x}{2} F^{1:1;2}_{1:0;1} \left[\begin{array}{ccc} 1: \frac{1}{2} \; ; \frac{1}{2}, \frac{1}{2} & ; & \\ & & x^2, \, x^2 \\ \frac{3}{2}: -; 1 & ; & \end{array} \right] \tag{4.6}$$

Again
$$\int_0^{\frac{\pi}{2}} \int_0^{\sin^{-1}x} \frac{d\theta d\phi}{\sqrt{(1-x^2\sin^2\phi - \sin^2\theta\cos^2\phi)}}$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\sin^{-1}x} (1-x^2\sin^2\phi - \sin^2\theta\cos^2\phi)^{-\frac{1}{2}} d\theta d\phi$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\sin^{-1}x} \Big(\sum_{p=0}^{\infty} \frac{(\frac{1}{2})_p (x^2\sin^2\phi + \sin^2\theta\cos^2\phi)^p}{p!} \Big) d\theta d\phi$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\sin^{-1}x} \sum_{m,n=0}^{\infty} \frac{(\frac{1}{2})_{m+n} (x^2\sin^2\phi)^m (\sin^2\theta\cos^2\phi)^n}{m! \, n!} d\theta d\phi$$

$$= \sum_{m,n=0}^{\infty} \frac{(\frac{1}{2})_{m+n} x^{2m}}{m! \, n!} \Big(\int_0^{\frac{\pi}{2}} \sin^{2m}\phi\cos^{2n}\phi d\phi \Big) \Big(\int_0^{\sin^{-1}x} \sin^{2n}\theta d\theta \Big)$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_{m+n} x^{2m}}{m! \, n!} \frac{\Gamma(\frac{2m+1}{2})\Gamma(\frac{2n+1}{2})}{2\Gamma(\frac{2m+2n+2}{2})} \Big[-\frac{(\frac{1}{2})_n x \sqrt{(1-x^2)}}{n!} \sum_{r=0}^{n-1} \frac{(1)_r \, x^{2r}}{(\frac{3}{2})_r} + \frac{(\frac{1}{2})_n \sin^{-1}x}{(1)_n} \Big]$$

$$= -\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_{m+n} x^{2m}}{m! \, n!} \frac{\Gamma(\frac{2m+1}{2})\Gamma(\frac{2n+1}{2})}{2\Gamma(\frac{2m+2n+2}{2})} \frac{(\frac{1}{2})_n x \sqrt{(1-x^2)}}{n!} \sum_{r=0}^{n-1} \frac{(1)_r \, x^{2r}}{(\frac{3}{2})_r} +$$
$$+ \frac{\pi\sin^{-1}x}{2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_{m+n} (\frac{1}{2})_m (\frac{1}{2})_n (\frac{1}{2})_n \, x^{2m}}{m! \, n! (1)_n (1)_{m+n}}$$

$$= -\frac{\pi \, x\sqrt{(1-x^2)}}{2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{r=0}^{n-1} \frac{(\frac{1}{2})_{m+n} (\frac{1}{2})_m (\frac{1}{2})_n (\frac{1}{2})_n (1)_r \, x^{2m+2r}}{m! \, n! (1)_n (1)_{m+n} (\frac{3}{2})_r} +$$
$$+ \frac{\pi\sin^{-1}x}{2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_{m+n} (\frac{1}{2})_m (\frac{1}{2})_n (\frac{1}{2})_n \, x^{2m}}{m! \, n! (1)_n (1)_{m+n}}$$

$$= -\frac{\pi \, x\sqrt{(1-x^2)}}{2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{(\frac{1}{2})_{m+n+r+1} (\frac{1}{2})_m (\frac{1}{2})_{n+r+1} (\frac{1}{2})_{n+r+1} (1)_r \, x^{2m} x^{2r}}{m! (1)_{n+r+1} (1)_{m+n+r+1} (1)_{n+r+1} (\frac{3}{2})_r} +$$
$$+ \frac{\pi\sin^{-1}x}{2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_{m+n} (\frac{1}{2})_m (\frac{1}{2})_n (\frac{1}{2})_n \, x^{2m}}{m! \, n! (1)_n (1)_{m+n}}$$

$$= -\frac{\pi\, x\sqrt{(1-x^2)}}{16}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\frac{(\frac{3}{2})_{m+n+r}(\frac{3}{2})_{n+r}(\frac{3}{2})_{n+r}(\frac{1}{2})_m(1)_r(1)_r(1)_n\; x^{2m}x^{2r}}{(2)_{n+r}(2)_{m+n+r}(2)_{n+r}(\frac{3}{2})_r m!n!r!}+$$

$$+\frac{\pi\sin^{-1}x}{2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_{m+n}(\frac{1}{2})_m(\frac{1}{2})_n(\frac{1}{2})_n\; x^{2m}}{m!\; n!(1)_n(1)_{m+n}}$$

$$= -\frac{\pi x\sqrt{(1-x^2)}}{16}\,F^{(3)}\left[\begin{array}{ll}\frac{3}{2}::-;\frac{3}{2},\frac{3}{2};-:\frac{1}{2};1;1,1 & ;\\ & \quad x^2,\;1,\;x^2\\ 2::-;2,\;2\;;-:-;-;\frac{3}{2} & ;\end{array}\right]+$$

$$+\frac{\pi\sin^{-1}x}{2}\,F^{1:1;2}_{1:0;1}\left[\begin{array}{ll}\frac{1}{2}:\frac{1}{2}\;;\frac{1}{2},\frac{1}{2} & ;\\ & \quad x^2,\;1\\ 1:-;\,1 & ;\end{array}\right] \tag{4.7}$$

Similarly we can prove

$$\frac{1}{2}\Bigg(\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{(1-\frac{(1+x)}{2}\sin^2\phi)}}\Bigg)^2-\frac{1}{2}\Bigg(\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{(1-\frac{(1-x)}{2}\sin^2\phi)}}\Bigg)^2$$

$$=\frac{\pi^2}{8}\left\{{}_2F_1\left[\begin{array}{ll}\frac{1}{2},\frac{1}{2} & ;\\ & \quad \frac{(1+x)}{2}\\ 1 & ;\end{array}\right]\right\}^2-\frac{\pi^2}{8}\left\{{}_2F_1\left[\begin{array}{ll}\frac{1}{2},\frac{1}{2} & ;\\ & \quad \frac{(1-x)}{2}\\ 1 & ;\end{array}\right]\right\}^2 \tag{4.8}$$

4.4 Evaluation of integral given by Erdélyi et. al.

$$\Pi^*(\phi,\psi,k)=\Bigg(\int_0^{\phi}\frac{k^2\cos\psi\sin\psi\sqrt{(1-k^2\sin^2\psi)}\;\sin^2 t}{(1-k^2\sin^2\psi\sin^2 t)\sqrt{(1-k^2\sin^2 t)}}\,dt\Bigg)=k^2\sin\psi\cos\psi\sqrt{(1-k^2\sin^2\psi)}\times$$

$$\times\int_0^{\phi}{}_1F_0\left[\begin{array}{ll}1 & ;\\ & \quad k^2\sin^2\psi\sin^2 t\\ - & ;\end{array}\right]{}_1F_0\left[\begin{array}{ll}\frac{1}{2} & ;\\ & \quad k^2\sin^2 t\\ - & ;\end{array}\right]\sin^2 t\,dt$$

$$=k^2\sin\psi\cos\psi\sqrt{(1-k^2\sin^2\psi)}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(1)_m(k^2\sin^2\psi)^m(\frac{1}{2})_n(k^2)^n}{m!n!}\int_0^{\phi}\sin^{2m+2n+2}t\,dt$$

$$=k^2\sin\psi\cos\psi\sqrt{(1-k^2\sin^2\psi)}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(1)_m(k^2\sin^2\psi)^m(\frac{1}{2})_n(k^2)^n}{m!n!}\times$$

$$\times\Bigg[-\Big(\frac{(\frac{1}{2})_{m+n+1}\sin\phi\cos\phi}{(1)_{m+n+1}}\sum_{r=0}^{m+n}\frac{(1)_r\sin^{2r}\phi}{(\frac{3}{2})_r}\Big)+\Big(\frac{\phi(\frac{1}{2})_{m+n+1}}{(1)_{m+n+1}}\Big)\Bigg]$$

$$=\;-\frac{k^2\sin(2\psi)\sin(2\phi)\sqrt{1-k^2\sin^2\psi}}{8}\times$$

$$\times\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{m+n}\frac{(1)_m(k^2\sin^2\psi)^m(\frac{1}{2})_n(k^2)^n(\frac{3}{2})_{m+n}(1)_r\sin^{2r}\phi}{(1)_m(1)_n(2)_{m+n}(\frac{3}{2})_r}+$$

$$+\frac{\phi\,k^2\sin(2\psi)\sqrt{1-k^2\sin^2\psi}}{4}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(1)_m(k^2\sin^2\psi)^m(\frac{1}{2})_n(k^2)^n(\frac{3}{2})_{m+n}}{(1)_m(1)_n(2)_{m+n}}$$

$$=\;-\frac{k^2\sin(2\psi)\sin(2\phi)\sqrt{1-k^2\sin^2\psi}}{8}\times$$

$$\times\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\frac{(k^2\sin^2\psi)^{m+r}(\frac{1}{2})_n(k^2)^n(\frac{3}{2})_{m+n+r}(1)_r\sin^{2r}\phi(1)_m(1)_r}{(2)_{m+n+r}(\frac{3}{2})_r\;m!n!r!}-$$

$$-\frac{k^2\sin(2\psi)\sin(2\phi)\sqrt{1-k^2\sin^2\psi}}{8}\times$$

$$\times\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\frac{(1)_m(k^2\sin^2\psi)^m(\frac{1}{2})_{n+r+1}(k^2)^{n+r+1}(\frac{3}{2})_{m+n+r+1}(1)_{r+m+1}\sin^{2(m+r+1)\phi}}{(1)_m(1)_{n+r+1}(2)_{m+n+r+1}(\frac{3}{2})_{m+r+1}}+$$

$$+\frac{\phi\, k^2\sin(2\psi)\sqrt{1-k^2\sin^2\psi}}{4}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(1)_m(k^2\sin^2\psi)^m(\frac{1}{2})_n(k^2)^n(\frac{3}{2})_{m+n}}{(2)_{m+n}m!n!}$$

$$= -\frac{k^2\sin(2\psi)\sin(2\phi)\sqrt{(1-k^2\sin^2\psi)}}{8}\times$$

$$\times F^{(3)}\left[\begin{array}{ll}\frac{3}{2}::-;-;-:1;\frac{1}{2};1,1; & \\ & k^2\sin^2\psi, k^2,\ k^2\sin^2\psi\sin^2\phi\\ 2::-;-;-:-;-;\frac{3}{2}; & \end{array}\right]-$$

$$-\frac{k^4\sin(2\psi)\sin(2\phi)\sqrt{(1-k^2\sin^2\psi)}}{32}\times$$

$$\times F^{(3)}\left[\begin{array}{ll}\frac{5}{2}::-;\frac{3}{2};2:1;1;1; & \\ & k^2\sin^2\psi\sin^2\phi, k^2,\ k^2\sin^2\phi\\ 3::-;2;\frac{5}{2}:-;-;-; & \end{array}\right]+$$

$$+\frac{\phi\, k^2\sin(2\psi)\sqrt{(1-k^2\sin^2\psi)}}{4}F^{1:1;1}_{1:0;0}\left[\begin{array}{ll}\frac{3}{2}:1\ ;\frac{1}{2}; & \\ & k^2\sin^2\psi,\ k^2\\ 2:-;-; & \end{array}\right] \tag{4.9}$$

4.5 Evaluation of (3.10)

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\frac{[k^2\cos^2\theta+(1-k^2)\cos^2\phi]\,d\theta d\phi}{\sqrt{(1-k^2\sin^2\theta)}\sqrt{(1-(1-k^2)\sin^2\phi)}}$$

$$=\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}(1-k^2\sin^2\theta)^{-\frac{1}{2}}(1-(1-k^2)\sin^2\phi)^{-\frac{1}{2}}(k^2\cos^2\theta+(1-k^2)\cos^2\phi)\,d\theta d\phi$$

$$= k^2\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_m(\frac{1}{2})_n\,k^{2m}(1-k^2)^n}{m!n!}\int_{\theta=0}^{\frac{\pi}{2}}\sin^{2m}\theta\cos^2\theta\,d\theta\int_{\phi=0}^{\frac{\pi}{2}}\sin^{2n}\phi\,d\phi+$$

$$+(1-k^2)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_m(\frac{1}{2})_n\,k^{2m}(1-k^2)^n}{m!n!}\int_{\theta=0}^{\frac{\pi}{2}}\sin^{2m}\theta\,d\theta\int_{\phi=0}^{\frac{\pi}{2}}\sin^{2n}\phi\cos^2\phi\,d\phi$$

$$=\frac{k^2\pi^2}{8}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_m(\frac{1}{2})_m(\frac{1}{2})_n(\frac{1}{2})_n\,k^{2m}(1-k^2)^n}{(2)_m(1)_n\,m!n!}+$$

$$+\frac{(1-k^2)\pi^2}{8}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_m(\frac{1}{2})_m(\frac{1}{2})_n(\frac{1}{2})_n\,k^{2m}(1-k^2)^n}{(2)_n(1)_m\,m!n!}$$

$$=\frac{k^2\pi^2}{8}\sum_{m=0}^{\infty}\frac{(\frac{1}{2})_m(\frac{1}{2})_m\,k^{2m}}{(2)_m m!}\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n(\frac{1}{2})_n(1-k^2)^n}{(1)_n n!}+$$

$$+\frac{(1-k^2)\pi^2}{8}\sum_{m=0}^{\infty}\frac{(\frac{1}{2})_m(\frac{1}{2})_m\,k^{2m}}{(1)_m\,m!}\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n(\frac{1}{2})_n\,(1-k^2)^n}{(2)_n\,n!}$$

$$=\frac{k^2\pi^2}{8}{}_2F_1\left[\begin{array}{ll}\frac{1}{2},\frac{1}{2}\ ; & \\ & k^2\\ 2\ ; & \end{array}\right]{}_2F_1\left[\begin{array}{ll}\frac{1}{2},\frac{1}{2}\ ; & \\ & (1-k^2)\\ 1\ ; & \end{array}\right]+$$

$$+\frac{(1-k^2)\pi^2}{8}{}_2F_1\left[\begin{array}{ll}\frac{1}{2},\frac{1}{2}\ ; & \\ & k^2\\ 1\ ; & \end{array}\right]{}_2F_1\left[\begin{array}{ll}\frac{1}{2},\frac{1}{2}\ ; & \\ & (1-k^2)\\ 2\ ; & \end{array}\right] \tag{4.10}$$

References

Agarwal, R. P. (1996). *Resonance of Ramanujan's Mathematics,* Vol. I, New Age International(P) Ltd., New Delhi.

Agarwal, R. P. (1996). *Resonance of Ramanujan's Mathematics,* Vol. II, New Age International(P) Ltd., New Delhi.

Agarwal, R. P. (1999). *Resonance of Ramanujan's Mathematics,* Vol. III, New Age International(P) Ltd., New Delhi.

Andrews, G. E. and Berndt, B. C. (2005). *Ramanujan's Lost Notebook*, Part I, Springer-Verlag, New York.

Berndt, B. C. (1985). *Ramanujan's Notebooks,* Part I, Springer-Verlag, New York.

Berndt, B. C. (1989). *Ramanujan's Notebooks,* Part II, Springer-Verlag, New York.

Berndt, B. C. (1991). *Ramanujan's Notebooks,* Part III, Springer-Verlag, New York.

Berndt, B. C. (1994). *Ramanujan's Notebooks,* Part IV, Springer-Verlag, New York.

Berndt, B. C. (1998). *Ramanujan's Notebooks,* Part V, Springer-Verlag, New York.

Denis, R. Y., Singh, S. N. and Singh, S. P. (2006). On Certain Elliptic Integrals of Ramanujan. *J. Indian Math. Soc.*, 37(3-4), 113-119.

Erdélyi, A., Magnus, W., Oberhettinger, F. and Tricomi, F. G. (1953). *Higher Transcendental Functions,* Vol. II(Bateman Manuscript Project), McGraw-Hill, Book Co. Inc., New York, Toronto and London.

Jahnke, E. and Emde, F. (1945). *Tables of Functions with Formulas and Curves*. Fourth Edition, Dover Publications, Inc., New York.

Jahnke, E., Emde, F. and Lösch, F. (1960). *Tables of Higher Functions*. Sixth Edition, McGraw-Hill, New York.

Kyrala, A. (1972). *Applied functions of a Complex Variable*. Wiley-Interscience, A division of John Wiley and Sons Inc., New York.

Pearson, K. (1934). *Tables of the Complete and Incomplete Elliptic Integrals*. Cambridge University Press, Cambridge, England.

Peirce, B. O. (1929). *A Short Table of Integrals*. Third Edition, Ginn, Boston.

Srivastava, H. M. and Karlsson, Per. W. (1985). *Multiple Gaussian Hypergeometric Series*. Halsted Press (Ellis Horwood Ltd., Chichester, Brisbane, U. K.) John Wiley and Sons, New York, Chichester, Brisbane and Toronto.

Srivastava, H. M. and Manocha, H. L. (1984). *A Treatise on Generating Functions*. Halsted Press (Ellis Horwood Ltd., Chichester, Brisbane, U. K.) John Wiley and Sons, New York, Chichester, Brisbane and Toronto.