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The g-analytic Function Theory and Wave Equation

Chein-Shan Liu

Department of Civil Engineering, National Taiwan University, Taipei 106-17, Taiwan. E-mail: liucs@ntu.edu.tw

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Abstract

In this paper we develop a g-analytic function and a g-harmonic function theory for one-dimensional wave equation in the Minkowski space. In terms of the Minkowskian polar coordinates we can derive a set of complete hyperbolic type Trefftz bases, which can be transformed to polynomials as the bases for a trial solution of wave equation. The Cauchy-Riemann equations and the Cauchy theorem for g-analytic functions are proved, and meanwhile the existence of Cauchy integral formula is disproved from the non-uniqueness of the Dirichlet problem for wave equation under the boundary conditions on whole boundary, which is also known as the backward wave problem (BWP). Examples are used to demonstrate these results.

Keywords: wave equation, Minkowskian polar coordinates, Minkowski space, g-analytic function, g-harmonic function, backward wave problem

1. Introduction

The one-dimensional wave equation is given by

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad (x, t) \in \Omega, \tag{1}$$

where c is the speed of wave propagation, and the domain Ω may be bounded or unbounded. The wave propagation problems have attracted a lot of attentions since early last century and have been studied theoretically, computationally and experimentally due to its vital role in physical and engineering applications (Bleistein, 1984).

It is known that Equation (1) has a general solution:

$$u(x, t) = f(x + ct) + g(x - ct), \tag{2}$$

where f and g are twice differentiable functions. It is easily verified that the necessary and sufficient conditions for u to be the form (2) it must satisfy the diamond rule:

$$u(A) + u(C) = u(B) + u(D) \tag{3}$$

for any characteristic rectangle $ABCD$ with A at the top corner, B at the left corner, C at the bottom corner, and D at the right corner. However, for the Dirichlet problem of wave equation, for example,

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \ 0 < x < \pi, \ 0 < t < \alpha\pi, \\
&u(0, t) = u(\pi, t) = 0, \ \ 0 \le t \le \alpha\pi, \\
&u(x, 0) = \phi(x), \ u(x, \alpha\pi) = \varphi(x), \ \ 0 \le x \le \pi,
\end{aligned}$$

it is a classical ill-posed problem, which has been studied by Bourgin & Duffin (1939); John (1941); Fox & Pucci (1958); Dunninger & Zachmanoglou (1967); Abdul-Latif & Diaz (1971); Papi Frosali (1979); Levine & Vessella (1985); Vakhania (1994); Kabanikhin & Bektemesov (2012). They asserted that when α is a rational number the solution is not unique.

Let

$$u_1 = \frac{\partial u}{\partial t}, \ \ u_2 = \frac{\partial u}{\partial x};$$

we also come to a coupled hyperbolic system of first-order partial differential equations:

$$\frac{\partial u_1}{\partial t} = \frac{\partial u_2}{\partial x}, \ \ \frac{\partial u_1}{\partial x} = \frac{\partial u_2}{\partial t}, \tag{4}$$

of which Sobolev (1956) has studied the ill-posed property under imposed boundary conditions.

For wave equation (1), we will verify that u has a counterpart v, and they satisfy the newly defined Cauchy-Riemann equations as the pair in Equation (4). The Dirichlet problem of wave equation will be discussed from a different viewpoint in this paper.

The remaining portion of this paper is arranged as follows. For the purpose of comparison we introduce the Laplace equation and the analytic function theory of complex function in Section 2, where we propose some new problems for wave equation. In Section 3 we introduce a new g-number, corresponding to the complex number, in the Minkowksi space as a frame to study the wave equation. The Minkowksian polar coordinates of wave equation are developed, and the new Trefftz bases are derived. In Section 4 we develop a new g-analytic function theory for the g-function, and the Cauchy-Riemann equations are introduced for wave equation. The Cauchy theorem is derived in Section 5 for the g-analytic function. The existence of Cauchy integral formula is disproved by using the result from the Dirichlet problem of wave equation. Finally, some conclusions and the analogies between the Laplace equation and the wave equation are addressed in Section 7.

2. The Laplace Equation and Analytic Function

For the two-dimensional Laplace equation:

$$\Delta u(x,y) = \frac{\partial u^2}{\partial x^2} + \frac{\partial u^2}{\partial y^2} = 0 \quad \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0\right), \tag{5}$$

it is well-known that

$$\{1, r\cos\theta, r\sin\theta, \ldots, r^k\cos(k\theta), r^k\sin(k\theta), \ldots\} \tag{6}$$

forms a set of complete Trefftz bases (Liu, 2007a;, Liu, 2007b; Liu, 2008). Here,

$$r := \sqrt{x^2+y^2}, \ \ \theta := \arctan\frac{y}{x} \tag{7}$$

are polar coordinates in the Euclidean space $\mathbb{R}^2$.

Let $z = x + iy$ be a complex number, and $f(z) = u(x,y) + iv(x,y)$ be an analytic function in a complex domain C. It is well known that the Cauchy-Riemann equations hold (Marsden, 1973):

$$\frac{\partial u(x,y)}{\partial x} = \frac{\partial v(x,y)}{\partial y}, \ \ \frac{\partial u(x,y)}{\partial y} = -\frac{\partial v(x,y)}{\partial x}. \tag{8}$$

They imply both $u(x,y)$ and $v(x,y)$ satisfying the Laplace equation (5). When u is usually called the harmonic function in complex theory, v is called the conjugate harmonic function.

In the present paper we propose the problems: Are that there exist an analogous basis to Equation (6) and the Cauchy-Riemann pair (8) for wave equation (1)? In order to reply these problems we need to recast wave equation (1) into the one in the Minkowskian space-time domain.

3. The Minkowski Space $\mathbb{M}^{1,1}$

3.1 Mathematical Preliminaries of g-Integral

Let us mention the Minkowski space $\mathbb{M}^{1,1}$ and the rotation in that space. Liu (2000) has introduced the g number $w = x + gy$, where 1 and g are bases of a Jordan algebraic system which obey the following binary product rule:

$$\begin{array}{c|cc} \cdot & 1 & g \\ \hline 1 & 1 & g \\ g & g & 1 \end{array} \tag{9}$$

Using $g^2 := g \cdot g = 1$ in the Taylor series expansion of $e^{g\theta}$,

$$e^{g\theta} = 1 + g\theta + \frac{1}{2}g^2\theta^2 + \cdots,$$

it is easy to deduce

$$e^{g\theta} = \cosh\theta + g\sinh\theta, \tag{10}$$

which corresponds to the famous formula:

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

where $i^2 = -1$ and $\theta \in \mathbb{R}$.

For the complex number $z = x+iy$ we can view it as a point in the Euclidean space $\mathbb{R}^2$, and $z' = x'+iy' = e^{i\theta}(x+iy)$ can be viewed as a rotation in $\mathbb{R}^2$ by using

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \tag{11}$$

In contrast, for the g number $w = x+gy$ we can view it as a point in the Minkowski space $\mathbb{M}^{1,1}$, and $w' = x'+gy' = e^{g\theta}(x+gy)$ can be deemed as a rotation in $\mathbb{M}^{1,1}$:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \tag{12}$$

As pointed out by Liu (2002) the g-number bears certain similarity with the complex number, and it forms a Jordan algebra, which is a special case of the double numbers introduced by Yaglom (1968). The applications of the Jordan algebra can refer (Iordanescu, 2007; Iordanescu, 2009).

In the complex theory, for $z, z_1, z_2 \in \mathbb{C}$, we have

$$\bar{z}z = x^2 + y^2, \tag{13}$$
$$z_1 z_2 = r_1 r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)], \tag{14}$$
$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)], \tag{15}$$
$$z^p = r^p[\cos(p\theta) + i\sin(p\theta)], \tag{16}$$

where $\bar{z} = x - iy$ is the conjugate of $z = x + iy$ and p is an integer.

Like the complex number, $\overline{w} = x - gy$ is the conjugate g-number of $w = x + gy$. Similarly, for $w, w_1, w_2 \in \mathbb{M}^{1,1}$ we can prove that

$$\overline{w}w = x^2 - y^2, \tag{17}$$
$$w_1 w_2 = r_1 r_2[\cosh(\theta_1+\theta_2) + g\sinh(\theta_1+\theta_2)], \tag{18}$$
$$\frac{w_1}{w_2} = \frac{r_1}{r_2}[\cosh(\theta_1-\theta_2) + g\sinh(\theta_1-\theta_2)], \tag{19}$$
$$w^p = r^p[\cosh(p\theta) + g\sinh(p\theta)], \tag{20}$$

by means of

$$\sinh(\theta_1 \pm \theta_2) = \sinh\theta_1\cosh\theta_2 \pm \cosh\theta_1\sinh\theta_2,$$
$$\cosh(\theta_1 \pm \theta_2) = \cosh\theta_1\cosh\theta_2 \pm \sinh\theta_1\sinh\theta_2.$$

Quite different from the Euclidean length $\bar{z}z = x^2+y^2 \geq 0$ for a point (x, y) in the Euclidean plane, the Minkowskian length $\overline{w}w = x^2-y^2$ may be positive, zero or negative. The formulae (17)-(20) may bear certain similarities with the formulae (13)-(16). Besides that we will explore the similarity between the Laplace equation and wave equation (1).

As the definition for the complex function, a function with $w = x + gy$ as an independent variable is called the g-function. For the later use we may need to execute the integral of $f(w)$ in the plane (x, y), which is called the g-integral. We use the following examples to demonstrate the integrals of g-function.

Example 1. Calculate the following complex integral along a unit circle:

$$\oint \bar{z}dz = \int_0^{2\pi} e^{-i\theta}e^{i\theta}id\theta = 2\pi i.$$

It can be seen that θ plays both the roles as the integral variable and also the variable to represent the complex number in the polar coordinates. In the Euclidean space they are the same variable θ.

However, for the g-integral along a unit circle we cannot use the Minkowskian polar coordinates, because θ is a geometric variable in the Euclidean space to present the angle, not the Minkowskian polar angle variable as that used in the g-number. By using $x = \cos\theta$ and $y = \sin\theta$, which must be viewed as integral parameters for the g-integral, we can do

$$\begin{aligned}&\oint \overline{w}dw = \oint (x - gy)(dx + gdy) = \oint xdx - ydy + g(xdy - ydx)\\&= \int_0^{2\pi} -\sin 2\theta d\theta + g(\cos^2\theta + \sin^2\theta)d\theta = 2\pi g.\end{aligned}$$

The above two integrals are similar. However, the following two integrals are totally different.

Example 2. Along a unit circle we have

$$\oint \frac{dz}{z} = \oint \frac{\overline{z}dz}{\overline{z}z} = \oint \overline{z}dz = 2\pi i, \tag{21}$$

due to $\overline{z}z = 1$. For the corresponding g-integral of the same function we have

$$\begin{aligned}&\oint \frac{dw}{w} = \oint \frac{dx + gdy}{x + gy} = \int_0^{2\pi} \frac{(\cos\theta - g\sin\theta)(g\cos\theta - \sin\theta)}{(\cos\theta - g\sin\theta)(\cos\theta + g\sin\theta)} d\theta\\&= -\int_0^{2\pi} \tan(2\theta)d\theta + g\int_0^{2\pi} \frac{1}{\cos(2\theta)}d\theta\\&= -\frac{1}{2}\ln\cos(2\theta)|_0^{2\pi} + \frac{1}{2}\ln\left(\frac{1}{\cos(2\theta)} + \tan(2\theta)\right)\Bigg|_0^{2\pi} = 0.\end{aligned} \tag{22}$$

This example shows that the integrals of complex function and g-function have quite different integral behavior. In Section 5 we will demonstrate that the different behavior comes from the theory of singular point and analytic function.

3.2 The Wave Equation in the Minkowski Space $\mathbb{M}^{1,1}$

Through a suitable transformation of the t coordinate in Equation (1) it is always that $(ct)^2 > x^2$ for all $(x, t) \in \Omega$ with $|x|$ bounded. For wave equation we prefer to employ the future cone in the time-like space as the problem domain, because time is always towards future with $(ct)^2 > x^2$.

Let

$$y := ct, \tag{23}$$

$$(r, \theta) := \left(\sqrt{y^2 - x^2}, \ln\sqrt{\frac{y + x}{y - x}}\right), \tag{24}$$

$$x := r\sinh\theta, \quad y := r\cosh\theta. \tag{25}$$

We must emphasize that $y^2 - x^2 > 0$ in the domain Ω, which meaning that (x, y) is a time-like vector in $\mathbb{M}^{1,1}$, and then the definitions (24) and (25) make sense. So a time-like point can be expressed as

$$w = y + gx = re^{g\theta}. \tag{26}$$

The pair (r, θ) may be named the Minkowskian polar coordinates, which are totally different from that in Equation (7).

According to Equation (23), the wave equation (1) can be written as

$$\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial x^2} = 0. \tag{27}$$

By using Equations (24) and (25) we have

$$\frac{\partial r}{\partial x} = -\sinh\theta, \quad \frac{\partial r}{\partial y} = \cosh\theta, \tag{28}$$

$$\frac{\partial \theta}{\partial x} = \frac{\cosh\theta}{r}, \quad \frac{\partial \theta}{\partial y} = -\frac{\sinh\theta}{r}. \tag{29}$$

Then through some operations we can derive

$$\frac{\partial^2 u}{\partial x^2} = \sinh^2\theta \frac{\partial^2 u}{\partial r^2} + \frac{\cosh^2\theta}{r^2}\frac{\partial^2 u}{\partial \theta^2} - \frac{\cosh\theta\sinh\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta} - \frac{\cosh^2\theta}{r}\frac{\partial u}{\partial r} + \frac{\cosh\theta\sinh\theta}{r^2}\frac{\partial u}{\partial\theta}, \tag{30}$$

$$\frac{\partial^2 u}{\partial y^2} = \cosh^2\theta \frac{\partial^2 u}{\partial r^2} + \frac{\sinh^2\theta}{r^2}\frac{\partial^2 u}{\partial \theta^2} - \frac{\cosh\theta\sinh\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta} - \frac{\sinh^2\theta}{r}\frac{\partial u}{\partial r} + \frac{\cosh\theta\sinh\theta}{r^2}\frac{\partial u}{\partial\theta}. \tag{31}$$

Inserting them into Equation (27) and using $\cosh^2\theta - \sinh^2\theta = 1$ we arrive to

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} = 0. \tag{32}$$

It is interesting to note that the above equation bears certain similarity with the second equation in Equation (5). However, the third terms in these two equations are different with a minus sign.

3.3 The Trefftz Bases

Similar to that $r^k\cos(k\theta)$ and $r^k\sin(k\theta)$ satisfy Equation (5), we can prove that $r^k\cosh(k\theta)$ and $r^k\sinh(k\theta)$ identically satisfy Equation (32). We have

$$\frac{\partial}{\partial r} r^k \cosh(k\theta) = kr^{k-1}\cosh(k\theta),$$
$$\frac{\partial^2}{\partial r^2} r^k \cosh(k\theta) = k(k-1)r^{k-2}\cosh(k\theta),$$
$$\frac{\partial^2}{\partial \theta^2} r^k \cosh(k\theta) = k^2 r^k\cosh(k\theta).$$

Inserting them into Equation (32), ends the proof that $r^k\cosh(k\theta)$ is a solution of Equation (32). Similarly, we have

$$\frac{\partial}{\partial r} r^k \sinh(k\theta) = kr^{k-1}\sinh(k\theta),$$
$$\frac{\partial^2}{\partial r^2} r^k \cosh(k\theta) = k(k-1)r^{k-2}\sinh(k\theta),$$
$$\frac{\partial^2}{\partial \theta^2} r^k \cosh(k\theta) = k^2 r^k\sinh(k\theta).$$

Inserting them into Equation (32), we prove that $r^k\sinh(k\theta)$ is a solution of Equation (32).

So we have the following set of hyperbolic-type bases for wave equation (32):

$$\{1, r\cosh\theta, r\sinh\theta, \ldots, r^k\cosh(k\theta), r^k\sinh(k\theta), \ldots\}, \tag{33}$$

and we may call the method that employs the above bases to expand the trial solution of u the *Trefftz method for wave equation*.

4. The g-Analytic Function and Cauchy-Riemann Equations

The main results are given in the following theorems. For simple notation we let $\mathbb{M}^+$ be the future cone in $\mathbb{M}^{1,1}$ with $(x,y) \in \mathbb{M}^{1,1}$, $y^2 - x^2 > 0$ and $y > 0$. The results also hold for the space-like vector $(x,y) \in \mathbb{M}^{1,1}$, $x^2 - y^2 > 0$.

Theorem 1. Let $f := u + gv : \mathcal{D} \subset \mathbb{M}^+ \mapsto \mathbb{M}^+$ be a given g-function, with $\mathcal{D}$ an open set in the future cone. Then $f'(w_0)$ exists if and only if f is a differentiable function in the sense of real variable and, at $(x_0, y_0) = w_0$, u and v satisfy the Cauchy-Riemann equations:

$$\frac{\partial u(x,y)}{\partial x} = \frac{\partial v(x,y)}{\partial y}, \quad \frac{\partial u(x,y)}{\partial y} = \frac{\partial v(x,y)}{\partial x}. \tag{34}$$

Thus, if $\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$ and $\partial v/\partial y$ exist, then f is g-analytic on $\mathcal{D}$. If $f'(w_0)$ does exist, then

$$f'(w_0) = \frac{\partial u(x,y)}{\partial x} + g\frac{\partial v(x,y)}{\partial x} = \frac{\partial v(x,y)}{\partial y} + g\frac{\partial u(x,y)}{\partial y}. \tag{35}$$

Proof. For a separate and more direct proof is given here to show that if $f'(w_0)$ exists, then u and v satisfy the Cauchy-Riemann equations (34). In the limit

$$f'(w_0) = \lim_{w \to w_0} \frac{f(w) - f(w_0)}{w - w_0},$$

let us take the special case that $w = x + gy_0$. Then

$$\begin{aligned} \frac{f(w) - f(w_0)}{w - w_0} &= \frac{u(x, y_0) + gv(x, y_0) - u(x_0, y_0) - gv(x_0, y_0)}{x - x_0} \\ &= \frac{u(x, y_0) - u(x_0, y_0)}{x - x_0} + g\frac{v(x, y_0) - v(x_0, y_0)}{x - x_0}. \end{aligned}$$

As $x \to x_0$ we obtain the limit $f'(w_0) = \partial u/\partial x + g\partial v/\partial x$.

On the other hand, let $w = x_0 + gy$. Then

$$\begin{aligned} \frac{f(w) - f(w_0)}{w - w_0} &= \frac{u(x_0, y) + gv(x_0, y) - u(x_0, y_0) - gv(x_0, y_0)}{g(y - y_0)} \\ &= \frac{u(x_0, y) - u(x_0, y_0)}{g(y - y_0)} + \frac{v(x_0, y) - v(x_0, y_0)}{y - y_0}. \end{aligned}$$

As $y \to y_0$ we obtain the limit

$$\frac{1}{g}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = g\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y},$$

where $g^2 = 1$ was used. Thus, since $f'(w_0)$ exists and has the same value regardless of how w approaches w_0, we can obtain

$$g\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} + g\frac{\partial v}{\partial x}.$$

By equating the real and g parts of this equation, we can derive the Cauchy-Riemann equations (34), and two formulas for $f'(w)$ in Equation (35). □

The analytic function is well-developed in the complex theory. In order to distinct the present analytic function with that in the complex theory, we call it the g-analytic function with its real and g parts satisfying the Cauchy-Riemann equations (34). Next we prove that

Theorem 2. Let $f : \mathcal{D} \subset \mathbb{M}^+ \mapsto \mathbb{M}^+$ be a g-analytic function, with $f(w) = u(x, y) + gv(x, y)$. Then u and v satisfy wave equation (27):

$$\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial x^2} = 0, \tag{36}$$

$$\frac{\partial^2 v}{\partial y^2} - \frac{\partial^2 v}{\partial x^2} = 0. \tag{37}$$

Proof. From Equation (34) it follows that

$$\begin{aligned} \frac{\partial^2 u(x, y)}{\partial y^2} &= \frac{\partial^2 v(x, y)}{\partial x \partial y}, \\ \frac{\partial^2 u(x, y)}{\partial x^2} &= \frac{\partial^2 v(x, y)}{\partial y \partial x}. \end{aligned}$$

Subtracting the above two equations we obtain Equation (36). On the other hand, Equation (34) renders

$$\begin{aligned} \frac{\partial^2 v(x, y)}{\partial y^2} &= \frac{\partial^2 u(x, y)}{\partial x \partial y}, \\ \frac{\partial^2 v(x, y)}{\partial x^2} &= \frac{\partial^2 u(x, y)}{\partial y \partial x}. \end{aligned}$$

A same procedure leads to Equation (37). □

In the complex analytic function theory u and v are usually called harmonic function and conjugate harmonic function. By the same token, u and v in the g-analytic function theory might be called g-harmonic function and conjugate g-harmonic function.

For analytic function and g-analytic function, the Cauchy-Riemann equations in terms of (Euclidean and Minkowskian) polar coordinates are, respectively,

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}, \tag{38}$$

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = \frac{1}{r}\frac{\partial u}{\partial \theta}. \tag{39}$$

From Equation (38), the second equation in Equation (5) follows. Similarly, from Equation (39), Equation (32) follows straightforward.

The most simple analytic function and g-analytic function are polynomial functions $f(z) = z^k$ and $f(w) = w^k$.

Let $z = x + iy = re^{i\theta}$, and it is easy to derive

$$r^k \cos(k\theta) = \mathrm{Re}(x + iy)^k, \quad r^k \sin(k\theta) = \mathrm{Im}(x + iy)^k. \tag{40}$$

Let $w = y + gx = re^{g\theta}$. From Equation (24) it is easy to derive

$$r^k \cosh(k\theta) = \frac{1}{2}[(y + x)^k + (y - x)^k], \tag{41}$$

$$r^k \sinh(k\theta) = \frac{1}{2}[(y + x)^k - (y - x)^k]. \tag{42}$$

We can express the Trefftz bases (33) for wave equation directly in terms of the above polynomials. However, the Trefftz bases (6) for the Laplace equation cannot be directly expressed in terms of polynomials; they must take the real and imaginary parts of z^k.

We can decompose the wave equation (27) into

$$\left(\frac{\partial}{\partial y} + g\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial y} - g\frac{\partial}{\partial x}\right)u = 0. \tag{43}$$

By analogy to the Laplace equation we can derive a neater form:

$$\frac{\partial^2 u}{\partial w \partial \overline{w}} = 0, \tag{44}$$

where $w = y + gx$ and $\overline{w} = y - gx$. Hence, the general solution of Equation (44) is $u = \phi(w) + \psi(\overline{w})$; that is, $u = \phi(y + gx) + \psi(y - gx)$. Since ψ is arbitrary we can take $\psi(w) = \overline{\phi}(w)$. Then $\psi(\overline{w}) = \overline{\phi}(\overline{w}) = \overline{\phi(w)}$, and $u = 2\mathrm{Re}[\phi(w)]$. If we take ϕ as a g-analytic function and replace it by $\phi/2$, we see that the real part of any g-analytic function is a g-harmonic function of x and y. The converse is also true: any g-harmonic function is the real part of a g-analytic function.

5. The Cauchy Theorem

The Cauchy theorem in complex theory is closely related to the exact function on the plane (x, y). Again we let $\mathbb{M}^+$ be the future cone in $\mathbb{M}^{1,1}$ with $(x, y) \in \mathbb{M}^{1,1}$, $y^2 - x^2 > 0$ and $y > 0$.

Theorem 3. Suppose that $f(w)$ is a g-analytic function, with f' continuous, on and inside/outside the simple closed curve $\Gamma \in \mathbb{M}^+$. Then

$$\oint_\Gamma f(w)dw = 0. \tag{45}$$

Proof. Upon setting $f(w) = u + gv$ and $w = y + gx$ we have

$$\begin{aligned}\oint_\Gamma f(w)dw &= \oint_\Gamma (u + gv)(dy + gdx) \\ &= \oint_\Gamma (udy + vdx) + g\oint_\Gamma (udx + vdy).\end{aligned}$$

Due to the first Cauchy-Riemann equation (34) with $u_x = v_y$ there exists a potential function $d\phi(x,y) = \phi_x dx + \phi_y dy = vdx + udy$; similarly, another potential function ψ exists, such that $d\psi(x,y) = \psi_x dx + \psi_y dy = udx + vdy$ due to $u_y = v_x$. The subscripts above denote the partial differentials. Then we have

$$\oint_\Gamma f(w)dw = \oint_\Gamma d\phi + g\oint_\Gamma d\psi = 0. \tag{46}$$

Also we can prove it by using the Green's Theorem to each integral:

$$\oint_\Gamma f(w)dw = \int\int_{\mathcal{D}} \left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right] dxdy + g\int\int_{\mathcal{D}} \left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right] dxdy = 0. \tag{47}$$

Both terms are zero by using the Cauchy-Riemann equations (34). □

Example 2 (continued). Now it is a good position to demonstrate that the contour integral in Equation (21) for $\frac{1}{z}$ is non-zero, but the contour integral in Equation (22) for $\frac{1}{w}$ is zero. It is that

$$\frac{1}{z} = \frac{1}{x+iy} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2} = u + iv \tag{48}$$

is not analytic at the singular point $z = 0$ which is inside the domain. Although the above u and v satisfy the Cauchy-Riemann equations (8):

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{y^2-x^2}{(x^2+y^2)^2} = \frac{\partial v}{\partial y},\\ \frac{\partial u}{\partial y} &= \frac{-2xy}{(x^2+y^2)^2} = -\frac{\partial v}{\partial x}.\end{aligned}$$

On the other hand,

$$\frac{1}{w} = \frac{1}{x+gy} = \frac{x}{x^2-y^2} - g\frac{y}{x^2-y^2} = u + gv \tag{49}$$

is a g-analytic function, due to $y^2 - x^2 > 0$ (or $x^2 - y^2 > 0$) and

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{-x^2-y^2}{(x^2-y^2)^2} = \frac{\partial v}{\partial y},\\ \frac{\partial u}{\partial y} &= \frac{2xy}{(x^2-y^2)^2} = \frac{\partial v}{\partial x}.\end{aligned}$$

The domain is an exterior one which is outside the unit circle, such that the singular point $w = 0$ is not in the domain.

Corollary 1. Suppose that u is a g-harmonic function, on and inside/outside the simple closed curve $\Gamma \in \mathbb{M}^+$. Then

$$\oint_\Gamma u_x dy + u_y dx = 0. \tag{50}$$

Proof. Upon setting $f = u_x + gu_y$ and $w = y + gx$ and reminding that f is a g-analytic function because u_x and u_y satisfying the Cauchy-Riemann equations, by Cauchy Theorem 3 we have

$$\begin{aligned}0 &= \oint_\Gamma (u_x + gu_y)(dy + gdx)\\ &= \oint_\Gamma u_x dy + u_y dx + g\oint_\Gamma u_x dx + u_y dy,\end{aligned}$$

where the g part is zero due to $\oint_\Gamma u_x dx + u_y dy = \oint_\Gamma du = 0$. Then we obtain Equation (50). □

Can we find a g-analytic function $f = u+gv$, such that its real part u is a given g-harmonic function? Indeed we have

Theorem 4. In a simply connected region $\mathcal{D} \in \mathbb{M}^+$, if we have a given g-harmonic function u then we can find a g-analytic function $f = u + gv$ at most differencing by a constant g part.

Proof. We have

$$dv = v_x dx + v_y dy = u_y dx + u_x dy,$$

by assuming a g-analytic function $f = u + gv$. Then,

$$v(x,y) = \int_{(x_0,y_0)}^{(x,y)} (u_y dx + u_x dy) + c_0, \tag{51}$$

where (x_0, y_0) is a fixed point in $\mathcal{D}$ and c_0 is a constant. We need to check that v is the g-part of f. That is, we need to check that u and v satisfy the Cauchy-Riemann equations (34). By Equation (51) it is trivial.

We give an example to demonstrate Theorem 4.

Example 3. Let $u = x^2 + y^2 + xy$ be a g-harmonic function, which satisfies wave equation (27). Then we take $f = u + gv$ with

$$\begin{aligned} & dv = v_x dx + v_y dy, \\ & v = \int u_y dx + u_x dy + c_0 = \int (2y + x)dx + (2x + y)dy + c_0 \\ & = \frac{x^2}{2} + 2xy + \frac{y^2}{2} + c_0. \end{aligned}$$

Hence, this g-analytic function $f = u + gv$ is given by

$$\begin{aligned} & f = x^2 + y^2 + xy + g\left(\frac{x^2}{2} + 2xy + \frac{y^2}{2} + c_0\right) = \left(1 + \frac{g}{2}\right)(y + gx)^2 + gc_0 \\ & = \left(1 + \frac{g}{2}\right)w^2 + gc_0 := f(w). \end{aligned}$$

As a continuation we give two examples to demonstrate the Cauchy Theorem 3.

Example 4. First we take

$$f = u + gv = x^2 + y^2 + xy + g\left[\frac{x^2}{2} + 2xy + \frac{y^2}{2}\right]$$

as that given in the above with $c_0 = 0$. The closed curve is given by $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 = \{0 \le x \le 1,\ y = 0\} \cup \{x = 1,\ 0 \le y \le 1\} \cup \{0 \le x \le 1,\ y = 1\} \cup \{x = 0,\ 0 \le y \le 1\}$. Inserting them into the following integral we have

$$\begin{aligned} & \oint_\Gamma f(w)dw = \oint_\Gamma \left(x^2 + y^2 + xy + g\left[\frac{x^2}{2} + 2xy + \frac{y^2}{2}\right]\right)(dy + gdx) \\ & = \int_{\Gamma_1} \left(x^2 + g\frac{x^2}{2}\right) gdx + \int_{\Gamma_2} \left(1 + y^2 + y + g\left[\frac{1}{2} + 2y + \frac{y^2}{2}\right]\right) dy \\ & - \int_{\Gamma_3} \left(x^2 + 1 + x + g\left[\frac{x^2}{2} + 2x + \frac{1}{2}\right]\right) gdx - \int_{\Gamma_4} \left(y^2 + g\frac{y^2}{2}\right) dy \\ & = \frac{g}{3} + \frac{1}{6} + \frac{11}{6} + \frac{10g}{6} - \frac{11g}{6} - \frac{10}{6} - \frac{1}{3} - \frac{g}{6} = 0. \end{aligned}$$

Although the integral domain $\mathcal{D}$ passes across the boundary $x = y$ of future cone, we still obtain the correct value of line integral. Indeed, the first two lines are in the space-like region, while the last two lines fall into the time-like region. At the crossed point it gives no contribution to the integral, and both in the space-like region and time-like region the Cauchy theorem holds. Thus we can say that the Cauchy theorem is applicable in the whole space $\mathbb{M}^{1,1}$.

Example 5. Then we take

$$f = u + gv = \sin y \cos x + g \cos y \sin x,$$

and Γ is the same that in Example 4. It is easy to check that the above f is a g-analytic function. We have

$$\begin{aligned}
&\oint_\Gamma f(w)dw = \oint_\Gamma (\sin y \cos x + g \cos y \sin x)(dy + gdx) \\
&= \int_{\Gamma_1} g \sin xgdx + \int_{\Gamma_2} (\sin y \cos 1 + g \cos y \sin 1)dy \\
&- \int_{\Gamma_3} (\sin 1 \cos x + g \cos 1 \sin x)gdx - \int_{\Gamma_4} \sin ydy \\
&= 1 - \cos 1 + (1 - \cos 1)\cos 1 + g\sin^2 1 + (\cos 1 - 1)\cos 1 - g\sin^2 1 + \cos 1 - 1 = 0.
\end{aligned}$$

6. The Non-existence of Cauchy Integral Formula and Backward Wave Problem

In complex theory the Cauchy integral is given by

$$f(z) = \frac{1}{2\pi i}\oint_\Gamma \frac{f(\zeta)}{\zeta - z}d\zeta, \tag{52}$$

where $f(\zeta)$ is an analytic function, and z is a domain point inside a multiply-connected domain with boundary Γ.

For the g-analytic function $f(w)$ we do not have a similar result. First we prove that

Corollary 2. For $\mathcal{D} \in \mathbb{M}^+$, $w \in \mathcal{D}$ being a g-number, consider a box with a center $w_0 = y_0 + gx_0$ and each side having a length 2ℓ such that the closed curve along the box is given by $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 = \{x_0 - \ell \le x \le x_0 + \ell,\ y = y_0 - \ell\} \cup \{x = x_0 + \ell,\ y_0 - \ell \le y \le y_0 + \ell\} \cup \{x_0 - \ell \le x \le x_0 + \ell,\ y = y_0 + \ell\} \cup \{x = x_0 - \ell,\ y_0 - \ell \le y \le y_0 + \ell\}$. Then we have

$$\oint_\Gamma \frac{1}{w - w_0}dw = 0. \tag{53}$$

Proof. We have

$$\begin{aligned}
&\oint_\Gamma \frac{1}{w - w_0}dw = \oint_\Gamma \frac{dy + gdx}{y + gx - w_0} \\
&= \int_{\Gamma_1} \frac{gdx}{y_0 - \ell + gx - w_0} + \int_{\Gamma_2} \frac{dy}{y + g(x_0 + \ell) - w_0} \\
&- \int_{\Gamma_3} \frac{gdx}{y_0 + \ell + gx - w_0} - \int_{\Gamma_4} \frac{dy}{y + g(x_0 - \ell) - w_0} \\
&= \ln[x_0 + \ell - g(\ell + gx_0)] - \ln[x_0 - \ell - g(\ell + gx_0)] \\
&+ \ln[\ell - gx_0 + g(x_0 + \ell)] - \ln[-\ell - gx_0 + g(x_0 + \ell)] \\
&+ \ln[x_0 - \ell + g(\ell - gx_0)] - \ln[x_0 + \ell + g(\ell - gx_0)] \\
&+ \ln[-\ell - gx_0 + g(x_0 - \ell)] - \ln[\ell - gx_0 + g(x_0 - \ell)] \\
&= \ln[\ell - g\ell] - \ln[-\ell - g\ell] + \ln[\ell + g\ell] - \ln[-\ell + g\ell] \\
&+ \ln[-\ell + g\ell] - \ln[\ell + g\ell] + \ln[-\ell - g\ell] - \ln[\ell - g\ell] = 0.
\end{aligned}$$

This ends the proof. □

The above situation is drastically different from that appears in the complex theory as given below.

Corollary 3. In a simply connected region $\mathcal{D} \in \mathbb{C}$, $z \in \mathcal{D}$ being a complex number, consider a box with a center $z_0 = x_0 + iy_0$ and each side having a length 2ℓ such that the closed curve along the box is given by $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$ $= \{x_0 - \ell \le x \le x_0 + \ell,\ y = y_0 - \ell\} \cup \{x = x_0 + \ell,\ y_0 - \ell \le y \le y_0 + \ell\} \cup \{x_0 - \ell \le x \le x_0 + \ell,\ y = y_0 + \ell\} \cup \{x = x_0 - \ell,\ y_0 - \ell \le y \le y_0 + \ell\}$. Then we have

$$\oint_\Gamma \frac{1}{z - z_0}dz = -2\pi i, \tag{54}$$

$$\oint_{-\Gamma} \frac{1}{z - z_0}dz = 2\pi i. \tag{55}$$

Proof. We have

$$\begin{aligned}
&\oint_\Gamma \frac{1}{z-z_0}dz = \oint_\Gamma \frac{x+idy}{x+iy-z_0}\\
&= \int_{\Gamma_1} \frac{dx}{x-x_0-i\ell} + \int_{\Gamma_2} \frac{dy}{y-y_0-i\ell} - \int_{\Gamma_3} \frac{dx}{x-x_0+i\ell} - \int_{\Gamma_4} \frac{dy}{y-y_0+i\ell}\\
&= \ln(\ell-i\ell) - \ln(-\ell-i\ell) + \ln(\ell-i\ell) - \ln(-\ell-i\ell) + \ln(-\ell+i\ell)\\
&- \ln(\ell+i\ell) + \ln(-\ell+i\ell) - \ln(\ell+i\ell) = 4\ln\frac{1-i}{1+i} = 4\ln(-i) = -2\pi i.
\end{aligned}$$

This ends the proof of Equation (54). For a clockwise integral we can obtain Equation (55) directly. □

Equation (55) is a key point to prove the Cauchy integral formula (52) in a multiply-connected domain for the complex analytic function. Similarly, Equation (53) is a key point to prove the following result in a multiply-connected domain for the g-analytic function.

Theorem 5. Let $\mathcal{D} \in \mathbb{M}^+$ be a multiply-connected region with an outer/inner contour Γ_0 counter-clockwise and some inner/outer contours $\Gamma_i,\ i = 1,\dots,n$ clockwise. If $f(w)$ is continuous in $\overline{\mathcal{D}}$ and g-analytic in $\mathcal{D}$, then for every point $w_0 = y_0 + gx_0 \in \mathcal{D}$ we have

$$\oint_\Gamma \frac{f(w)}{w-w_0}dw = 0, \tag{56}$$

where $\Gamma = \Gamma_0 \cup \Gamma_1 \ldots \cup \Gamma_n$.

Proof. Let

$$F(w) = \frac{f(w)}{w-w_0},$$

which is g-analytic in $\mathcal{D}$ besides at the point w_0. Let w_0 be a center of the box as constructed in Corollary 2, and meanwhile we let Γ_ℓ be a box clockwise, which is inside $\mathcal{D}$ if ℓ is small enough. We apply Theorem 3 to the region enclosed by Γ and Γ_ℓ, such that

$$\oint_\Gamma \frac{f(w)}{w-w_0}dw = \oint_{\Gamma_\ell} \frac{f(w)}{w-w_0}dw. \tag{57}$$

As shown in Corollary 2 the right-hand side is independent to ℓ, and when $\ell \to 0$ we have

$$\underset{\ell\to 0}{\text{limit}} \oint_{\Gamma_\ell} \frac{f(w)}{w-w_0}dw = f(w_0) \underset{\ell\to 0}{\text{limit}} \oint_{\Gamma_\ell} \frac{1}{w-w_0}dw. \tag{58}$$

In view of Corollary 2 this theorem is proven. □

The situation of Equation (56) is drastically different from the Cauchy integral formula (52) in the complex theory. To demonstrate this theorem we give the following example.

Example 6. Consider a unit box $[0,1]^2$ with a contour $\Gamma := \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 = \{0 \le x \le 1,\ y = 0\} \cup \{x = 1,\ 0 \le y \le 1\} \cup \{0 \le x \le 1,\ y = 1\} \cup \{x = 0,\ 0 \le y \le 1\}$, and let $f(w) = w = y + gx$ be a g-analytic function. For any point w_0 outside the box we can evaluate

$$\begin{aligned}
&\oint_\Gamma \frac{w}{w-w_0}dw = \oint_\Gamma \frac{(y+gx)(dy+gdx)}{y+gx-w_0}\\
&= \int_{\Gamma_1} \frac{dx}{gx-w_0} + \int_{\Gamma_2} \frac{(y+g)dy}{y+g-w_0} - \int_{\Gamma_3} \frac{(x+g)dx}{1+gx-w_0} - \int_{\Gamma_4} \frac{ydy}{y-w_0}\\
&= g + w_0[\ln(g-w_0) - \ln(-w_0)] + 1 + w_0[\ln(1+g-w_0) - \ln(g-w_0)]\\
&-g - w_0[\ln(1+g-gw_0) - \ln(g-gw_0)] - 1 + w_0[\ln(-w_0) - \ln(1-w_0)] = w_0\ln 1 = 0.
\end{aligned}$$

We further demonstrate the different situations of the Cauchy integral formulae in the complex theory and in the g-number theory. Let us consider the one-dimensional wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2},\ \ (x,t) \in \Omega := \{0 < x < \ell,\ 0 < t \le t_f\}, \tag{59}$$

$$u(0,t) = u_0(t),\ \ u(\ell,t) = u_\ell(t),\ \ 0 \le t \le t_f, \tag{60}$$

$$u(x,0) = f(x),\ \ u_t(x,0) = g(x),\ \ 0 \le x \le \ell. \tag{61}$$

For the direct problem we specify boundary conditions and initial conditions on $\Gamma = \{x = 0,\ 0 \le t \le t_f\} \cup \{x = \ell,\ 0 \le t \le t_f\} \cup \{0 \le x \le \ell,\ t = 0\}$ as that specified in Equations (60) and (61). *This sort problem is an initial value problem of wave equation, which is more like the exterior problem of the Laplace equation.*

For the backward wave problem (BWP) we specify the following conditions:

$$\begin{aligned} &u(0,t) = u_0(t),\ \ u(\ell,t) = u_\ell(t),\ \ 0 \le t \le t_f, \\ &u(x,0) = f(x),\ \ u(x,t_f) = h(x),\ \ 0 \le x \le \ell. \end{aligned} \tag{62}$$

The BWP is a boundary value problem in space-time domain for wave equation, which is more like the interior problem of the Laplace equation.

From the aspect of wave control, we have formulated an inverse wave problem by solving the problem that given what initial condition of $u_t(x,0)$ the wave at a time t_f will vibrate with the desired quantity $h(x)$. The BWP as pointed out by Ames and Straugham (1997) has important applications in geophysics and optimal control theory. Bourgin and Duffin (1939) and Abdul-Latif and Diaz (1971) have proved that the BWP has a unique solution only when ct_f/ℓ=irrational. But when ct_f/ℓ=rational, the uniqueness is not satisfied. Liu (2010) has proposed a second-kind Fredholm integral regularization method to solve the BWP.

Consider a special solution of Equation (1) with $c = 1$:

$$u(x,t) = \cos\pi\left(x + \frac{1}{2}\right)\sin\pi t,$$

where $(x,t) \in [0,1]^2$. It is obvious that $u(0,t) = u(1,t) = u(x,0) = u(x,1) = 0$. Thus the boundary values are all zero. If the Cauchy integral as that in Equation (52) holds for the g-analytic function, then we will arrive to a contradiction that $u = 0$ inside the unit square. Also if the Cauchy integral formula holds for the g-analytic function, then we will have a unique solution of the BWP, no matter what the value of ct_f/ℓ is. This contradicts to the above non-uniqueness of the BWP when ct_f/ℓ=rational.

The Cauchy integral formula is very important for the Laplace equation, which renders a famous Poisson integral formula, where u is fully determined by the boundary values of u on the circle. But for the boundary value problem of wave equation, i.e. BWP, we cannot derive a similar formula. The situation for the non-uniqueness of the BWP is peculiar. In Equation (2) there exist closed curves which are level sets. It is clear that the uniqueness does not hold for domains with this type of boundary (Vakhani, 1994).

We use the following example to show the efficiency of the Trefftz method to solve the Dirichlet problem of wave equation.

Example 7. Let

$$u(x,t) = \exp[-(x-t)^2] + \exp[-(x+t)^2],\ -1 < x < 1,\ 0.1 < t < 1 \tag{63}$$

be an exact solution of the wave equation with $c = 1$. We use the polynomial Trefftz bases to expand the solution, and then by using the collocation method on the boundary to satisfy the boundary conditions. The resultant linear algebraic equations are used to determine the expansion coefficients. In Fig. 1 we can see that the solution obtained by the Trefftz method is close to the exact initial velocity, whose maximum error is 1.57×10^{-2}, where a relative noise with intensity 0.01 is added on the final time data at $t_f = 1$.

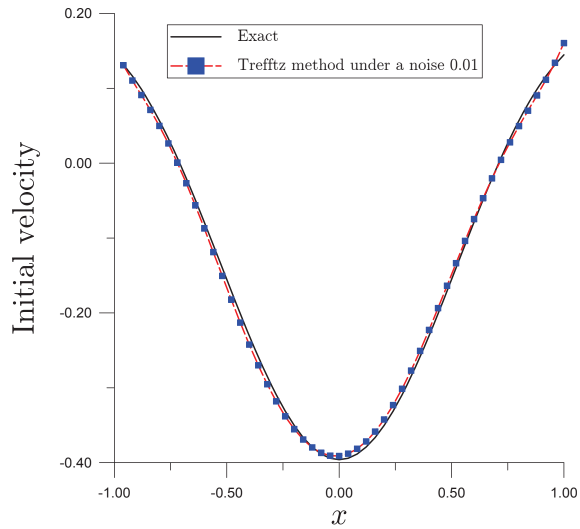

Figure 1. The recovery of a bell type initial velocity by using the Trefftz method to solve the Dirichlet problem of wave equation.

7. Conclusions

We have developed a new theory of g-analytic function in the Minkowski space $\mathbb{M}^{1,1}$, whose foundation is the Jordan algebra. The real and g parts of the g-analytic function are respectively the g-harmonic function and conjugate g-harmonic function, satisfying the Cauchy-Riemann equations in the space $\mathbb{M}^{1,1}$ and both being the solutions of wave equation. In summary, the analogies between the Laplace equation and wave equation are summarized as follows:

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$	$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0$
$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$	$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$
$\frac{\partial u(x,y)}{\partial x} = \frac{\partial v(x,y)}{\partial y}$	$\frac{\partial u(x,y)}{\partial x} = \frac{\partial v(x,y)}{\partial y}$
$\frac{\partial u(x,y)}{\partial y} = -\frac{\partial v(x,y)}{\partial x}$	$\frac{\partial u(x,y)}{\partial y} = \frac{\partial v(x,y)}{\partial x}$
$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta},\ \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$	$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta},\ \frac{\partial v}{\partial r} = \frac{1}{r}\frac{\partial u}{\partial \theta}$
$\{1, r^k \cos(k\theta), r^k \sin(k\theta)\}$	$\{1, r^k \cosh(k\theta), r^k \sinh(k\theta)\}$
$\frac{\partial^2 u}{\partial z \partial \bar{z}} = 0$	$\frac{\partial^2 u}{\partial w \partial \bar{w}} = 0$
Analytic function	g-Analytic function
Harmonic function	g-Harmonic function
$f(z) = \frac{1}{2\pi i}\oint \frac{f(\zeta)}{\zeta - z} d\zeta$	$0 = \oint \frac{f(\zeta)}{\zeta - w} d\zeta.$

We proved that the non-existence of the Cauchy integral formula in the space $\mathbb{M}^{1,1}$ is closely related to the non-uniqueness of the solution of the Dirichlet problem for wave equation, which is an interior problem. However, the Cauchy integral formula in the space $\mathbb{R}^2$ guarantees the unique solution of the interior problem for the Laplace equation. In the present theory the direct wave problem is therefore equivalent to finding in the specified domain a g-analytic function whose real part takes the given initial values and boundary values.

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