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Levinson's Theorem and the Nonlocal Saito Potential

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Abstract

We consider Levinson's theorem for the nonlocal Saito potential. We find that the phase of the Jost function gives a correct result for the zero-energy phase shift of π associated with the additional node in the scattering wavefunction. This analysis takes into account the zero-energy continuum bound state of the potential. A comparison is given with previous results which do not consider the possibility that such a state is present.

Keywords: Levinson's Theorem, Nonlocal potentials, Continuum bound states, Spurious states, Jost functions, Fredholm determinants, Definition of the phase shift

1. Introduction

In nucleon-nucleus and nucleus-nucleus scattering, antisymmetrization results in nonlocal effective two-body interactions which may be very tedious to obtain and quite elaborate in their final form (Special Supplement of the Progress of Theoretical Physics, 1977). It was first suggested by Saito (1968, 1969) that such complicated nonlocal kernels could be approximated by simplied nonlocal kernels which retained the fundamental features associated with the Pauli exclusion principle. In particular, Saito showed that his approach could produce the extra nodes in the two-body relative scattering wave-function required for orthogonality of that wavefunction with respect to already occupied single particle states.

Because the Saito model has a variety of applications, it has been studied extensively. One aspect of such studies has been to understand how the Saito model produces extra nodes in the scattering wavefunction. It is now known (B. Bagchi, 1978) that the mechanism by which the Saito model achieves an extra node (or nodes) is that of using a kernel with a continuum bound state (A. Martin, 1958; B. Mulligan, 1976) at zero energy. When placed at zero energy, a continuum bound state affects the scattering spectrum at positive energies only through the presence of the additional node. This additional node can be explicitly demonstrated (B. Bagchi, 1980) as forcing the scattering wavefunction to be orthogonal to the wavefunction of the continuum bound state. Thus Saito's approach can orthogonalize a scattering wavefunction with respect to any state or, by including several continuum bound states at zero energy, with respect to any set of states. This leads to a scattering wavefunction with the required number of extra nodes.

When studying the nodal structure of a scattering wavefunction, it is customary to relate the nodes in the zero-energy wavefunction to the zero-energy phase shift. Such a relationship was first established in the case of a local potential by Levinson (N. Levinson, 1949). The generalization of Levinson's theorem to nonlocal potentials has been investigated by many authors (B. Mulligan and S.B. Qadri). However, attempts to discuss Levinson's theorem in the context of the Saito model (N.J. Englefield, 1974; W. Glockle, 1976) have failed to explain the mechanism by which the extra nodes are produced. In particular, both Refs. (N.J. Englefield, 1974) and (W. Glockle, 1976) explicitly exclude the effects on Levinson's theorem resulting from a zero-energy continuum bound state. As has been discussed in detail by Newton

(R.G. Newton, 1960), the question of a boundstate at zero energy requires special attention. Let us focus, for example, on the $\ell = 0$ radial equation. As Newton shows (R.G. Newton, 1960; 1977, P1348) for a local potential this radial equation cannot have a bound state at zero energy although a half-bound state is possible. On the other hand, a nonlocal potential can have many zero-energy bound states (R.G. Newton, 1977, P1582). In any such circumstance, a slight adjustment of the potential parameters can move the bound states into the continuum. Thus, we easily recognize a bound state of this kind as a zero-energy continuum bound state.

In the present paper, we explain the relationship between Levinson's theorem and the extra nodes generated in Saito's model. We will restrict our considerations to the $\ell = 0$ case and to the particular example from the class of nonlocal potentials associated with the Saito model which has become known as the Saito potential (S. Okai, 1972). In the following sections, first we describe the radial equation for $\ell = 0$ case and discuss the integral equations associated with various solutions and their corresponding Fredholm determinants. By calculating the Fredholm determinants for Saito potential, we show the origin of the extra node by contour integration. This work will have impact in nuclear scattering theory for nonlocal potentials.

2. The saito potential and the jost function

The $\ell = 0$ radial equation with the Saito potential is

$$\Big(\frac{d^2}{dr^2} + k^2\Big) u(k,\ r) = \xi(r) \int_0^\infty \xi(s) \frac{d^2}{ds^2} u(k,\ s) ds \tag{1}$$

where

$$\xi(r) = (4a^3)^{1/2} r e^{-\alpha r} \tag{2}$$

The procedure we use for discussing Levinson's theorem for the Saito potential is based on the Jost function $\mathfrak{L}^+(k)$, and follows the approach of Swan (P. Swan, 1955). The Jost function is defined as the Jost solution evaluated at $r = 0$.

$$\mathfrak{L}^+(k) = f^+(k,\ r)|_{r=0} \tag{3}$$

Swan noted that on the positive imaginary axis in the complex k plane the Jost function $\mathfrak{L}^+(k)$ for a local potential has a zero associated with each bound state (H.M. Nussenzveig, 1972). Thus Swan was able to make use of a theorem from complex analysis, known as the argument principle, which states that if a function is analytic inside and on a simple contour, except for a finite number of poles inside the contour, and if the function has no zeros on the contour, then the integral of the logarithmic derivative of that function around the contour is equal to $2\pi i$ times the number of zeros inside the contour minus the number of poles inside the contour. For a central local potential it therefore is possible to construct an integral around a path in the upper half plane 3 such that this integral is $2\pi i$ times the number of bound states of the potential. Since for a local potential the scattering phase shift is given by the negative of the phase of $\mathfrak{L}^+(k)$, Levinson's theorem follows immediately. As indicated by Swan, his procedure can be generalized for use with nonlocal potentials. For a symmetric nonlocal potential, the definition of the Jost function given by Eq. (3) can be applied without ambiguity (B. Bagchi, 1979, P1251). However, the Saito potential is not symmetric. Difficulties associated with the definition of the Jost function in the case of a nonsymmetric nonlocal potential have been discussed in Ref. (B. Bagchi, 1979, p1973). It is shown in Ref. (B. Bagchi, 1978) that in the case of the Saito potential the difficulties can be overcome. In this connection, we replace the definition of the Jost function given in Eq. (3) by one in terms of Fredholm determinants.

3. Integral equations and fredholm determinants

It was first demonstrated by Jost and Pais (R. Jost, 1951) that for a local potential the Jost function is related to Fredholm determinants of integral equations for solutions of the radial equation. The same situation has been demonstrated to be the case for a symmetric nonlocal potential (C.S. Warke, 1971; Y Singh, 1971; S.S. Ahmed, 1974). For a nonsymmetric nonlocal potential, the appropriate relation is discussed in Ref. (B. Bagchi, 1979, P1973).

The integral equations of interest are those for the physical solution, the regular solution, and the Jost solution. The physical solution of Eq. (l) satisfies the integral equation

$$\psi^+(k,\ r) = \sin kr + \int_0^\infty G^+(k,\ r,\ r')\xi(r')dr' \int_0^\infty \xi(s) \frac{d^2}{ds^2} \psi^+(k,\ s) ds \tag{4}$$

Where

$$G^+(k,\ r,\ r') = -k^{-1} e^{ikr>} \sin kr_< \tag{5}$$

The regular solution of Eq. (1) satisfies the integral equation

$$\varphi(k,\ r) = k^{-1} \sin kr + \int_0^r G(k,\ r,\ r')\xi(r')dr' \int_0^\infty \xi(s) \frac{d^2}{ds^2} \varphi(k,\ s) ds \tag{6}$$

where

$$G(k,\ r,\ r') = k^{-1}\sin k(r-r') \tag{7}$$

The integral equation for the Jost solution is

$$f^{+}(k,\ r) = e^{ikr} - \int_{r}^{\infty} G(k,\ r,\ r')\xi(r')dr' \int_{0}^{\infty} \xi(s)\frac{d^2}{ds^2}f^{+}(k,\ s)ds \tag{8}$$

where G is given by Eq. (7).

The Fredholm determinants associated with the kernels of Eqs. (4), (6), and (8) are, respectively, $D^{+}(k)$, $D(k)$, and $\triangle(k)$. It is shown in Ref. 3 that for the Saito potential $D(k) = \triangle(k)$. Thus for the Saito potential, the Jost function can be defined (B. Bagchi, 1979, P1251; P1973) as

$$\mathfrak{L}^{+}(k) = \frac{D^{+}(k)}{D(k)} \tag{9}$$

4. Levinson's theorem for the saito potential

The Fredholm determinants $D^{+}(k)$ and $D(k)$ for the Saito potential are given in Ref. (B. Bagchi, 1978). From the expressions for $D^{+}(k)$ and $D(k)$, it is evident that there is a continuum bound state at $k = 0$. Using Eq. (9), the expression for $\mathfrak{L}^{+}(k)$ is

$$\mathfrak{L}^{+}(k) = \frac{(k+\alpha+2i\alpha)(k-\alpha+2i\alpha)(k-i\alpha)^2}{(k+i\alpha)^2(k+i\sqrt{3}\alpha)(k-i\sqrt{3}\alpha)} \tag{10}$$

In the upper half-plane, $\mathfrak{L}^{+}(k)$ has a double zero at $k = i\alpha$ and a pole at $k = i\sqrt{3}\alpha$. However, Eq. (10) contains no explicit information about the continuum bound state at $k = 0$. In taking the ratio of $D^{+}(k)$ and $D(k)$ to form $\mathfrak{L}^{+}(k)$, the double zero of $D^{+}(k)$ at $k = 0$ is canceled by the double zero of $D(k)$, leaving no zero of $\mathfrak{L}^{+}(k)$ at $k = 0$. [Note that for a half bound state at $k = 0$, a zero of $\mathfrak{L}^{+}(k)$ is required.]

Thus, as noted in Ref. (B. Mulligan, 1981), for a nonlocal potential the Jost function contains incomplete information. Nevertheless, it can still be used for a correct derivation of Levinson's theorem. It is demonstrated in Ref. (B. Bagchi, 1977) that if the phase shift is taken as the negative of the phase of $\mathfrak{L}^{+}(k)$, then Levinson's theorem as derived from $\mathfrak{L}^{+}(k)$ will yield a zero-energy phase shift of π for each extra node in the zero-energy wavefunction due to a continuum bound state. It is shown in Ref. (B. Bagchi, 1978) that the zero-energy scattering wavefunction for the Saito potential has an extra node. Thus we would expect a derivation of Levinson's theorem using $\mathfrak{L}^{+}(k)$ to yield a zero-energy phase shift of π for the Saito potential.

That this is the case can be easily demonstrated. Using the logarithmic derivative of $\mathfrak{L}^{+}(k)$, the argument principle for the contour C of Fig. 1 yields

$$\int_{C} \frac{\mathfrak{L}^{+}(k)'}{\mathfrak{L}^{+}(k)}dk = \ln \mathfrak{L}^{+}(k)|_{c} = 2\pi i \tag{11}$$

Taking the phase shift to be defined as the negative of the phase of $\mathfrak{L}^{+}(k)$, that is, taking

$$\mathfrak{L}^{+}(k) = |\mathfrak{L}^{+}(k)|e^{-i\delta(k)} \tag{12}$$

and making use of the fact that $\delta(-k) = -\delta(k)$, Eq. (11) becomes

$$2i[\delta(0) - \delta(\infty)] = 2\pi i \tag{13}$$

Thus we get

$$\delta(0) - \delta(\infty) = \pi \tag{14}$$

5. Comparison with previous work

The discussion of the properties of the Saito potential by Englefield and Shoukry (1974) is based upon a formulation of the problem in momentum space. Using this approach, Englefield and Shoukry conclude that the effect of the Saito potential in adding an extra node to the scattering wavefunction is to increase the zero-energy phase shift by π. However, they do not attribute this extra π to a bound state at zero energy. Rather, they point out that their conclusions might need to be modified if a bound state at zero energy were to be included.

Glockle and Le Tourneux (1976) analyze the Saito potential by the more conventional method of contour integration in k space. The function which they select for forming a logarithmic derivative for the application of the argument principle yields the extra π at $k = 0$ in the phase shift due to the Saito potential. However, this is because the function which they choose [the function $D(k)$ defined on page 19 of Ref. (W. Glockle, 1976) happens to be constructed such that the double zero on the real k axis due to the continuum bound state has been removed. Furthermore, they explicitly point out that they do not consider the possibility of a continuum bound state at $k = 0$, and that their conclusions would have to be modified were such a state to be present. In particular, they point out (W. Glockle, 1976, P30) that, in the presence of a continuum bound state, an additional π must be added.

6. Conclusion

In the present paper we have shown that the standard approach initiated by Swan (1981), when applied in the customary manner using the Jost function $\mathfrak{L}^{+}(k)$, gives results which are consistent with the extra node present in the scattering wavefunction for the Saito potential. We also reemphasize a conclusion presented in Ref. (B. Mulligan, 1981): Although a calculation of the phase shift at zero-energy using the Jost function gives a correct result in the presence of a continuum bound state, the Jost function does not itself have zeros on the real k axis associated with the continuum bound state. The Fredholm determinant $D^{+}(k)$, on the other hand, does have a pair of zeros on the real k axis for each such state.

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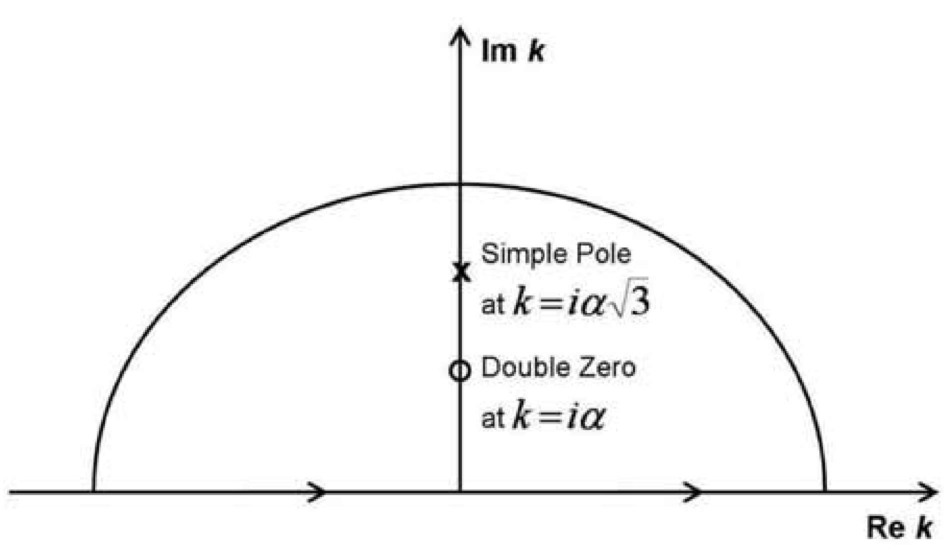

Figure 1. Poles and zeros of $\mathfrak{L}^{+}(k)$ in the upper half plane for the Saito potential. Poles are indicated by , zeros by o. There are no poles or zeros of $\mathfrak{L}^{+}(k)$ on the real axis.

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Equivalent Bilevel Programming Form for the Generalized Nash Equilibrium Problem

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Abstract

Generalized Nash Equilibrium problem is widely applied but hard to solve. In this paper, we transform the generalized Nash game into a special bilevel programming with one leader and multi-followers by supposing a suppositional leader, that is an upper decision maker. The relations between their solutions are discussed. We also discuss the further simplification of the bilevel programming. Many conclusions and the further research are drawn at last.

Keywords: Generalized Nash equilibrium point, Bilevel Programming, Efficient solution, Optimal solution

1. Introduction

Game theory is the study of problem of conflict and cooperation among independent decision-makers. And it is a mathematical framework that describes interactions between multi-agents and allows for their incomes(Newmann and Morgenstern, 1944; Osborne and Rubinstein, 1994; Samuelson, 1997). A game defines an interaction between some agents. Each agent has a series of available strategies, where a strategy determines an action of the agent in the game. Game theory has played a substantial role in economics and has been applied to many application areas such as biology, transportation(Sun and Gao, 2006), sociology, political sciences(Schelling, 1960), psychology(Scharlemann *et al.*, 2001), management science(Patriksson and Rockafellar, 2002), warfare and so on.

Games appear in normal form(strategic form), extensive form and coalitional form. The first two are close relatives, they constitute the basic paradigm of non-cooperative game theory. The coalitional form is the basic paradigm of cooperative game theory(Nash, 1951). Most of the game researchers pay their attentions to non-cooperative finite game with perfect information, i.e., each player in the game enjoys complete information and he/she independently selects a strategy and receives a corresponding payoff value that depends on the strategies selected by all players. Players choose their best strategies to maximize their payoffs respectively.

In this paper, we will consider the static generalized Nash equilibrium game, a kind of non-cooperative finite game with perfect information. It is also called social equilibrium game or pseudo-Nash equilibrium game (Ichiishi, 1983). In this kind of game, players affect each other when they make decisions not only on their utility functions but also on their feasible strategy sets. And it is a basic assumption that any player, when taking his decision, either does so simultaneously or without knowing the choice of the other players. Researchers of game theory are generally aware that solving Nash equilibrium problem can be a tedious, error-prone affair, even when the game is very simple, and they also know that the need to solve a game arises with fair frequency. It is by now a well-known fact that the Nash equilibrium problem where each player solves a convex parameter program can be formulated and solved as a finite-dimensional variational inequality (Facchinei and Pang, 2003; Harker and Pang, 1990). The generalized Nash game is a Nash game in which each player's strategy set depends on the other players' strategies. The connection between the generalized Nash games and quasi-variational inequalities (QVIs) was recognized by Bensoussan (Bensoussan, 1974) as early as 1974 who studies these problems with quadratic functions in Hilbert space.

As for the generalized Nash equilibrium model, Ichiishi (Ichiishi, 1983) proved the existence of equilibrium point under the conditions that utility function u^i is quasi-concave and the mapping K^i is continuous for all $i \in \boldsymbol{I}$. A general assumption is that the utility function is concave and even is quasi-concave in most of the study. Similar to normal Nash equilibrium problem, the generalized Nash equilibrium problem can be transformed into a quasi-variational inequality problem. But the calculation of quasi-variational inequality problem is intractable. Under some given conditions, Ichiishi (Ichiishi, 1983) calculated a quasi-variational inequality problem in virtue of a general variational inequality problem equivalently. Many effective algorithms(Pang and Yang, 1998; Daniele and Maugeri, 2002; Pang, 2002)have been established to solve the general variational inequality problem such as Newton method and diagonalization method and so on.

Bi-level programming has been the focus of hierarchical system for many years and many excellent results have been founded (Vicente and Calamai, 1994; Zhu *et al*, 2004; Dussault *et al*, 2006; Shi *et al*, 2006).In this paper, we transform the generalized Nash equilibrium problem into a special bi-level programming with multi-follower.

The remainder of the paper is organized as follows. Section 2 introduces the basic concepts and notations, and the generalized Nash equilibrium problem is described. Section 3 deals with the transformation between the generalized Nash equilibrium game and Bi-level programming. Section 4 presents another equivalent form for GNEP, and some conclusions are drawn in the last section.

2. Concepts and properties

To describe a generalized Nash equilibrium game, we need to specify three factors, that is the number of players, the set of strategies available to each player and their payoff functions which determine each player's payoff as a function of the strategies choosed by all players(Stinchcombe, 2005).

Consider a finite n-person generalized Nash game in normal form.

Let $\boldsymbol{I}$ denote the finite set of players, $\boldsymbol{I} = \{1, 2, \cdots, n\}$.

For any player $i \in \boldsymbol{I}$, its *strategy set* $S^{\mathbf{i}} = \{s^i_1, s^i_2, \cdots, s^i_{n_i}\}$, consisting of n_i possible actions called *pure strategies*. A player's *mixed strategy* is a probability distribution over his space of pure strategies. In other words, a mixed strategy consists of a random draw of a pure strategy. It can be represented by a nonnegative vector $\boldsymbol{x}^i = (x^i_1, x^i_2, \cdots, x^i_{n_i})$, Where $\sum_{k=1}^{n_i} x^i_k = 1$. Then the mixed strategy set of player i is $\boldsymbol{X}^i = \{\boldsymbol{x}^i = (x^i_1, x^i_2, \cdots, x^i_{n_i})| \sum_{k=1}^{n_i} x^i_k = 1,\ x^i_k \geq 0,\ k = 1, 2, \cdots, n_i\}$. In particular, for some mixed strategy $\boldsymbol{S}^i = \{s^i_1, s^i_2, \cdots, s^i_{n_i}\}$, if there exists some component $x^i_k = 1$, this strategy is just the k th pure strategy. So we can see the pure strategy is a special case of the mixed strategy and then we denote the strategy set of player i as $\boldsymbol{X}^i \subseteq \boldsymbol{R}^{n_i}$ and the feasible strategy space of the game as $\boldsymbol{X} = \prod_{i\in\boldsymbol{I}} \boldsymbol{X}^i \subseteq \boldsymbol{R}^m$, where $m = \sum_{i\in\boldsymbol{I}} n_i$. And then we indicate $\boldsymbol{X}^{-i} = \prod_{i\in\boldsymbol{I}\setminus\{i\}} \boldsymbol{X}^i \subseteq \boldsymbol{R}^{m-n_i}$ as the Descartes product of all players' strategy sets except for the strategy set of player i.

Let $K^i : \boldsymbol{X}^{-i} \to \boldsymbol{X}^i$ be a point-to-set mapping, that is, $\forall \boldsymbol{x}^{-i} \in \boldsymbol{X}^{-i}$, $K^i(\boldsymbol{x}^{-i}) \subseteq \boldsymbol{X}^i$, where $\boldsymbol{x}^{-i} = (\boldsymbol{x}^1, \cdots, \boldsymbol{x}^{i-1}, \boldsymbol{x}^{i+1}, \cdots, \boldsymbol{x}^n)$. Here the mapping may portray the influence ability of the other $n - 1$ players to player i . Let $u^i : grK^i \times \boldsymbol{X}^{-i} \to \mathbf{R}$ be the *utility (or payoff) function* for player i , where grK^i is the value region of mapping K^i .

Given the above factors, we can portray the generalized Nash equilibrium game as the ternary group $\{\boldsymbol{X}^i,\ K^i,\ u^i\}_{i\in\boldsymbol{I}}$.

In game theory, what is emphasized is individual rationality (Cruz and Simaan, 2000). Every player will choose the strategy which optimize his/her utility function under the condition of other players fixed their strategies. That is, $\forall i \in \boldsymbol{I}$, if other $n - 1$ players chosen their optimal strategies as $\boldsymbol{x}^{-i*} = (\boldsymbol{x}^{1*}, \cdots, \boldsymbol{x}^{(i-1)*}, \boldsymbol{x}^{(i+1)*}, \cdots, \boldsymbol{x}^{n*})$, then player i should optimize his utility function $u^i(\boldsymbol{x}^i, \boldsymbol{x}^{-i*})$ on the feasible strategy set $K^i(\boldsymbol{x}^{-i*})$. This course can be described as the following parameter programming denoted as $(EP_{(\boldsymbol{x}^{-i*})})$:

$$\begin{aligned} &\max_{\boldsymbol{x}^i} u^i(\boldsymbol{x}^i, \boldsymbol{x}^{-i*}) \\ &s.t.\ \ \boldsymbol{x}^i \in K^i(\boldsymbol{x}^{-i*}) \end{aligned} \tag{2.1}$$

Then the generalized Nash equilibrium problem (GNEP) can be presented by the following series of parameter programming, that is,

For $i \in \boldsymbol{I}$, player i solves the parameter programming (2.2)

$$\begin{aligned} &\max_{\boldsymbol{x}^i} u^i(\boldsymbol{x}^i, \boldsymbol{x}^{-i*}) \\ &s.t.\ \ \boldsymbol{x}^i \in K^i(\boldsymbol{x}^{-i*}) \end{aligned} \tag{2.2}$$

where$\boldsymbol{x}^{-i*}$ is the optimal decisions of the players except for player i.

A Nash equilibrium point is a strategy profile such that there is no agent's interest to deviate unilaterally. So is the generalized Nash equilibrium point. We may portray it with the mathematical formulation.

Definition 2.1 *A* ***Generalized Nash Equilibrium Point*** *is defined as a point* $\boldsymbol{x}^* = (\boldsymbol{x}^{1*}, \boldsymbol{x}^{2*}, \cdots, \boldsymbol{x}^{n*})$ *such that* $\forall i \in \boldsymbol{I}$ *, the following conditions hold,* $\boldsymbol{x}^{i*} \in K^i(\boldsymbol{x}^{-i*})$*and* $u^i(\boldsymbol{x}^*) \geq u^i(\boldsymbol{y}^i, \boldsymbol{x}^{-i*})$, $\forall \boldsymbol{y}^i \in K^i(\boldsymbol{x}^{-i*})$.

The generalized Nash equilibrium conditions show that no player can increase his/her expected reward by unilaterally changing his/her strategy only. Here $\boldsymbol{x}^{i*} \in K^i(\boldsymbol{x}^{-i*})$ means that the optimal strategy of every player must be in the region decided by other players at the generalized Nash equilibrium point. This is just the difference between the generalized Nash equilibrium game and the normal form game.

From the above model and definition we can see the calculation of generalized Nash equilibrium point is hard enough. Because solving the parameter programming is not easy. An effective solution is transforming the GNEP and using the known algorithm. In the next section, we transform the generalized Nash equilibrium problem into a special bi-level programming problem and discuss the relations between their solutions.

3. Equivalent Bilevel Programming Form for GNEP

In the game portrayed above, all the players is evenness but they influence each other by their decision variables. So we must resolve all of the n parameter programmes at the same time in order to solve the generalized Nash equilibrium point. We know there is another effective model to describe the complex interactive influence among all the players, that is the multi-level programming. Bilevel programming is the simplest multi-level programming and it describes the delicate hierarchical relations between the upper leader and the lower follower.

Suppose there is an upper leader in the generalized Nash equilibrium problem, and it is endowed with the corresponding decision variable, constraints and objective function. We get a bilevel programming. So we may transform the transverse relations among all the decision-makers in the game into the lengthways relationship between the upper leader and the lower follower.

Denote the upper leader's decision variable as $\boldsymbol{x} = (\boldsymbol{x}^1, \boldsymbol{x}^2, \cdots, \boldsymbol{x}^n) \in \boldsymbol{R}^m$, where $\boldsymbol{x}^i \in \boldsymbol{R}^{n_i}$, $i \in \boldsymbol{I}$ and $\sum_{i\in \boldsymbol{I}} n_i = m$ such that $\forall i \in \boldsymbol{I}$ the condition $\boldsymbol{x}^i \in K^i(\boldsymbol{x}^{-i})$ holds. His objective is to maximize the function $-(\boldsymbol{y}-\boldsymbol{x})^T(\boldsymbol{y}-\boldsymbol{x})$ where $\boldsymbol{y} = (\boldsymbol{y}^1, \boldsymbol{y}^2, \cdots, \boldsymbol{y}^n) \in \boldsymbol{R}^m$, $\boldsymbol{y}^i \in \boldsymbol{R}^{n_i}$, $i \in \boldsymbol{I}$ and $\sum_{i\in \boldsymbol{I}} n_i = m$.

Then we get the following bilevel programming:

$$
\left(BP_1\right) \qquad \begin{array}{l} \max\limits_{(\boldsymbol{x},\boldsymbol{y})} \; f(\boldsymbol{x},\boldsymbol{y}) = -(\boldsymbol{y}-\boldsymbol{x})^T(\boldsymbol{y}-\boldsymbol{x}) \\ s.t. \begin{cases} \boldsymbol{x}^i \in K^i(\boldsymbol{x}^{-i}) \quad i \in \boldsymbol{I} \\ \text{and } \boldsymbol{y} \text{ is an efficient solution of the programming } V(\boldsymbol{x})\text{:} \\ \mathbf{V}(\boldsymbol{x}) \qquad \max\limits_{\boldsymbol{y}} \left(u^1(\boldsymbol{y}^1, \boldsymbol{x}^{-1}), \cdots, u^n(\boldsymbol{y}^n, \boldsymbol{x}^{-n})\right) \\ \qquad\qquad s.t.\ \boldsymbol{y}^i \in K^i(\boldsymbol{x}^{-i}) \ \ \forall i \in \boldsymbol{I} \end{cases} \end{array}
$$

The relations between the generalized Nash equilibrium point of GNEP and the optimal solutions of (BP_1) is satisfied. We have the following conclusions to describe the equivalence.

Lemma 3.1 *For the given* $\mathbf{x}^* \in \mathbf{R}^m$*, the multi-objective programming* $V(\mathbf{x}^*)$ *has efficient solution if and only if* $\forall i \in \mathbf{I}$*,* $EP_{(\mathbf{x}^{-i*})}$ *has optimal solution.*

Proof:

If for some given $\boldsymbol{x}^* \in \boldsymbol{R}^m$, the multi-objective programming $V(\boldsymbol{x}^*)$ has efficient solution, there exists some $j \in \boldsymbol{I}$, such that the correspond $EP_{(\boldsymbol{x}^{-j*})}$ has no optimal solution.

Because the programming $V(\boldsymbol{x}^*)$ has efficient solution, we can see the feasible region $EP_{(x^{-j*})}$, that is the set $K^j(\boldsymbol{x}^{-j*})$, is not empty. Since $EP_{(\boldsymbol{x}^{-j*})}$ has no optimal solution on $K^j(\boldsymbol{x}^{-j*})$, we can know that the function $u^j(y^j, x^{-j*})$ is not upper-bounded on the set $K^j(\boldsymbol{x}^{-j*})$.

Further, we suppose $\boldsymbol{y}^* = (\boldsymbol{y}^{1*}, \boldsymbol{y}^{2*}, \cdots, \boldsymbol{y}^{n*})$ is an efficient solution of $V(\boldsymbol{x}^*)$. It is right to know that $\boldsymbol{y}^{j*} \in K^j(\boldsymbol{x}^{-j*})$. Because the function u^j is unbounded on set $K^j(\boldsymbol{x}^{-j*})$, we know there must exist some $\bar{\boldsymbol{y}}^j \in K^j(\boldsymbol{x}^{-j*})$ such that $u^j(\bar{\boldsymbol{y}}^j, \boldsymbol{x}^{-j*}) > u^j(\boldsymbol{y}^{j*}, \boldsymbol{x}^{-j*})$.

Let $\bar{\boldsymbol{y}} = (\boldsymbol{y}^{1*}, \cdots, \boldsymbol{y}^{(j-1)*}, \bar{\boldsymbol{y}}^j, \boldsymbol{y}^{(j+1)*}, \cdots, \boldsymbol{y}^{n*})$. It is easy to know that $\bar{\boldsymbol{y}}$ is a feasible solution to $V(\boldsymbol{x}^*)$ and we have the following inequalities:

$$
u^i(\boldsymbol{y}^{i*}, \boldsymbol{x}^{-i*}) = u^i(\boldsymbol{y}^{i*}, \boldsymbol{x}^{-i*}),\ \forall i \in \boldsymbol{I},\ i \neq j,\ \ and\ \ u^j(\bar{\boldsymbol{y}}^j, \boldsymbol{x}^{-j*}) > u^j(\boldsymbol{y}^{j*}, \boldsymbol{x}^{-j*})
$$

This is a contradiction to the existence of the efficient solution to $V(\boldsymbol{x}^*)$. So $\forall i \in \boldsymbol{I}$, $EP_{(\boldsymbol{x}^{-i*})}$ has optimal solution.

Conversely, if for every $i \in \boldsymbol{I}$, the programming $EP_{(\boldsymbol{x}^{-i*})}$ has optimal solution, denoted as $\boldsymbol{y}^{i*}$. Let $\boldsymbol{y}^* = (\boldsymbol{y}^{1*}, \boldsymbol{y}^{2*}, \cdots, \boldsymbol{y}^{n*})$. Then we have, $\forall i \in \boldsymbol{I}$, $\boldsymbol{y}^{i*} \in K^i(\boldsymbol{x}^{-i*})$. So the vector $\boldsymbol{y}^*$ is a feasible solution to $V(\boldsymbol{x}^*)$.

Now we will prove it is an efficient solution to $V(\boldsymbol{x}^*)$.

In fact, for every feasible solution $\boldsymbol{y} = (\boldsymbol{y}^1, \boldsymbol{y}^2, \cdots, \boldsymbol{y}^n)$ to $V(\boldsymbol{x}^*)$, because $\boldsymbol{y}^{i*}$ is the optimal solution to $EP_{(\boldsymbol{x}^{-i*})}$, $i \in \boldsymbol{I}$. So ,it is obviously that $\forall i \in \boldsymbol{I}, \boldsymbol{y}^i$ is a feasible solution to $EP_{(\boldsymbol{x}^{-i*})}$ and the following inequalities hold:

$$u^i(\boldsymbol{y}^{i*}, \boldsymbol{x}^{-i*}) \geq u^i(\boldsymbol{y}^i, \boldsymbol{x}^{-i*}), \ \forall i \in \boldsymbol{I}$$

So we can see $\boldsymbol{y}^*$ is not only an effective solution but also an optimal solution to $V(\boldsymbol{x}^*)$.

Theorem 3.2 $\mathbf{y}^* \in \mathbf{R}^m$ *is a generalized Nash equilibrium point to GNEP if and only if there exists an* $\mathbf{x}^* \in \mathbf{R}^m$ *such that* $(\mathbf{x}^*, \mathbf{y}^*)$ *solves* (BP_1) *and the optimal value is zero.*

Proof:

First, we must notice the fact that $\boldsymbol{y}^*$ is a generalized Nash equilibrium point to GNEP shows that: $\forall i \in \boldsymbol{I}$, $\boldsymbol{y}^{i*}$ is an optimal solution to $EP_{(\mathbf{y}^{-i*})}$.

Let $\boldsymbol{x}^* = \boldsymbol{y}^*$, then we have:

$$\boldsymbol{y}^{i*} \in K^i(\boldsymbol{x}^{-i*}), \quad \forall i \in \boldsymbol{I}$$

To prove the conclusion, we must prove $\boldsymbol{y}^*$is an effective solution to $V(\boldsymbol{x}^*)$ firstly.

If it is not right, then there exists some $\bar{\boldsymbol{y}} \in \boldsymbol{R}^m$ such that

$$\bar{\boldsymbol{y}}^i \in K^i(\boldsymbol{x}^{-i*}) \ and \ u^i(\bar{\boldsymbol{y}}^i, \boldsymbol{x}^{-i*}) \geq u^i(\boldsymbol{y}^{i*}, \boldsymbol{x}^{-i*}), \quad \forall i \in \boldsymbol{I}$$

and there at least one inequality is strictly, w.l.o.g., suppose the i_0's inequality is strictly.

It is easy to know that $\bar{\boldsymbol{y}}^{i_0}$ is a feasible solution to $EP_{(y^{-i_0*})}$. Moreover, according the above supposition, we have $u^{i_0}(\bar{\boldsymbol{y}}^{i_0}, \boldsymbol{y}^{-i_0^*}) > u^{i_0}(\boldsymbol{y}^{i_0^*}, \boldsymbol{y}^{-i_0^*})$ It is a contradiction to $\boldsymbol{y}^{i_0^*}$ is the optimal solution to $EP_{(\mathbf{y}^{-i_0^*})}$.

So the above supposition is not right and we can say $\boldsymbol{y}^*$ is an efficient solution to $V(\boldsymbol{x}^*)$.

Because of $\boldsymbol{x}^* = \boldsymbol{y}^*$, the fact $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ is a feasible solution to (BP_1) is obvious.

So $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ is an optimal solution to (BP_1), and the optimal value is zero, because of $f(\boldsymbol{x}, \boldsymbol{y}) \leq 0, \ \ \forall(\boldsymbol{x}, \boldsymbol{y})$ and $f(\boldsymbol{x}^*, \boldsymbol{y}^*) = 0$.

Conversely, if $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ is an optimal solution to (BP_1), and the optimal value is zero, we can say $\boldsymbol{x}^* = \boldsymbol{y}^*$ right now because of the special upper objective function f of (BP_1). Also we have $\boldsymbol{x}^{i*} \in K^i(\boldsymbol{x}^{-i*}), \ \ \forall i \in \boldsymbol{I}$ and $\boldsymbol{y}^*$ is an efficient solution to $V(\boldsymbol{x}^*)$.

Now we will prove $\boldsymbol{y}^*$ is an generalized Nash equilibrium point to GNEP.

If it is not so, that means there exists an $i \in \boldsymbol{I}$, such that $\boldsymbol{y}^{i*}$ is not an optimal solution to $EP_{(\mathbf{y}^{-i*})}$. According to Lemma 3.1, we know $EP_{(\mathbf{y}^{-i*})}$ has optimal solution, we denote it as $\bar{\boldsymbol{y}}^i$. Then we get the following conclusions, that is:

$$\bar{\boldsymbol{y}}^i \in K^i(\boldsymbol{x}^{-i*}) \ and \ u^i(\bar{\boldsymbol{y}}^i, \boldsymbol{x}^{-i*}) > u^i(\boldsymbol{y}^{i*}, \boldsymbol{x}^{-i*})$$

Let $\bar{\boldsymbol{y}} = (\boldsymbol{y}^{1*}, \cdots, \boldsymbol{y}^{(i-1)*}, \bar{\boldsymbol{y}}^i, \boldsymbol{y}^{(i+1)*}, \cdots, \boldsymbol{y}^{n*})$. It is easy to know that $\bar{\boldsymbol{y}}$ is a feasible solution to $V(\boldsymbol{x}^*)$ and we have the following inequalities:

$$u^j(\boldsymbol{y}^{j*}, \boldsymbol{x}^{-j*}) = u^i(\boldsymbol{y}^{j*}, \boldsymbol{x}^{-j*}), \ \forall j \in \boldsymbol{I}, \ j \neq i, \ \ and \ \ u^i(\bar{\boldsymbol{y}}^i, \boldsymbol{x}^{-i*}) > u^j(\boldsymbol{y}^{j*}, \boldsymbol{x}^{-j*})$$

This is a contradiction to $\boldsymbol{y}^{i*}$ is an efficient solution to $V(\boldsymbol{x}^*)$. So the conclusion is right.

4. Other equivalent forms for GNEP

Notice that the lower multi-objective programming probe $V(\boldsymbol{x})$ of the bilevel programming problem (BP_1) is separable, that is, the decision variables of every programming are independent each other not only in objective functions but also in the constrains (Chen and Craven, 1994; Mehrdad,1996). So we know the effective solution to $V(\boldsymbol{x})$ is just the aggregate of the optimal solution to every single-objective programming, that is,

$$\max_{\boldsymbol{y}^i} u^i(\boldsymbol{y}^i, \boldsymbol{x}^{-i})$$

$$s.t. \ \ \boldsymbol{y}^i \in K^i(\boldsymbol{x}^{-i})$$

Then, we can transform the problem $V(\boldsymbol{x})$ into a single-objective programming problem equivalently by combination the multi-objective programmes linearly. Without loss of generality, let all the weighting coefficient be one (Ashry, 2006; Balbas and Guerrs, 1996). We get the transformed single-objective programming $P(\boldsymbol{x})$ as following:

$$P(\boldsymbol{x}) \qquad \begin{array}{l} \max\limits_{\boldsymbol{y}} \sum\limits_{i=1}^{n} u^i(\boldsymbol{y}^i, \boldsymbol{x}^{-i}) \\ s.t. \boldsymbol{y}^i \in K^i(\boldsymbol{x}^{-i}), \forall i \in \boldsymbol{I} \end{array}$$

And the corresponding bilevel programming problem (BP_1) is transformed into the following problem (BP_2) correspondingly:

$$\left(BP_2\right) \qquad \begin{aligned} &\max_{(x,y)} \ f(\boldsymbol{x},\boldsymbol{y}) = -(\boldsymbol{y}-\boldsymbol{x})^T(\boldsymbol{y}-\boldsymbol{x}) \\ &s.t. \begin{cases} \boldsymbol{x}^i \in K^i(\boldsymbol{x}^{-i}) \quad i \in \boldsymbol{I} \\ \text{and } \boldsymbol{y} \text{ is an optiaml solution of the programming } P(\boldsymbol{x}): \\ \mathbf{P}(\boldsymbol{x}) \qquad \begin{aligned} &\max_{y} \left(u^1(\boldsymbol{y}^1,\boldsymbol{x}^{-1}) + \cdots + u^n(\boldsymbol{y}^n,\boldsymbol{x}^{-n})\right) \\ &s.t.\, \boldsymbol{y}^i \in K^i(\boldsymbol{x}^{-i}) \ \ \forall i \in \boldsymbol{I} \end{aligned} \end{cases} \end{aligned}$$

Similar to the above discuss in section 3, we have the following conclusions, we list them without detail proofs.

Lemma 4.1 *For the given* $\mathbf{x}^* \in \mathbf{R}^m$, $P(\mathbf{x}^*)$ *has optimal solution if and only if* $\forall i \in \mathbf{I}$, $EP_{(\mathbf{x}^{-i*})}$ *has optimal solution.*

Proof: We just need to prove that $P(\boldsymbol{x}^*)$ has optimal solution is equivalent with the multi-objective programming $V(\boldsymbol{x}^*)$ has efficient solution.

If $P(\boldsymbol{x}^*)$ has optimal solution $\boldsymbol{y}^*$, it is obviously that $\boldsymbol{y}^*$ is an efficient solution to $V(\boldsymbol{x}^*)$.

Conversely, if $V(\boldsymbol{x}^*)$ has an efficient solution $\boldsymbol{y}^*$, it means that $\forall \boldsymbol{y} = (\boldsymbol{y}^1, \boldsymbol{y}^2, \cdots, \boldsymbol{y}^n) \in \boldsymbol{R}^m$ such that $\forall i \in \boldsymbol{I}$, $\boldsymbol{y}^i \in K^i(\boldsymbol{x}^{-i})$, the following inequalities hold:

$$u^i(\boldsymbol{y}^i, \boldsymbol{x}^{-i*}) \le u^i(\boldsymbol{y}^{i*}, \boldsymbol{x}^{-i*}), \quad \forall i \in \boldsymbol{I}$$

By adding them up, we have the following inequality:

$$\sum_{i=1}^{n} u^i(\boldsymbol{y}^i, \boldsymbol{x}^{-i*}) \le \sum_{i=1}^{n} u^i(\boldsymbol{y}^{i*}, \boldsymbol{x}^{-i*})$$

It just shows that $\boldsymbol{y}^*$ is an optimal solution to $P(\boldsymbol{x}^*)$. Thus we complete the conclusion.

Theorem 4.2 $\mathbf{y}^* \in \mathbf{R}^m$ *is a generalized Nash equilibrium point to GNEP if and only if there exists* $\mathbf{x}^* \in \mathbf{R}^m$ *such that* $(\mathbf{x}^*, \mathbf{y}^*)$ *solves* (BP_2) *and the optimal value is zero.*

5. Conclusion

In this paper, we consider the transforming form of generalized Nash equilibrium problem. We construct a special bilevel programming model and establish some results about the relations between solutions of the two models with strictly proof. Due to the special structure of the bilevel programming problem in this paper, the properties and the calculation of the general Nash equilibrium point becomes possible and easy. Of course, much more research is needed in order to provide algorithmic tools to effectively solve GNEP. In this regard we feel it deserves further investigations in the special bilevel programming (BP_1) and (BP_2).

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Forecasting Low-Cost Housing Demand in Johor Bahru, Malaysia Using Artificial Neural Networks (ANN)

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Abstract

There is a need to fully appreciate the legacy of Malaysia urbanization on affordable housing since the proportions of urban population to total population in Malaysia are expected to increase up to 70% in year 2020. This study focused in Johor Bahru, Malaysia one of the highest urbanized state in the country. Monthly time-series data have been used to forecast the demand on low-cost housing using Artificial Neural Networks approach. The dependent indicator is the low-cost housing demand and nine independents indicators including; population growth; birth rate; mortality baby rate; inflation rate; income rate; housing stock; GDP rate; unemployment rate and poverty rate. Principal Component Analysis has been adopted to analyze the data using SPSS package. The results show that the best Neural Network is 2-22-1 with 0.5 learning rate and momentum rate respectively. Validation between actual and forecasted data show only 16.44% of MAPE value. Therefore Neural Network is capable to forecast low-cost housing demand in Johor Bahru, Malaysia.

Keywords: Low-cost housing, Artificial neural networks, Principal component analysis

1. Introduction

Accurate predictions of the level of aggregate demand for construction are of vital importance to all sectors of this industry such as developers, builders and consultants. Empirical studies have shown that accuracy performance varies according to the types of forecasting technique and the variables to be forecast. Hence, there is a need to identify different techniques, in terms of accuracy, in the prediction of needs for facilities (Goh B. H., 1998).

Under the Seventh Malaysia Plan (1996-2000) and Eight Malaysia Plan (2001-2005), Malaysian government is committed to provide adequate, affordable and quality housing for all Malaysia, particularly the low income group. This in line with Istanbul Declaration on Human Settlement and Habit Agenda (1996) to ensure adequate shelter for all (Syafiee Shuid, 2004). The total number of housing units targeted was 800,000 units under Seventh Malaysia Plan and 782,300 units of housing is targeted to be construct under Eighth Malaysia Plan (Chapter 18, Eight Malaysia Plan, 2006). During the Ninth Malaysia Plan, requirement for new houses is expected to be about 709,400 units of which 19.2% will be in Selangor followed by Johor at 12.9%, Sarawak 9.4% and Perak 8.2%(Ninth Malaysia Plan, 2006).

Unfortunately, in 2004 there are 100,000 of low-cost houses in Selangor, Malaysia overhang (The Sun, 2004). The over construction of the low-cost at Selangor had cause million of lost while at the same time the money can be use to provide low-cost houses in other states in Malaysia. Based on Draft Kuala Lumpur Structure Plan 2020, Kuala Lumpur still lacks of 20,595 units of houses. In spite of the pricing of low-cost houses may be too high, one of the other reasons why the houses remain unsold is because they were built in undesirable locations (Salleh Buang, News Straits Time, 2004).

Therefore, there is a vital need to have a model to forecast low-cost housing demand in Malaysia so that there will be no more under or over construction of low-cost houses. At the same time, budget, time and manpower can be saved.

2. Independent and dependent indicators

The methodologies of this study are including finding out the significant indicators using Principal Component Analysis (PCA) adapted from SPSS 10.0, series of trial and error process to find out the suitable number of hidden neurons, learning rate, and momentum rate for the network and screening the result using the best Neural Network (NN) model.

PCA is used to derive new indicators; that is the significant indicators from the nine selected indicators. The indicators are: (1) population growth; (2) birth rate; (3) mortality baby rate; (4) inflation rate; (5) income rate; (6) housing stock; (7) GDP rate; (8) unemployment rate; and (9) poverty rate. The dependent indicator is the monthly time series data on low cost housing demand starting from January 2000 to December 2003.

3. Significant indicators

The determinant of the correlation matrix, R is 2.84x 10^{-14} that is very close to zero. It shows that linear dependencies are exist among the response indicators. Therefore, the PCA method can be performed. By testing from the hypothesis, populations of the correlation matrix are equal to identity matrix, which considered all the data are multivariate normal while the indicators are uncorrelated. For this case, there are nine indicators within 36 data therefore, p= 9 and N = 36.

$$\begin{aligned} -a.\ln(v) &= -(N-1-(2p+5)/6)\ln(R) \\ &= -(36-1-(2x9+5)/6)\ln(2.84x10^{-14}) \\ &= 972.16 \end{aligned}$$

Therefore, the value for the test statistic for these data is 972.16 and the critical point of the chi-square distribution with $p(p-1)/2 = 36$ for the degree of freedom, = 0.001, the critical point is 71.64. Clearly it shows that the hypothesis is rejected at the 0.001 significant levels because of 972.16 > 71.64. From the scree plot (refer Figure 1), eigenvalue for the principal component (PC) three to nine are close to zero which they can be ignored. Since the eigenvalue for PC one to two are greater than one, total variation for the two PCs is 98.0%. Therefore, two PCs are used for the analysis. According to Johnson (1998), the number of component is to be equal to the number of eigenvalue of R, which is 1. Therefore, the significant indicators for each component are with the value of component score coefficient matrix nearest to 1. The other indicators are still considered but they give less effect compared to the significant indicators. Table 1 show that the most significant indicators for PC1 are income rate and PC2 is population growth.

4. Model development

According to Cattan (1994), a network is required to perform two tasks; (1) reproduce the patterns it was trained on and (2) predict the output given patterns it has not seen before, which involves interpolation and extrapolation. In order to perform these tasks, a backpropagation network with one hidden layer is used. To find out the best number of hidden neurons for the network, the default setting of backpropagation algorithm in Neuroshell2 is applied. In this study, the learning rate and momentum rate is determined by means of trial-and-error, following four phases as shown in Table 2. These rates have been stated by SPSS Inc (1995) according to experiences in various fields using neural network. This method also has been used by Sobri Harun (1999) and Khairulzan (2002). The learning process is divided into four phases and in each phase, the learning and momentum rate will be change. The average error used is 0.001 and 40,000 learning epochs. The number for the input node for Johor Bahru district is two since it have two PCs as the input. The number of the output neuron for this task is one which is the housing demand. Figure 2 shows the Neural Network topology with 2 inputs and one output. Using the training and testing data, a series of trial and error process is conducted by varying the number of hidden neurons in order to find the suitable number of hidden neurons. The process started by applying the smallest number of hidden neurons.

In this study, the hidden neuron varies from 1 to 40. Training and testing are conducted by increasing hidden neurons after each training and testing process. The network will minimize the difference between the given output and the prediction output monitored by the minimum average error while the training process is conducted. When the value is reducing, the error also will be minimizing. This process continues until 40,000 cycles of test sets were presented after the minimum average error or the minimum average reaches the convergence rate, which comes first.

Figure 3 shows the performance of testing when different number of neurons is applied in hidden layer with different phases. From the figure, value of r shows that the performance of training and testing network almost similar with each other using different learning and momentum rate. It also shows that the network performance are good where all the values of r are uniform between 0.49 to 0.55 except using 3, 5, 22,30, 33 and 39 numbers of neurons in all phases and 22 neurons in phase 2. Phase 3 and 4 show that the lowest network performance is when using 3 numbers of hidden neuron while the highest network performance is when using 22 numbers of neurons. Thus, the best Neural Network for Johor Bahru district to forecast low cost housing demand is 2-22-1, which is 2 numbers of neurons in input layer, 22 numbers of neurons in hidden layer and 1 number of neuron in output layer.

Evaluation using Mean Absolute Percentage Error (MAPE) shows that MAPE value using 0.5 learning rate and 0.5 momentum rate (Phase 3) has the best performance with 13.71% rather than using 0.4 learning rate and 0.6 momentum rate with 16.44% (refer Table 3 and Figure 4). The ability of forecasting is very good if MAPE value is less than 10% while MAPE for less than 20% is good (Sobri Harun, 1999). Therefore, the best Neural Network for Johor Bahru district to forecast demand on low cost housing is 2-22-1 with using 0.5 learning rate and 0.5 momentum rate.

5. Discussion

Out of nine indicators, PCA has derived two PCs with significant indicator for PC1 is income rate and PC2 is population growth. The best NN model to forecast low cost housing demand in Johor Bahru is 2-22-1 using 0.5 learning and momentum rate respectively. Comparison between the actual and forecasted data shows that NN capable to forecast low cost housing demand in Johor Bahru with the best value of MAPE is 13.71%.

6. Conclusions

In conclusion, NN is capable to forecast low cost housing demand in Johor Bahru, Malaysia. Currently, low cost housing which offered is not enough and cannot afford the increasing demand. Therefore, by developing this model, it is hoped it can be helpful to the related agencies such as developer or any other relevant government agencies in making their development planning for low cost housing demand in urban area in Malaysia towards the future as there is no model have been created yet. Furthermore, a lot of advantages if a better planning of low cost housing construction is done such as save in budget, time, manpower and also paper less.

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Table 1. Component Score Coefficient matrix for Johor Bahru

	Component	
	1	2
Population growth	-.006	.634
Birth rate	-.134	-.151
Mortality baby rate	.109	.390
Inflation rate	.136	-.012
Income rate	.137	.038
Housing stock	.129	-.090
GDP rate	-.136	.096
Unemployment rate	.133	-.164
Poverty rate	-.135	.115

Table 2. Determination of learning and momentum rate.

	Phase 1	Phase 2	Phase 3	Phase 4
Learning rate	0.9	0.7	0.5	0.4
Momentum rate	0.1	0.4	0.5	0.6

Table 3. Actual and forecasted demand on low cost housing for October, November and December 2003 in Johor Bahru district

Time series	Actual data	Forecasted data	MAPE (%)	
			Phase 3	Phase 4
October 2003	203	165	18.47	20.42
November 2003	131	113	13.71	16.44
December 2003	99	81	18.22	21.29

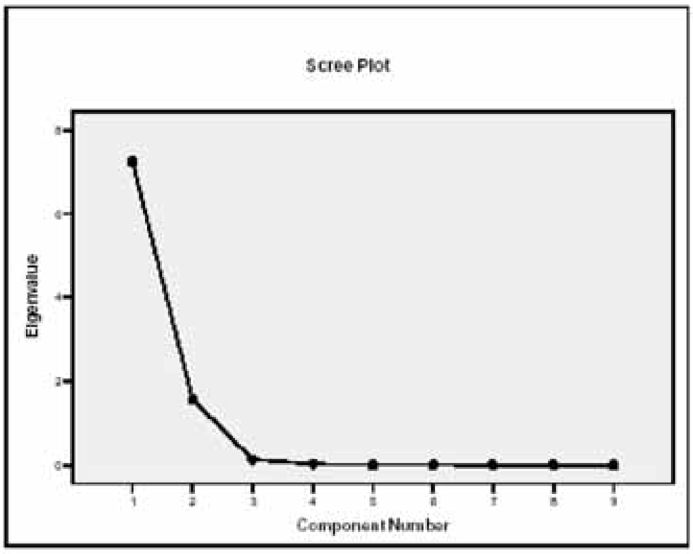

Figure 1. Scree plot for Johor Bahru

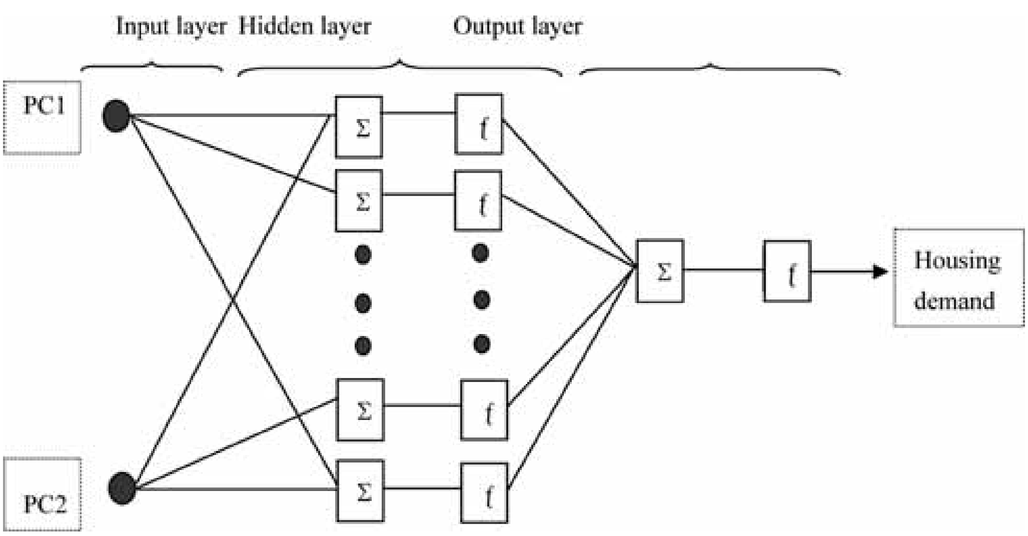

Figure 2. Neural Network topology with 2 inputs for Johor Bahru

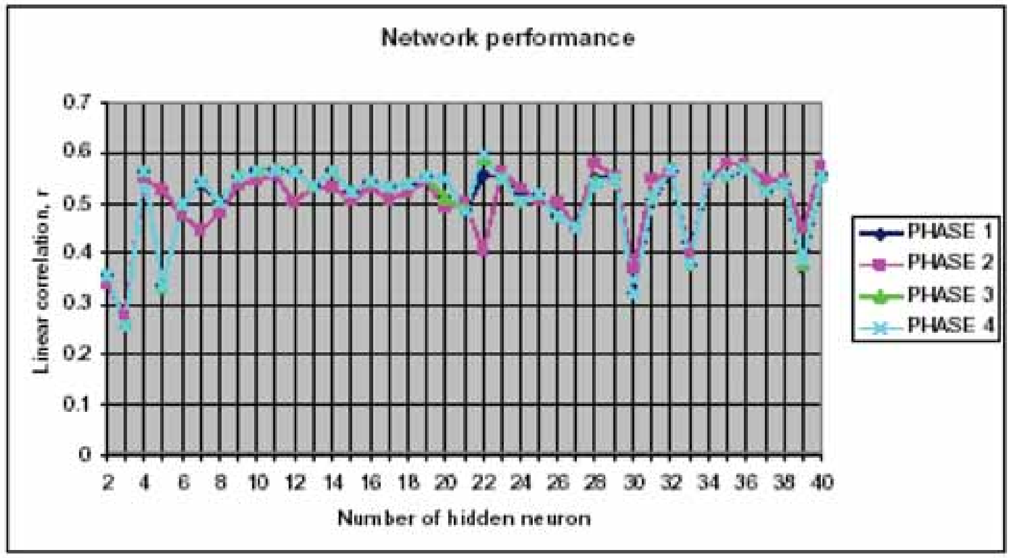

Figure 3. Network performance of testing with different number of neurons and phases

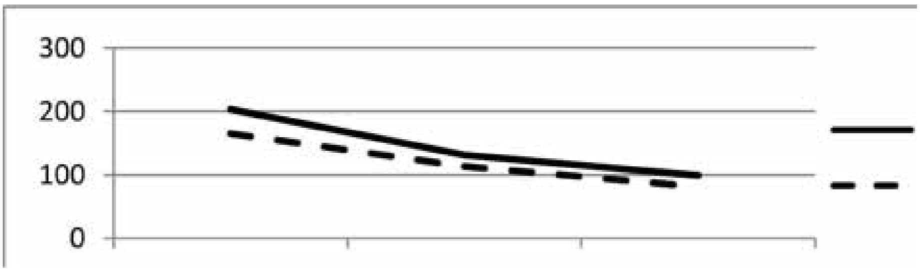

Figure 4. Graph of actual and forecasted demand data

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Existence of Positive Periodic Solutions of a Lotka-Volterra System with Multiple Time Delays

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Abstract

In this paper, a class of Lotka-Volterra system with multiple time delays is considered. By using the continuation theorem of coincidence degree theory, we derive a set of easily verifiable sufficient conditions that guarantees the existence of at least a positive periodic solution.

Keywords: Lotka-Volterra system, Periodic solution, Multiple time delay, Continuation theorem, Topological degree

1. Introduction

Lotka-Volterra system is an important population system and has been studied by many authors, see (Chen Shihua, et.al, 2004, Fan Meng, et.al,1999, Li Yongkun, et.al, 2001) and the reference therein. but most of the previous results focused on the stability, attractiveness, persistence and periodicity of solution to the ordinary differential systems or time delay systems with constant delays. Rare work has been done for the systems with varying delays and varying coefficients. In 1991, Weng di Wang et. al (Wang Wending,et. al, 1991) had considered a two-dimensional predator-prey system with a finite constant number discrete delays

$$\begin{cases} \dot{x}(t) = x(t)\left[r_1 - \sum_{j=1}^{m} a_{1j}x(t-\tau_{1j}) - \sum_{j=1}^{m} b_{1j}y(t-\rho_{1j})\right], \\ \dot{y}(t) = y(t)\left[r_2 + \sum_{j=1}^{m} a_{2j}x(t-\tau_{2j}) - \sum_{j=1}^{m} b_{2j}y(t-\rho_{2j})\right], \end{cases} \tag{1}$$

with initial conditions

$$x(s) = \varphi(s) \geq 0, s \in [-\tau, 0]; \varphi(0) > 0,$$

$$y(s) = \psi(s) \geq 0, s \in [-\tau, 0]; \psi(0) > 0,$$

where r_1, r_2 are real constants with $r_1 > 0$; $a_{ij},\ b_{ij},\ \tau_{ij},\ \rho_{ij}$ $(i = 1, 2; j = 1, 2, ..., m)$ are non-negative constants. Not all of a_{1j} and not all b_{1j} $(j = 1, 2, ..., m)$ are zero; Both $\varphi(s)$ and $\psi(s)$ are continuous on the interval $[-\tau, 0]$ in which $\tau = max\{\tau_{ij}, \rho_{ij} : i = 1, 2; j = 1, 2, ..., m\}$. And obtained the conclusion that the time delays are harmless for uniform persistence of the solutions to the system.

We note that any biological or environmental parameters are naturally subject to fluctuation in time. It is necessary and important to consider models with periodic ecological parameters or perturbations which might be naturally exposed (for example, those due to seasonal effects of weather, food supply, mating habits, hunting or harvesting seasons, etc.). Thus, the assumption of periodicity of the parameters is a way of incorporating the periodicity of the environment.

In this paper, we are concerned with the effects of periodicity of ecological and environmental parameters and time delays. Then system (1) can be modified as the form:

$$\begin{cases} \dot{x}(t) = x(t)\left[r_1(t) - \sum_{j=1}^{m} a_{1j}(t)x(t-\tau_{1j}(t)) - \sum_{j=1}^{m} b_{1j}(t)y(t-\rho_{1j}(t))\right], \\ \dot{y}(t) = y(t)\left[r_2(t) + \sum_{j=1}^{m} a_{2j}(t)x(t-\tau_{2j}(t)) - \sum_{j=1}^{m} b_{2j}(t)y(t-\rho_{2j}(t))\right], \end{cases} \tag{2}$$

with initial conditions

$$\begin{cases} x(s) = \varphi(s) \geq 0, s \in [-\tau, 0]; \varphi(0) > 0, \\ y(s) = \psi(s) \geq 0, s \in [-\tau, 0]; \psi(0) > 0, \end{cases} \tag{3}$$

where $r_1(t), r_2(t)$ are real functions with $r_i(t) > 0, (i = 1, 2)$; $a_{ij}(t), b_{ij}(t), \tau_{ij}(t), \rho_{ij}(t)(i = 1, 2; j = 1, 2, ..., m)$ are non-negative functions. Not all of $a_{1j}(t)$ and not all $b_{1j}(t)(j = 1, 2, ..., m)$ are zero; Both $\varphi(s)$ and $\psi(s)$ are continuous on the interval $[-\tau, 0]$ in which $\tau = \max_{t\in R} \max\{\tau_{ij}(t), \rho_{ij}(t) : i = 1, 2; j = 1, 2, ..., m\}$.

Throughout the paper, we always assume that
(H_1) $r_i(t),\ a_{ij}(t),\ b_{ij}(t),\ \tau_{ij}(t),\ \rho_{ij}(t)(i = 1, 2; j = 1, 2, ..., m)$ are ω periodic, i.e.,

$$r_i(t+\omega) = r_i(t),\ a_{ij}(t+\omega) = a_{ij}(t),\ b_{ij}(t+\omega) = b_{ij}(t),$$

$$\tau_{ij}(t+\omega) = \tau_{ij}(t),\ \rho_{ij}(t+\omega) = \rho_{ij}(t)$$

for any $t \in R$.
(H_2) $r_i(t),\ a_{ij}(t),\ b_{ij}(t),\ \tau_{ij}(t),\ \rho_{ij}(t)\ (i = 1, 2; j = 1, 2, ..., m)$ are all positive, i.e.,

$$r_i(t),\ a_{ij}(t),\ b_{ij}(t),\ \tau_{ij}(t),\ \rho_{ij}(t)\ (i = 1, 2; j = 1, 2, ..., m) > 0.$$

The principle object of this article is to find a set of sufficient conditions that guarantees the existence of at least a positive periodic solution for system (2) (3).

2. Basic lemma

In order to explore the existence of positive periodic solutions of (2) (3) and for the reader·s convenience, we shall first summarize below a few concepts and results without proof, borrowing from (Yang Zhihui, et. al, 2007).

Let X, Y be normed vector spaces, $L : DomL \subset X \to Y$ is a linear mapping, $N : X \to Y$ is a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $dimKerL = codimImL < +\infty$ and ImL is closed in Y. If L is a Fredholm mapping of index zero and there exist continuous projectors $P : X \to X$ and $Q : Y \to Y$ such that $ImP = KerL, ImL = KerQ = Im(I - Q)$, it follows that $L \mid DomL \cap KerP : (I - P)X \to ImL$ is invertible. We denote the inverse of that map by K_P. If Ω is an open bounded subset of X, the mapping N will be called $L-$compact on $\bar{\Omega}$ if $QN(\bar{\Omega})$ is bounded and $K_P(I - Q)N : \bar{\Omega} \to X$ is compact. Since ImQ is isomorphic to $KerL$, there exist isomorphisms $J : ImQ \to KerL$.

Lemma 2.1. (Robert E. Gaines et. al, 1991)(Continuation Theorem) *Let L be a fredholm mapping of index zero and let N be $L-$compact on $\bar{\Omega}$. Suppose*
(a) for each $\lambda \in (0, 1)$, every solution x of $Lx = \lambda Nx$ is such that $x \notin \partial\Omega$;
(b) $QNx \neq 0$ for each $x \in KerL \bigcap \partial\Omega$, and $deg\{JQN, \Omega \bigcap \partial KerL, 0\} \neq 0$;
Then the equation $Lx = Nx$ has at least one solution lying in $DomL \bigcap \bar{\Omega}$.

Lemma 2.2. $R_+^2 = \{((x(t), y(t))^T \in R^2 \mid x(t) > 0, y(t) > 0\}$ *is positive invariant with respect to system (2) (3).*

Proof. In fact,

$$x(t) = \varphi(0)exp \int_0^t \left[r_1(s) - \sum_{j=1}^{m} a_{1j}(s)x(s-\tau_{1j}(s)) - \sum_{j=1}^{m} b_{1j}(s)y(s-\rho_{1j}(s))\right] ds,$$

$$y(t) = \psi(0)exp \int_0^t \left[r_2(s) + \sum_{j=1}^{m} a_{2j}(s)x(s-\tau_{2j}(s)) - \sum_{j=1}^{m} b_{2j}(s)y(s-\rho_{2j}(s))\right] ds.$$

In view of $\varphi(0) > 0, \psi(0) > 0, (i = 1, 2)$, obviously, the conclusion follows.

3. Existence of positive periodic solutions

For convenience and simplicity in the following discussion ,we always use the notations below throughout the paper:

$$\bar{g} = \frac{1}{\omega}\int_0^\omega g(t)dt,\ g^L = \min_{t\in[0,\omega]} g(t),\ g^M = \max_{t\in[0,\omega]} g(t),$$

where $g(t)$ is a ω continuous periodic function. Let $\sigma_{ij}(t) = t - \tau_{ij}(t),\ \theta_{ij}(t) = t - \rho_{ij}(t),\ t \in R,\ i = 1,2;\ j = 1,2,...,m$. Assume that

$$(H_3)\ \tau'_{ij}(t) < 1,\ \rho'_{ij}(t) < 1,\ (i = 1,2; j = 1,2,...,m)$$

Then $\sigma_{ij}(t)$ and $\theta_{ij}(t)$ have inverse functions denoted by $\mu_{ij}(t), \varepsilon_{ij}(t), (i = 1,2; j = 1,2,...,m)$, respectively. Obviously, $\mu_{ij}(t+\omega) = \mu_{ij}(t) + \omega, \varepsilon_{ij}(t+\omega) = \varepsilon_{ij}(t) + \omega$.

In the following, we will ready to state and prove our result.

Theorem 3.1. *Suppose that* $(H_1), (H_2), (H_3), (H_4)\ D_{22}^L \overline{r_1}^L > D_{12}^M \overline{r_2}^M$ *and* $(H_5)\ \overline{r_1}\sum_{j=1}^m \overline{b_{2j}} > \overline{r_2}\sum_{j=1}^m \overline{b_{1j}}$ *hold, where*

$$D_{22} = \sum_{j=1}^m \frac{b_{2j}(t)\varepsilon_{2j}(t)}{1-\rho'_{2j}(\varepsilon_{2j}(t))},\ D_{12} = \sum_{j=1}^m \frac{b_{1j}(t)\varepsilon_{1j}(t)}{1-\rho'_{1j}(\varepsilon_{1j}(t))},$$

then the system (2) (3) has at least a ω periodic solution.

Proof. Since solutions of (2) (3) remained positive for all $t \geq 0$, we let

$$u_1(t) = ln[x(t)],\ u_2(t) = ln[y(t)]. \tag{4}$$

Substituting (4) into (2), we obtain

$$\begin{cases} \dot{u}_1(t) = r_1(t) - \sum_{j=1}^m a_{1j}(t)exp\{u_1(t-\tau_{1j}(t))\} - \sum_{j=1}^m b_{1j}(t)exp\{u_2(t-\rho_{1j}(t))\}, \\ \dot{u}_2(t) = r_2(t) + \sum_{j=1}^m a_{2j}(t)exp\{u_1(t-\tau_{2j}(t))\} - \sum_{j=1}^m b_{2j}(t)exp\{u_2(t-\rho_{2j}(t))\}. \end{cases} \tag{5}$$

It is easy to see that if system (5) has one ω periodic solution $(u_1^*(t), u_2^*(t))^T$, then $(x^*(t), y*(t))^T = (exp[u_1^*(t), exp[u_2^*(t)])^T$ is a positive solution of system (2). Therefore, to complete the proof, it suffices to show that system (5) has at least one ω periodic solution.

Let $X = Z = \{u(t)\} = \{(u_1(t), u_2(t))^T \mid u(t) \in C(R, R^2), u(t+\omega) = u(t)\}$, and define $\|u\| = \|(u_1(t), u_2(t))^T\| = \max_{t\in[0,\omega]}|u_1(t)| + \max_{t\in[0,\omega]}|u_2(t)|$. Then X and Z are Banach spaces when they are endowed with the norm $\|.\|$.

Let $L : DomL \subset X \to Z$ and $N : X \to Z$ be the following:

$$Lu = x'(t),$$

$$Nu = \begin{pmatrix} r_1(t) - \sum_{j=1}^m a_{1j}(t)exp\{u_1(t-\tau_{1j}(t))\} - \sum_{j=1}^m b_{1j}(t)exp\{u_2(t-\rho_{1j}(t))\} \\ r_2(t) + \sum_{j=1}^m a_{2j}(t)exp\{u_1(t-\tau_{2j}(t))\} - \sum_{j=1}^m b_{2j}(t)exp\{u_2(t-\rho_{2j}(t))\} \end{pmatrix}. \tag{6}$$

Define continuous projective operators P and Q:

$$Pu = \frac{1}{\omega}\int_0^\omega u(t)dt, Qu = \frac{1}{\omega}\int_0^\omega u(t)dt, u \in X, u \in Z.$$

We can see that $KerL = \{u \in X \mid u = h \in R^2\}, ImL = \{u \in Z \mid \int_0^\omega u(t)dt = 0\}$ is closed in X and $dim(KerL) = 2 = codim(ImL)$, then it follows that L is a fredholm mapping of index zero. Moreover, it is easy to check that

$$QNx = \begin{pmatrix} \frac{1}{\omega}\int_0^\omega \left[r_1(t) - \sum_{j=1}^m a_{1j}(t)exp\{u_1(t-\tau_{1j}(t))\} - \sum_{j=1}^m b_{1j}(t)exp\{u_2(t-\rho_{1j}(t))\} \right] dt \\ \frac{1}{\omega}\int_0^\omega \left[r_2(t) + \sum_{j=1}^m a_{2j}(t)exp\{u_1(t-\tau_{2j}(t))\} - \sum_{j=1}^m b_{2j}(t)exp\{u_2(t-\rho_{2j}(t))\} \right] dt \end{pmatrix}.$$

By easily computation, we have

$$K_P(z) = \int_0^\omega z(u)du - \frac{1}{\omega}\int_0^\omega \left[\int_0^t z(u)du\right]dt,$$

$K_P(I-Q)Nu =$

$$\begin{pmatrix} \int_0^t F_1(s)ds \\ \int_0^t F_2(s)ds \end{pmatrix} - \begin{pmatrix} \frac{1}{\omega}\int_0^\omega \int_0^t F_1(s)dsdt \\ \frac{1}{\omega}\int_0^\omega \int_0^t F_2(s)dsdt \end{pmatrix} - \begin{pmatrix} (\frac{t}{\omega}-\frac{1}{2})\int_0^\omega F_1(s)ds \\ (\frac{t}{\omega}-\frac{1}{2})\int_0^\omega F_2(s)ds \end{pmatrix}, \tag{7}$$

where

$$\begin{aligned} F_1(s) &= r_1(s) - \sum_{j=1}^m a_{1j}(s)exp\{u_1(s-\tau_{1j}(s))\} - \sum_{j=1}^m b_{1j}(s)exp\{u_2(s-\rho_{1j}(s))\}, \\ F_2(s) &= r_2(s) + \sum_{j=1}^m a_{2j}(s)exp\{u_1(s-\tau_{2j}(s))\} - \sum_{j=1}^m b_{2j}(s)exp\{u_2(s-\rho_{2j}(s))\}. \end{aligned}$$

Obviously, QN and $K_P(I-Q)N$ are continuous. Since X is a finite-dimensional Banach space, using the Ascoli-Arzela theorem, it is not difficult to show that $\overline{K_P(I-Q)N(\bar{\Omega})}$ is compact for any open bounded set $\Omega \subset X$. Moreover, $QN(\bar{\Omega})$ is bounded. Thus, N is L–compact on $\bar{\Omega}$ with any open bounded set $\Omega \subset X$.

Now we are at the point to search for an appropriate open, bounded subset Ω for the application of the continuation theorem. Corresponding to the operator equation $Ly = \lambda Ny, \lambda \in (0,1)$, we have

$$\begin{cases} \dot{u_1}(t) = \lambda\left[r_1(t) - \sum_{j=1}^m a_{1j}(t)exp\{u_1(t-\tau_{1j}(t))\} - \sum_{j=1}^m b_{1j}(t)exp\{u_2(t-\rho_{1j}(t))\}\right], \\ \dot{u_2}(t) = \lambda\left[r_2(t) + \sum_{j=1}^m a_{2j}(t)exp\{u_1(t-\tau_{2j}(t))\} - \sum_{j=1}^m b_{2j}(t)exp\{u_2(t-\rho_{2j}(t))\}\right]. \end{cases} \tag{8}$$

Suppose that $u(t) = (u_1(t), u_2(t))^T \in X$ is an arbitrary solution of system (8) for a certain $\lambda \in (0,1)$, integrating both sides of (8) over the interval $[0,\omega]$ with respect to t , we obtain

$$\begin{cases} \int_0^t \left[\sum_{j=1}^m a_{1j}(t)exp\{u_1(t-\tau_{1j}(t))\} + \sum_{j=1}^m b_{1j}(t)exp\{u_2(t-\rho_{1j}(t))\}\right]dt = \overline{r_1}\omega, \\ \int_0^t \left[\sum_{j=1}^m a_{2j}(t)exp\{u_1(t-\tau_{2j}(t))\} - \sum_{j=1}^m b_{2j}(t)exp\{u_2(t-\rho_{2j}(t))\}\right]dt = -\overline{r_2}\omega. \end{cases} \tag{9}$$

In view of the following:

$$\begin{aligned} &\int_0^\omega \sum_{j=1}^m a_{ij}(t)exp\{u_1(t-\tau_{ij}(t))\}dt \\ &= \int_{-\tau_{ij}(0)}^{\omega-\tau_{ij}(\omega)} \sum_{j=1}^m \frac{a_{ij}(\mu_{ij}(s))}{1-\tau'_{ij}(\mu_{ij}(s))} exp\{u_1(s)\}ds, \\ &= \int_{-\tau_{ij}(0)}^{\omega-\tau_{ij}(\omega)} \sum_{j=1}^m \frac{a_{ij}(\mu_{ij}(s))}{1-\tau'_{ij}(\mu_{ij}(s))} exp\{u_1(s)\}ds \\ &= \int_0^\omega \sum_{j=1}^m \frac{a_{ij}(\mu_{ij}(s))}{1-\tau'_{ij}(\mu_{ij}(s))} exp\{u_1(s)\}ds, (i=1,2; j=1,2,...,m), (10) \end{aligned}$$

$$\begin{aligned} &\int_0^\omega \sum_{j=1}^m b_{ij}(t)exp\{u_2(t-\rho_{ij}(t))\}dt \\ &= \int_{-\rho_{ij}(0)}^{\omega-\rho_{ij}(\omega)} \sum_{j=1}^m \frac{b_{ij}(\varepsilon_{ij}(s))}{1-\rho'_{ij}(\varepsilon_{ij}(s))} exp\{u_2(s)\}ds \\ &= \int_{-\rho_{ij}(0)}^{\omega-\rho_{ij}(\omega)} \sum_{j=1}^m \frac{b_{ij}(\varepsilon_{ij}(s))}{1-\rho'_{ij}(\varepsilon_{ij}(s))} exp\{u_2(s)\}ds \\ &= \int_0^\omega \sum_{j=1}^m \frac{b_{ij}(\varepsilon_{ij}(s))}{1-\rho'_{ij}(\varepsilon_{ij}(s))} exp\{u_2(s)\}ds, (i=1,2; j=1,2,...,m).(11) \end{aligned}$$

From (9) (10) (11), we can obtain

$$\int_0^{\omega}\sum_{j=1}^{m}\frac{a_{1j}(\mu_{1j}(s))}{1-\tau'_{1j}(\mu_{1j}(s))}exp\{u_1(s)\}ds+\int_0^{t}\sum_{j=1}^{m}\frac{b_{1j}(\varepsilon_{1j}(s))}{1-\rho'_{1j}(\varepsilon_{1j}(s))}exp\{u_2(s)\}ds=\overline{r_1}\omega, \tag{12}$$

$$\int_0^{\omega}\sum_{j=1}^{m}\frac{a_{2j}(\mu_{2j}(s))}{1-\tau'_{2j}(\mu_{2j}(s))}exp\{u_1(s)\}ds-\int_0^{\omega}\sum_{j=1}^{m}\frac{b_{2j}(\varepsilon_{2j}(s))}{1-\rho'_{2j}(\varepsilon_{2j}(s))}exp\{u_2(s)\}ds=-\overline{r_2}\omega. \tag{13}$$

By the mean value theorem for improper integral, there exist $\xi_{ikj}\in[0,\omega]$ $(i=1,2;k=1,2;j=1,2,...,m)$ such that

$$A_{11}\int_0^{\omega}exp\{u_1(s)\}ds+A_{12}\int_0^{\omega}exp\{u_2(s)\}ds=\overline{r_1}\omega, \tag{14}$$

$$A_{21}\int_0^{\omega}exp\{u_1(s)\}ds-A_{22}\int_0^{\omega}exp\{u_2(s)\}ds=-\overline{r_2}\omega, \tag{15}$$

where

$$A_{11}=\sum_{j=1}^{m}\frac{a_{1j}(\xi_{11j})\mu_{1j}(\xi_{11j})}{1-\tau'_{1j}(\mu_{1j}(\xi_{11j}))},$$

$$A_{12}=\sum_{j=1}^{m}\frac{b_{1j}(\xi_{12j})\varepsilon_{1j}(\xi_{12j})}{1-\rho'_{1j}(\varepsilon_{1j}(\xi_{12j}))},$$

$$A_{21}=\sum_{j=1}^{m}\frac{a_{2j}(\xi_{21j})\mu_{2j}(\xi_{21j})}{1-\tau'_{2j}(\mu_{2j}(\xi_{21j}))},$$

$$A_{22}=\sum_{j=1}^{m}\frac{b_{2j}(\xi_{22j})\varepsilon_{2j}(\xi_{22j})}{1-\rho'_{2j}\varepsilon_{2j}((\xi_{22j}))}.$$

Then,

$$\int_0^{\omega}exp\{u_1(s)\}ds=\frac{(A_{22}\overline{r_1}-A_{12}\overline{r_2})\omega}{A_{11}A_{22}+A_{21}A_{12}}, \tag{16}$$

$$\int_0^{\omega}exp\{u_2(s)\}ds=\frac{(A_{21}\overline{r_1}+A_{11}\overline{r_2})\omega}{A_{11}A_{22}+A_{21}A_{12}}. \tag{17}$$

So we have

$$\frac{(D_{22}^{L}\overline{r_1}^{L}-D_{12}^{M}\overline{r_2}^{M})\omega}{D_{11}^{M}D_{22}^{M}+D_{21}^{M}D_{12}^{M}}\le\int_0^{\omega}exp\{u_1(s)\}ds\le\frac{(D_{22}^{M}\overline{r_1}^{M}-D_{12}^{L}\overline{r_2}^{L})\omega}{D_{11}^{L}D_{22}^{L}+D_{21}^{L}D_{12}^{L}}, \tag{18}$$

$$\frac{(D_{21}^{L}\overline{r_1}^{L}+D_{11}^{M}\overline{r_2}^{M})\omega}{D_{11}^{M}D_{22}^{M}+D_{21}^{M}D_{12}^{M}}\le\int_0^{\omega}exp\{u_2(s)\}ds\le\frac{(D_{21}^{M}\overline{r_1}^{M}+D_{11}^{L}\overline{r_2}^{L})\omega}{D_{11}^{L}D_{22}^{L}+D_{21}^{L}D_{12}^{L}}, \tag{19}$$

where

$$D_{11}=\sum_{j=1}^{m}\frac{a_{1j}(t)\mu_{1j}(t)}{1-\tau'_{1j}(\mu_{1j}(t))},$$

$$D_{12}=\sum_{j=1}^{m}\frac{b_{1j}(t)\varepsilon_{1j}(t)}{1-\rho'_{1j}(\varepsilon_{1j}(t))},$$

$$D_{21}=\sum_{j=1}^{m}\frac{a_{2j}(t)\mu_{2j}(t)}{1-\tau'_{2j}(\mu_{2j}(t))},$$

$$D_{22}=\sum_{j=1}^{m}\frac{b_{2j}(t)\varepsilon_{2j}(t)}{1-\rho'_{2j}(\varepsilon_{2j}(t))}.$$

By the condition of theorem 3.1, there exist $t_i\in[0,\omega], i=1,2$ such that

$$u_1(t_1)=\ln\left[\frac{(A_{22}\overline{r_1}-A_{12}\overline{r_2})\omega}{A_{11}A_{22}+A_{21}A_{12}}\right], \tag{20}$$

$$u_2(t_2)=\ln\left[\frac{(A_{21}\overline{r_1}+A_{11}\overline{r_2})\omega}{A_{11}A_{22}+A_{21}A_{12}}\right]. \tag{21}$$

By the condition (H_4) of theorem 3.1, there exist $B_1, B_2 > 0$ such that

$$|u_1(t_1)| \le B_1, |u_2(t_2)| \le B_2. \tag{22}$$

In view of the following:

$$\begin{aligned}
\int_0^\omega |\dot{u}_1(t)|dt &= \lambda \int_0^\omega \left|\left[r_1(t) - \sum_{j=1}^m a_{1j}(t)exp\{u_1(t-\tau_{1j}(t))\}\right.\right. \\
&\quad \left.\left. - \sum_{j=1}^m b_{1j}(t)exp\{u_2(t-\rho_{1j}(t))\}\right]\right| dt \\
&\le 2\int_0^\omega |r_1(t)|dt = 2\int_0^\omega r_1(t)dt = 2\overline{r_1}\omega := B_3,
\end{aligned} \tag{23}$$

$$\begin{aligned}
\int_0^\omega |\dot{u}_2(t)|dt &= \lambda \int_0^\omega \left|\left[r_2(t) + \sum_{j=1}^m a_{2j}(t)exp\{u_1(t-\tau_{2j}(t))\}\right.\right. \\
&\quad \left.\left. - \sum_{j=1}^m b_{2j}(t)exp\{u_2(t-\rho_{2j}(t))\}\right]\right| dt \\
&\le \overline{r_2}\omega + \int_0^\omega \left|\sum_{j=1}^m a_{2j}(t)exp\{u_1(t-\tau_{2j}(t))\}\right| dt \\
&\quad + \int_0^\omega \left|\sum_{j=1}^m b_{2j}(t)exp\{u_2(t-\rho_{2j}(t))\}\right| dt \\
&\le \overline{r_2}\omega + \sum_{j=1}^m \frac{a_{2j}(\mu_{2j}(s))\omega}{1-\mu'_{2j}(\mu_{2j}(s))} \int_0^\omega exp\{u_1(s)\}ds \\
&\quad + \sum_{j=1}^m \frac{b_{2j}(\varepsilon_{2j}(s))\omega}{1-\rho'_{2j}(\varepsilon_{2j}(s))} \int_0^\omega exp\{u_2(s)\}ds. \\
&\le \overline{r_2}\omega + \sum_{j=1}^m \frac{a_{2j}(\mu_{2j}(s))\omega}{1-\mu'_{2j}(\mu_{2j}(s))} \frac{(A_{22}^M\overline{r_1}^M - A_{12}^L\overline{r_2}^L)\omega}{A_{11}^L A_{22}^L + A_{21}^L A_{12}^L} \\
&\quad + \sum_{j=1}^m \frac{b_{2j}(\varepsilon_{2j}(s))\omega}{1-\rho'_{2j}(\varepsilon_{2j}(s))} \frac{(A_{21}^M\overline{r_1}^M + A_{11}^L\overline{r_2}^L)\omega}{A_{11}^L A_{22}^L + A_{21}^L A_{12}^L} := B_4,
\end{aligned} \tag{24}$$

then it follows from (22) (23) (24) that

$$|u_1(t)| \le |u_1(t_1)| + \int_0^\omega |\dot{u}_1(t)|dt \le B_1 + B_3 := B_5, \tag{25}$$

$$|u_2(t)| \le |u_2(t_2)| + \int_0^\omega |\dot{u}_2(t)|dt \le B_2 + B_4 := B_6. \tag{26}$$

Obviously, B_1, B_2, B_3, B_4 are independent of $\lambda \in (0,1)$. By the condition (H_5) of Theorem 3.1, it is easy to show that the algebraic equations

$$\begin{cases} \sum_{i=1}^m \overline{a_{1j}}u_1 + \sum_{i=1}^m \overline{b_{1j}}u_2 = \overline{r_1}, \\ \sum_{i=1}^m \overline{a_{2j}}u_1 - \sum_{i=1}^m \overline{b_{2j}}u_2 = -\overline{r_2}, \end{cases} \tag{27}$$

has a unique positive solution $(\tilde{u_1}, \tilde{u_2})^T \in R^2$. Take $M = max\{B_5, B_6\} + B_0$, where B_0 is taken sufficiently large such that the unique positive solution $(\tilde{u}_1, \tilde{u}_2)^T \in R^2$ satisfies $\max_{t\in[0,\omega]} |\tilde{u}_1| + \max_{t\in[0,\omega]} |\tilde{u}_2| < B_0$,

Let $\Omega := \{u = \{u(t)\} \in X : \|u\| < M\}$, then it is easy to see that Ω is an open, bounded set in X and verifies requirement (a) of Lemma 2.1. When $(u_1(t), u_2(t))^T \in \partial\Omega \cap KerL = \partial\Omega \cap R^2, u = \{(u_1, u_2)^T\}$ is a constant vector in R^2 with $\|u\| =$

$\|(u_1(t), u_2(t))^T\| = \max_{t\in[0,\omega]} |u_1(t)| + \max_{t\in[0,\omega]} |u_2(t)| = M$. Then we have

$$QNy = \begin{pmatrix} \overline{r_1} - \sum_{i=1}^{m} \overline{a_{1j}} u_1 - \sum_{i=1}^{m} \overline{b_{1j}} u_2 \\ \overline{r_2} + \sum_{i=1}^{m} \overline{a_{2j}} u_1 - \sum_{i=1}^{m} \overline{b_{2j}} u_2 \end{pmatrix} \neq 0. \tag{28}$$

Letting J be the identity mapping and by direct calculation, we get

$$\begin{aligned} & deg\left\{JQN(u_1, u_2)^T; \partial\Omega \bigcap kerL; 0\right\} \\ = \;& deg\left\{QN(u_1, u_2)^T; \partial\Omega \bigcap kerL; 0\right\} \\ = \;& sign\left\{det\begin{pmatrix} -\sum_{i=1}^{m} \overline{a_{1j}} & -\sum_{i=1}^{m} \overline{b_{1j}} \\ \sum_{i=1}^{m} \overline{a_{2j}} & -\sum_{i=1}^{m} \overline{b_{2j}} \end{pmatrix}\right\} \\ = \;& sign\left\{\sum_{i=1}^{m} \overline{a_{1j}} \sum_{i=1}^{m} \overline{b_{2j}} + \sum_{i=1}^{m} \overline{a_{2j}} \sum_{i=1}^{m} \overline{b_{1j}}\right\} = 1 \neq 0. \end{aligned}$$

This proves that condition (b) in Lemma 2.1 is satisfied. By now, we have proved that Ω verifies all requirements of Lemma 2.1, then it follows that $Lu = Nu$ has at least one solution $(u_1(t), u_2(t))^T$ in $DomL \cap \overline{\Omega}$, that is to say, (5) has at least one ω periodic solution in $DomL \cap \overline{\Omega}$. Then we know that $((x(t), y(t))^T = (exp\{u_1(t)\}, exp\{u_2(t)\})^T$ is an ω periodic solution of system (2) (3) with strictly positive components. This completes the proof.

Remark 3.1. Theorem 3.1 remains valid if some or all terms are replaced by corresponding terms with discrete time delays, distribute delays (finite or infinite), or deviating arguments respectively. At this point, we would like to point out that, when one applies the continuation theorem from the coincidence degree theory to explore the existence of periodic solutions to the system of differential equations or difference equations, time delays of any type or the deviating arguments have no effect on the existence of positive solutions.

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Applications of Maximal Network Flow Problems in Transportation and Assignment Problems

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Abstract

This paper presents some modifications of Ford-Fulkerson's labeling method for solving the maximal network flow problem with application in solving the transportation and assignment problems. The modifications involve the tree representation of the nodes labeled and the edges used them. It is shown that after each flow adjustment some of the labels can be retained for the next labeling process. Through certain computational aspects it has been suggested that to indicate that with theses the primal-dual approach for solving the transportation and assignment problems is improved to certain extent.

Keywords: Maximal network flow, Labeling process, Transportation & assignment problems

Mathematics Subject Classification 2000: 76M15, 76M35, 60G20.

1. Introduction

New solution techniques have been suggested to solve transportation, mainly in the tree representation of a basis, location of a better adjacent basis, and labeling and relabeling of trees (Glover, 1982; Klingman, 1983). In this paper some of theses ideas are incorporated into the primal-dual approach to the transportation and assignment problems. Each label computed in the labeling method of Ford-Fulkerson (Ford, L.R., 1962) for the maximum flow problems consists of two element. The second element is used for flow augmentation. After each flow adjustment, all labels are discarded and the labeling process starts from the source again. At each labeling process the nodes labeled and the edges used to label them are represented by a forest on the transportation network. It is shown that by using either the predecessor and distance labels (Singh, Vinai K., 2007) or the triple labels (Johnoson, E.I., 1986) on the nodes of the tree, several of the labeled nodes can be retained as "labeled and unscanned" nodes for the next labeling process.

Based on the Ford-Fulkerson's primal- dual approach for solving the transportation problems, it is found the primal-dual method can be improved considerably making it once again competitive with the MODI method for solving the transportation problems. For the assignment problem our computational experience with these modifications has been very encouraging and since the primal-dual approach is similar in the spirit to Hungarian Method, it appears that we have accelerated the Hungarian Method. It should be pointed out that we have not in any sense attempted to "optimize" our codes. Presumably, the computational times can be reduced by appropriate modifications to the codes.

There are a variety of other problems for which the ideas developed in this paper can be useful. For instance; algorithms for solving the Bottleneck Assignment and Bottleneck Transportation problems (Garfinkel, R.S., 1981) involve the solution of maximal flow problems.

2. Definition and notations

Let G =(N, A) be a finite directed network with N representing the node set and A the arc set Let x $=w_0$, w_1, ..., w_k = y be a sequence of distinct nodes having the property that either (w_{j-1}, w_j) ∈A or(w_j, w_{j-1}) ∈A for j=1,2, ... ,k. Singling out, for each i, one of theses two possibilities, the resulting sequence of nodes is called a path from x to y. Arcs (w $_{j-1}$, w

$_j$) that belong to the path are referred to as forward arcs while arcs (w $_{j+1}$, w $_j$)belonging to the same path referred to as reverse arcs. A cycle is a path from a node to itself. A tree in G = (N,A) is any set of | N | - 1 arcs without cycles. Each tree is pictured vertically in the plane and extending downwards with the highest node as root. In such a rooted tree, for each node x, a distance label d(x) and the triple- label [consisting of predecessor P(x), right- neighbour B(x), and leftmost successor L(x)] can be defined.

3. Algorithms for Solving the Maximal Flow Problem

Let G =(N,A) be a directed network, where each (x, y) ∈ A has associated with it as non negative number c(x,y) which is the capacity of the arc (x, y). The maximal flow problem from source to sink can be stated as:

$$\text{Max } Q = \sum_{y\in\alpha(x)} f(x,y) - \sum_{y\in\beta(x)} f(y,x) = \begin{cases} Q & if \quad x = s \\ 0 & if \quad x \neq s,t \\ -Q & if \quad x = t \end{cases}$$

0≤f(x,y)≤c(x,y), for all (x,y) ∈A

Where s= source node of he network.

t =sink node of the network.

f (x,y)=flow in arc (x,y)

α(x) ={ y ∈ N : (x,y)∈ A}

β(x)={y ∈ N : (y,x)∈ A}

We define the following characterization of the structure of the labeled nodes and the edges used to label them in a particular application of the labeling process (i.e. routine A):

N_k= {y_1,y_2,......,y_q} as the set of nodes labeled on the kth application of the labeling

process.

[Note that s ∈ N $_k$. further, t ∈ N $_k$ if there is a breakthrough.]

E $_k$ ={(y $_{j-1}$, y $_j$): y$_{j-1}$, y $_j$ ∈ N $_k$, P(y $_j$)= y $_{j-1}$ and either (y $_{j-1}$, y $_j$) ∈ A or (y $_j$, y $_{j-1}$) ∈A}

The connected graph (N_k, E_k) is obviously a tree. We define the rooted tree T_k=(N_k, E_k) as the tree formed by the graph $(N_k, E_k$) with s as the root.

Let the flow augmenting path obtained by the kth labeling process be represented by

$$F_k = \{s = y_1,\ (y_1,\ y_2),,\ y_2,\ (y_2,\ y_3),\ y_3.,......, y_{n-1},\ (y_{n-1},\ y_n),\ y_n = t\}$$

Where (y$_i$, y$_{i+1}$) ∈ E $_k$ for i= 1, 2,. . . . . . , n-1.

Define $A_k^+ = \{set\ of\ forward\ arcs\ in\ F_k\}$

$A_k^- = \{set\ of\ reverse\ arcs\ in\ F_k\}$

After the kth flow adjustment, there exists at least one (y $_{j-1}$, y $_j$) ∈ F_k such that either:

(i) f (y $_{j-1}$, y $_j$)=c(y $_{j-1}$, y $_j$)for (y $_{j-1}$, y $_j$) ∈ A_k^+

Or (ii) f (y $_j$, y $_{j-1}$)=0for (y $_j$, y $_{j-1}$) ∈ A_k^-

Let J_k = {(y $_{j-1}$, y $_j$): (y$_{j-1}$, y $_j$)a binding edge after he kth flow adjustment}

A backtracking algorithm 2 is given below for flow adjustment at a breakthrough. This algorithm does not use the second element (E) of the label which is computed in Ford-Fulkerson's routine A. The algorithm also identifies the set J_k which is later utilized to identify those nodes whose labels remain valid even after the flow adjustment. One computational experience discussed later in the paper and the results given in (Barr, R., F. Glover, 1982) indicate that there is some computational saving in using a backtracking procedure for flow adjustment instead of the∈ function. Furthermore, we note that the improved labeling algorithms 2 and 3 become computationally less attractive if the ∈ function of Ford-Fulkerson is utilized instead of the backtracking procedure.

Algorithm 1: for flow adjustment along F_k using a backtracking routine.

Step 0: Breakthrough occurs in the kth labeling process.

Step 1 (Backtrack): Let $\epsilon = \{\epsilon_1, \epsilon_2\}$

$$\epsilon_1 = Min\{c(x,y) - f(x,y)$$

$$(x, y) \in A_k^+$$

$$\epsilon_2 = Min\{f(x, y)\}$$

$$(x, y) \in A_k^-$$

Step 2 (Flow Augmentation): Let $J_k = \Phi$

(a) Let f(x, y)=f(x, y)+ ϵ for (x, y) $\in A_k^+$

if f ‘ (x, y)=c(x, y), then setting $J_k = J_k \cup$(x,y)

(b) Let f’ (y, x) =f(y, x) - ϵ for (y,x) $\in A_k^-$

if f’(y,x)=0, then setting $J_k = J_k \cup$ (y, x).

The adjusted flows are represented by f’(x, y). Stop

It will now be shown that with the use of either the predecessor – distance labels or the triple – labels, some of the labeled nodes can be retained for the next labeling process.

Lemma 1: after the k th breakthrough and flow adjustment let M = {$m_1, m_2, \ldots, m_l$} be the set of labeled nodes such that d (m_1) ≤d (x $_b$) for i= 1,2,. . . ..,l where d (x $_b$) = Min [d(x $_j$)]

$$(x_i,\ y_i) \in J_k$$

The predecessor and distance labels of the node $m_1 \in$ M remain valid and these nodes can be retained as labeled and unscanned for the next labeling process.

Proof: After the k th flow adjustment, all edges $(m_{j-1}, m_j) \in E_k$with

d (m_j) ≤d (x_b) have flows f(m_{j-1}, m_j) or f(m_j, m_{j-1})

such that f(m_{j-1}, m_j) < c(m_{j-1}, m_j) if m_j is labeled m_{j-1}^+

or f(m_{j-1}, m_j) > 0 if m_j is labeled m_{j-1}^-

This is true since $(m_{j-1}, m_j) \notin J_k$. On the next application of the labeling process, node m_j can be labeled from node m_{j-1} again. Since s∈ M, all m_i ∈M can be labeled in the same sequence again. The labels, therefore, remain valid, and the nodes m_i ∈M can be retained in the labeled and unscanned state for the next labeling process.

Corollary to Lemma 1: After the kth breakthrough and flow adjustment, if P(t)=x_b, the labels and the status (scanned or unscanned) of each labeled node except nodes x_b and t can be retained for the next labeling process. Nodex_b is set to the labeled and unscanned state while t is set to the unlabeled state.

Lemma 2: After the kth breakthrough and flow adjustment, we need to discard only the triple –label of the nodes in the tree T_k^* = (N_k^*, E_k^*) with root y_b, where y_b is such that

$$d(y_b) = Min[d(x_i)]$$

$$(x_i\ ,\ y_i) \in J_k$$

In addition, if L $(x_b) = y_b$ we have to set L(x_b) =B (y_b) , where node x_b is such that P$(y_b) = x_b$. Otherwise we have to set B(u)=B(y_b), where node u is such that B(u)= y_b.

Proof: The proof of the first part of lemma is similar to proof of lemma 1. The second part of the lemma then follows from the fact that the subtree with $y_b$as root has to be disconnected from the tree with s as the root.

Using the results of Lemmas 1 and 2, we propose the following improved labeling algorithms from solving the maximal flow problem.

Algorithm 2: Improved labeling algorithm for solving the maximal flow problem using the predecessor and distance functions.

Step 0: (i) At the start all nodes are in the unlabeled state

(ii) Node s receives labels (Φ, 0) (s is now labeled and unscanned).

Step 1 (labeling Process): Select any labeled unscanned node x with label ($z^{\pm}$, h). To each unlabeled node y such that f(x,y) < c(x,y), assign the label ($x^{\pm}$, h+1). To each unlabeled node y such that f(y,x)>0, assign the label (x^{-}, h+1) [y is now labeled and unscanned, x is labeled and scanned].Repeat until t is labeled [breakthrough] or until no more labels can be assigned and t is unlabeled. In the breakthrough case go to step 2, in the latter case, stop.

Step 2 (Flow Change):

(i) Apply Algorithm 1.

(ii) Let d(y_b)=Min [d(x_i)]. If P(t)= x_b, go to Step 2(iii); otherwise go to step 2(iv).

$$(x_i, y_i) \in J_k$$

(iii) Set $x_b$to the labeled and unscanned state. Erase the label on t. Go to step 1.

(iv) Set all labeled nodes y_j such that d(y_j) ≤ d(x_b) to the labeled and unscanned state and erase the labels on the other labeled nodes. Go to step 1.

Algorithm 3: Improved labeling algorithm for the maximal flow problem using the triple labels.

Step 0: (i) At the start all nodes are in the unlabeled state.

(ii) Node s receives the labels (Φ, 0) (s is now labeled and unscanned).

Step 1 (Labeling Process):

Select any labeled unscanned node x with label ($x^{\pm}$, B(x), L(x)). To each unlabeled node y such that f(x, y) < c(x, y), assign the label (x^{+}, B(y), L(y)) where B(y) = L(x) and L(y)=Φ Set L(x) =y. To each unlabeled node y such that f(x,y)>0, assign the label (B(y), L(y)) where B(y)=L(x) and L(y)=Φ [x is now labeled and unscanned, y is labeled and unscanned]. Repeat until t is labeled [breakthrough] or until no more labels can be assigned and t is unlabeled. In the breakthrough case, go to step 2; in the latter case, stop.

Step 2 (Flow Change):

(i) Apply algorithm 1.

(ii) Let d (y_b)=Min [d(y_i)].Let x = P (y_b). If L(x) = B (y_b)

$$(x_i, y_i) \in J_k$$

and go to step 2 (iv). Otherwise go to Step 2 (iii).

(iii) Let node u be such that B (u) = y_b. Let B(u) and go to Step 2 (iv).

(iv) Disconnect the subtree with y_b as root from the tree with s as root. Erase all labels on the subtree with y_b as root.

(v) Set all remaining labeled nodes to labeled and unscanned state; go to Step 1.

The predecessor- distance and triple-label algorithms use the same criterion as Ford-Fulkerson's procedure in the search for flow augmenting paths. Optimality of solution at termination and finiteness of the algorithms for integers capacity functions are not in question. The main computational advantage of the predecessor- distance and triple –label algorithms is that after each flow adjustment, the labels that remain valid after a flow adjustment, it requires the use of one more label than the predecessor-distance algorithm.

4. Application To Transportation And Assignment Problems

The transportation problem (Ford, 1962) can be stated as:

$$\min Z = \sum_{i \in U} \sum_{j \in V} c_{ij} x_{ij}$$

subject to $\sum_{j \in V} x_{ij} = a_i$ i ∈ U =1, 2, ..., m }, set of rows

$\sum_{i \in U} x_{ij} = b_j$ j ∈ V = {1, 2, ..., n }, set of columns

$x_{ij} \geq 0$, for all i∈ U and j∈V

with $\sum_{i \in U} a_i = \sum_{j \in V} b_j$

The network(N,A) representation of this problem is a bipartite graph with i∈ U , j ∈V forming the two sets of nodes and arcs (i, j) connecting each i ∈ U to each j∈V.

The dual to the transportation problem maybe stated as:

Max Z' $= \sum_{i\in U} a_i u_i + \sum_{j\in V} b_j v_j$

subject to $u_i + v_j \leq c_{ij}$, for all $i \in$U,j ∈ V

The primal- dual algorithm (Florian, M., 1990) solves a sequence of restricted primal problems. Each such problem is a maximum flow problem on a subset of the extended transportation network (N*, A*) which is formed from (N,A) by the addition of a source s*, sink t*, arcs (s*, i) with c (s*, i) =a_i and arcs(j, t*) with c(j, t*)= b_j. Thus it is clear that either Algorithm 2 or 3 can be used for solving the maximum flow problems. However, since the transportation network is a special type of multi-source, muti-sink network, and alternative algorithm is computationally interesting. This algorithm (predecessor- root algorithm) is especially suitable and computationally very efficient for the assignment problem.

Definition and Notation: Let x^*_{ij} represent a flow such that $0 \leq a^*_i = a_i - \sum_{j\in V} x^*_{ij}$ and $0\leq b^*_j = b_j - \sum_{i\in U} x^*_{ij}$. A row i with $a^*_i > 0$ is defined as a source row i.

It is clear that a source node i, together with its descendant labeled nodes and the edges used to label them, forms a tree T^j_k =(N^j_k, E^j_k). The source rows are not connected to each other by labeling. Thus, the labeled nodes (rows and columns), together with the edges used to label them, form a forest F $_k = \{T^1_k, T^2_k, ..., T^q_k\}$.

Lemma 3: if on the kth application of the labeling process the flow augmenting path k is formed by edges from the treeT^j_k=(N^j_k, E^j_k)then after the flow adjustment is sufficient to discard the labels only on nodes $y_r \in N^j_k$. The predecessor and root labels on the other labeled nodes (i.e$y_r \in \bigcup_{i=1}^{q} N^i_k, i \neq$ j) remain valid, and the nodes can be retained as labeled and unscanned nodes for the next labeling process.

Proof: The flow in any edge $(y_{r-1}, y_r) \in \bigcup_{i=1}^{q} E^i_k(i \neq$j) is not change at the kth flow adjustment. Therefore at the next labeling process all nodes $y_r \in \bigcup_{i=1}^{q} N^i_k(i \neq$j) can be labeled in the same sequence again; they can therefore be retained as labeled and unscanned nodes.

Lemma 4: Suppose that at the k th breakthrough a particular column g, say, with b^*_g >0 and F_k have been detected in the tree T^j_k=(N^j_k,E^j_k)with source row i as root. If Min [a^*_i, e(g)]> b^*_g, then the predecessor and root labels on nodes $y_r \in N^j_k$ remain valid for the next labeling process. Therefore, the labels and the status (scanned or unscanned) of all $y_r \in \bigcup_{j=1}^{q} N^j_k$ can be retained for the next labeling process.

Proof: All edges$(y_{r-1}, y_r) \in E^j_k$ are non- binding. Therefore, each $y_r \in N^j_k$ can be labeled in the same sequence again. Furthermore, from a labeled and scanned node, no further labeling is possible. Hence the labels and status of all $y_r \in \bigcup_{j=1}^{q} N^j_k$ can be retained for the next labeling process.

We propose the following predecessor-root labeling algorithm, based on the results of Lemmas 3 and 4.

Algorithm 4: Predecessor- root labeling algorithm for solving the transportation problem.

Step 0: Begin with any dual feasible solution (u*,v*). Let x^*_{ij} = 0, for all (i, j). At the start all i ∈U and j ∈ V are in the unlabeled state.

Step 1 (Labeling Process): Each unlabeled source row i with a^*_i >0 is labeled as (j, R(i)) where R(i) = i. Next select a labeled row, say row I, and scan it for all unlabeled columns such that $c_{ij} - u_i - v_j$ =0; label these columns as (i, R (j)) where R(j)= R(i). Repeat until all labeled rows have been scanned. The select labeled column, say column j, and scan it for all unlabeled rows i such that x_{ij} >0 ; label these rows as (j, R(i)) where R(i)=R(j). Repeat until all labeled columns have been scanned. Now revert to row scanning of new labeled rows and then the column and so on. If a column l with $b^*_l > 0$ is labeled (breakthrough) go to step 2. Otherwise continue until no more labels can be assigned (non-breakthrough) and go to step 3.

Step 2 (Flow Adjustment):

(i) Set s = R(l). Backtrack along the detected flow augmenting path F_k from s to l and determine e(l) = Min [x^*_{ij} (i, j)∈ A^*_k] where A^+_k and A^-_k are as defined in section 3.

(ii) If Min[e(l), a^*_s]> b^*_l, set e(l) and go to step 2(iii).Otherwise set e(l)= Min [e(l), a^*_s] and go to Step 2 (iv)

(iii) Let $x^*_{ij} = x^*_{ij} + \varepsilon(l); for (i, j) \in A^+_k$

$$x^*_{ij} = x^*_{ij} - \varepsilon(l); for (i, j) \in A^-_k$$

$a^*_s = a^*_s - \varepsilon(l)$ and $b^*_l = b^*_l - \varepsilon$(l)

Retain all labels and go to step 1.

(iv)Adjust flows as in Step 2 (iii) above. $\sum_{i\in U}\sum_{j\in V} x^*_{ij} = \sum_{j\in V} b_j$, stop.

Otherwise erase the labels of all labeled rows i and labeled columns j such that R(i) =R(j)=s.Go to Step 1.

Step 3 (Dual variable change):

1. Let I and J be the index set of labeled rows and columns and

$$I^* = U - I, J^* = V - J,$$

Set $u_i = \begin{cases} u_i + \delta & i \in I \\ u_i & i \in I^* \end{cases}$

$$v_j = \begin{cases} v_j - \delta & j \in J \\ v_j & j \in J^* \end{cases}$$

Where $\delta = \min_{i,j} (c_{ij} - u_i - v_j)$

Go to Step 1 with the new dual feasible solution.

After each flow change, the labels discarded by the predecessor-root method include some labels that can be retained. If the triple – label algorithm discussed in the previous section is adopted for the primal-dual approach to the transportation problem, all valid labels after a flow change would be retained. However, the triple –label algorithm require the use of one more element in the label then the predecessor-root algorithm.

The predecessor – root algorithm specializes very well to accelerate the primal- dual method for solving the Assignment Problem (Ford, 1962).

For this problem since the a_i^*'s and b_j^*'s are equal to one, at each breakthrough, the net availability at the root of the treeT_k^j= (N_k^j,E_k^j) which contains F_k, becomes zero after flow adjustment. Therefore, all the labels on nodes $y_r \in N_k^j$ have to be discarded. In the next section we present some computational results using the predecessor- root algorithm to solve the assignment Problem.

5. Computational Results

In this section we discuss some computational experience with the application of the predecessor-root algorithm to the transportation and assignment problems and also the triple label algorithm applied to the transportation problem.

In table 1, the mean solution times as a function of problem size is presented for comparison, the rim and cost parameters were kept identical of those tested in (Singh, 2007). Our mean solution times indicate that the modified primal-dual method is computationally as attractive as the primal method. Approximate mean solution times with the 1971 code 100 ×100 and 150×150 transportation problems reported in (Singh, 2007) are 1.9 and 4.81 seconds respectively compared to our mean solution times of 6.56 and 14.55 seconds respectively. For 100×100 transportation problems the range of solution times with our code was4.3 seconds to 8.5 seconds. In table 1, we also analyzed the breakdown of mean solution times for the transportation problems. In spite of our modifications about 70% of the mean solution time was still taken up in the labeling process. This explains why our attempts in reducing the number of labels to be discarded after each flow change had been effective in reducing the mean solution time. Since such a large proportion of the mean solution time is spent in the labeling process, any significant reduction in scanning during the labeling process would reduce the mean solution time substantially. Lemma 4 was found to be very effective in the sense. For example, in the 100×100 transportation problem it was found that out of 250 average number of breakthroughs, about 100(or about 40%) of them did not require the discarding of any labels.

We tested the effect, on the mean solution time, of the variation in range of the parameters of the transportation problem. We found that mean solution time was relatively independent of the range of the rim parameters, but dependent on the range of the cost parameters. This can be explained by the fact that for problems with a large range in the cost parameter, an initial dual feasible solution would probable not have a large number of admissible cells. Also, at each dual variable change, only a few new cells would probably become admissible. Therefore, a large number of iterations is required before optimality is attained. This fact is brought out in table 2. This suggests that for transportation problems with a large range in cost parameter, the maximum dual change rule as suggested in (Ford, L.R., 1962) may be more efficient than Ford-Fulkerson's routine C.

Table 3(a) gives the mean solution times, using the predecessor- root algorithm, for solving the assignment problem. The range for was maintained as integers between 0 and 50 in the order to compare with (Florian, M., 1990). For 50×50 and 100 ×100 problems, our mean solution times are 0.47 and 1.74 seconds respectively compared to 0.51 and 2.0 seconds, respectively as respectively as reported in (Florian, M., 1990). For 100 × 100 assignment problems the range of solution times with our code was 1.43 to 2.0 seconds. The computational saving was especially significant for large problems. For the 100×100 and 175×175 assignment problems [c_{ij} integers in the range 0 -99] our mean solution times are 2.01 and 5.85 seconds respectively as shown in Table 3(b).

Table 4 gives the effect, on the mean solution time, of the variation in range of the cost parameter for the assignment problem. Again we find that the number of nonbreakthrough and the mean solution times are dependent on the number of significant digits in the cost parameter. This again suggests that even for the assignment problem with a large number of significant digits in the cost parameter, the maximum dual change rule may be more efficient computationally then Ford-Fulkerson's routine C.

< Table 1 >

All solution times are exclusive of input/output and based on 10 randomly generated problems, each solved by using QSB_+ Computer Software. The a_i and b_j are integers drawn from a uniform distribution between 1 and 99 and the $c_{ij}$are integers drawn from a uniform distribution between 0 and 99.

< Table 2 >

Note: NBT =Nonbreakthroughs

DVC = Time for Dual Variable Change

MST =Mean Solution Time

All solution times are exclusive of input/output and based on 10 randomly generated problems, each solved by using QSB_+ Computer Software. The parameters are integers drawn from a distribution between the ranges specified.

< Table 3(a) >

All Solution times are exclusive of input/output and based on 10 randomly generated problems, each solved on by using QSB_+ Computer Software. The $c_{ij}$are integers drawn from a uniform distribution between 0 and 50.

< Table 3(b) >

All solution times are exclusive of input/output and based on 10 randomly generated problem, each solved by using QSB_+ Computer Software. The c_{ij}re integers drawn from a uniform distribution between 0 and 99.

< Table 4 >

Note: DVC = Dual Variable Change

All solution times are exclusive of input/output and based on 10 randomly generated problems each solved on by using QSB_+ Computer software. The c_{ij} are integers drawn from a uniform distribution between the ranges specified.

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Table 1. Breakthrough of Mean Solution Time for solving the Transportation Problem as a Function of Problem Size

Problem Size	Average Number of nonbreak-Throughs	Average Number of break-throughs	Time for initialization (Sec.)	Time for labeling (Sec.)	Time for flow augmentation (Sec)	Time for dual vaiable Change (Sec.)	Mean solution time (Sec.)
30× 30	19	79	0.02	0.50	0.08	0.10	0.70
50×50	15	125	0.13	1.23	0.25	0.37	1.98
80×80	10	182	0.33	2.99	0.49	0.58	4.39
100×100	10	250	0.52	4.52	0.74	0.78	6.56
130×130	8	331	0.86	8.49	1.13	1.00	11.48
150×150	6	376	1.13	10.93	1.25	1.24	14.55
175×175	7	452	1.53	17.33	2.04	1.55	22.45

Table 2. Effect on the Mean Solution Time, of the Variation in range of the Parameters of the Transportation Problem (Size 100 x100)

Range for c_{ij}	0 -9	0 – 99	0 -999	0 -9999
Range fora_i and b_j	Average DVC % MST Number MST (Sec.) Of NBT	Average DVC % MST Number MST (Sec.) Of NBT	Average DVC % MST Number MST (Sec.) Of NBT	Average DVC % MST Number MST (Sec.) Of NBT
1-9	0 1.0% 0.96	10 16% 4.99	62 44.5% 13.97	138 58.0% 28.16
1-99	1 1.5% 1.36	10 12% 6.56	69 35.5% 18.27	132 48.5% 28.51
1-999	1 1.6% 1.29	10 11.6% 6.85	68 35.0% 18.07	132 46.5% 29.87

Table 3(a). Breakdown of Mean Solution Time for Solving The Assignment Problem as a Function of Problem Size

Average problem size	Time for number of nonbreak-throughs	Initialization (Sec.)	Time for labeling (Sec.)	Time for flow augmentation (Sec.)	Mean dual variable Change (Sec.)	Solution time (Sec.)
30×30	5	0.02	0.10	0.08	0.04	0.24
50×50	4	0.05	0.16	0.18	0.08	0.47
80×80	3	0.10	0.68	0.20	0.15	1.13
100×100	3	0.16	1.08	0.26	0.24	1.74

Table 3 (b). Breakthrough of Mean Solution Time for Solving the assignment Problem as a Function of Problem Size

Average Problem Size	Time for number of nonbreak-throughs	Initialization (Sec.)	Time for labeling (Sec.)	Time for flow augmentation (Sec.)	Mean dual variable Change (Sec.)	Solution time (Sec.)
100×100	5	0.17	1.20	0.20	0.44	2.01
130×130	4	0.29	2.15	0.25	0.61	3.30
150×150	3	0.36	2.74	0.35	0.64	4.09
175x175	3	0.49	3.83	0.47	0.79	5.58

Table 4. Effect on Mean Solution Time of the Variation in Range of the Cost Parameters of the Assignment Problem (Size 100x100)

Range for c_{ij}	Average number of non-breakthrough	Time for DVC Mean Solution Time	Mean Solution Time (Sec.)
0- 9	0	1.0%	0.69
0-99	5	21.9%	2.01
0-999	40	57.0%	7.13
0-9999	86	68.5%	13.34

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Some Geometric Transformations on New Trees

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Abstract

In this paper, we will introduce the tree of the new nested graph.The regular and irregular trees will be defined.

The effects of the folding and retraction on it are deduced. Some geometric transformations of the new tree into itself are discussed. The incidence and adjacent matrices of the new graph after and before these transformations are achieved.

Keywords: Graph, Tree

1. Introduction

A tree is a connected graph that contains no cycles. Trees sprout up as effective models in a wide variety of applications. Every connected graph G admits a spanning tree, which is a tree that contains every vertex of G and whose edges are edges of G. Every connected graph even admits a normal spanning tree.

In graph theory, tree is a connected cyclic graph. A cyclic graph which is not necessarily connected is sometimes called a forest (because it consists of trees).

We mention brief examples:

(1) Trees are useful for modeling the possible outcomes of an experiment. For example, consider an experiment in which a coin is flipped and a 6-sides die is rolled. The leaves in the tree correspond to the outcomes in the probability space for this experiment.

(2) Programmers often use tree structures to facilitate searches and sorts and to model the logic of algorithms. For instance, the logic for a program that finds the maximum of four numbers(w, x, y, z) can be represented by a tree.

(3) Chemists can use trees to represent, among other things, saturated hydro-carbons chemical compounds of the form C_nH_{2n+2} (propane, for example). The bonds between the carbons and hydrogen atoms are depicted in a tree.

2. Definitions and Background:

(1) Let M and N be two Riemannian manifolds (not necessarily of the same dimension), a map $F : M \longrightarrow N$ is said to be an "*Isometric folding*" of M into N if, for each piecewise geodesic path $\gamma : I \longrightarrow M$ ($I = [0, 1] \subset R$) , the induced path $F \circ \gamma : I \longrightarrow N$ is piecewise geodesic and of the same length as γ . If F not preserves lengths, then F is a topological folding (A.T.White, 1973).

(2) A subset A of a topological space X is called a "*Retract*" if there exist a continuous map $R : X \longrightarrow A$ (called retraction) such that $R(a) = a$, $\forall\, a \in A$ (M.El-Ghoul, 2006).

(3) Two or more edges joining the same pair of vertices are called " *multiple edges*" and an edge joining a vertex to itself is called a "*loop*"(William S.Massey, 1967).

(4) A "*connect*" graph is a graph that in one piece, whereas one which splits into several pieces is " *disconnected*" (William S.Massey, 1967).

3. The main result

We can define new types of trees:

3.1 New tree with loop

It is a connected graph that contains number of loops, see Fig.(1)

< Figure 1 >

Where,

$$A(G)=\begin{bmatrix}0&1&0&0&0\\1&1&1&1&1\\0&1&1&0&0\\0&1&0&1&0\\0&1&0&0&1\end{bmatrix}$$

and

$$I(G)=\begin{bmatrix}1&0&0&0\\1&1&1&1\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix}$$

We can make some geometric transformations on its loops:

(1) Folding:

We can make folding on loops which gives us the original tree

< Figure 2 >

Such that

$$A(G)=\begin{bmatrix}0&1&0&0&0\\1&1&1&1&1\\0&1&1&0&0\\0&1&0&1&0\\0&1&0&0&1\end{bmatrix}\overset{F_1}{\longrightarrow}\begin{bmatrix}0&1&0&0&0&0&0&0\\1&0&1&1&1&0&0&0\\0&1&0&0&0&0&0&1\\0&1&0&0&0&0&1&0\\0&1&0&0&0&1&0&0\\0&0&0&0&1&0&0&0\\0&0&0&1&0&0&0&0\\0&0&1&0&0&0&0&0\end{bmatrix}$$

and

$$I(G)=\begin{bmatrix}1&0&0&0\\1&1&1&1\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix}\overset{F_1}{\longrightarrow}\begin{bmatrix}1&0&0&0&0&0&0\\1&1&1&1&0&0&0\\0&1&0&0&0&0&1\\0&0&1&0&0&1&0\\0&0&0&1&1&0&0\\0&0&0&1&0&0&0\\0&0&1&0&0&0&0\\0&1&0&0&0&0&0\end{bmatrix}$$

From the above folding F_1, we find that F_1 reduce the number of loops (vanishes) and increase the number of vertices(v_5, v_6, v_7) and the number of edges (v_2v_7, v_3v_6, v_4v_5).

(2) Retraction:

By retraction, the terminal loops vanish.

Let $R_1 : (T - P_i) \longrightarrow A$, where P_i represent the loops

< Figure 3>

Where

$$A(G)=\begin{bmatrix}0&1&0&0&0\\1&1&1&1&1\\0&1&1&0&0\\0&1&0&1&0\\0&1&0&0&1\end{bmatrix}\overset{R_1}{\longrightarrow}\begin{bmatrix}0&1&0&0&0\\1&1&1&1&1\\0&1&0&0&0\\0&1&0&0&0\\0&1&0&0&0\end{bmatrix}$$

$$I(G)=\begin{bmatrix}1&0&0&0\\1&1&1&1\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix}\xrightarrow{R_1}\begin{bmatrix}1&0&0&0\\1&1&1&1\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix}$$

Note that the loops are vanished and $A(G) \neq A(R(G))$ but $I(G) = I(R(G))$.

(3) Dispersion:

< Figure 4 >

Where

$$A(G)=\begin{bmatrix}0&1&0&0&0\\1&1&1&1&1\\0&1&1&0&0\\0&1&0&1&0\\0&1&0&0&1\end{bmatrix}\xrightarrow{D_1}\begin{bmatrix}0&1&0&0&0&0&0&0&0&0&0\\1&0&1&1&1&0&0&0&0&0&0\\0&1&0&0&0&1&1&0&0&0&0\\0&1&0&0&0&0&0&1&1&0&0\\0&1&0&0&0&0&0&0&0&1&1\\0&0&1&0&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0&0&0&0\\0&0&0&1&0&0&0&0&0&0&0\\0&0&0&1&0&0&0&0&0&0&0\\0&0&0&0&1&0&0&0&0&0&0\\0&0&0&0&1&0&0&0&0&0&0\end{bmatrix}$$

and

$$I(G)=\begin{bmatrix}1&0&0&0\\1&1&1&1\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix}\xrightarrow{D_1}\begin{bmatrix}1&0&0&0&0&0&0&0&0&0\\1&1&1&1&0&0&0&0&0&0\\0&1&0&0&0&1&1&0&1&1\\0&0&1&0&0&0&1&1&0&0\\0&0&0&1&1&1&0&0&0&0\\0&0&0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1&0&0\\0&0&0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0&0&0\\0&0&0&0&1&0&0&0&0&0\end{bmatrix}$$

The dispersion increases the number of edges and vertices.

3.2 Pure looped tree

Consider a tree every edge in it is a loop, see Fig.(5)

< Figure 5 >

$$A(G)=\begin{bmatrix}0&2&0&0&0\\2&0&2&2&2\\0&2&1&0&0\\0&2&0&1&0\\0&2&0&0&1\end{bmatrix}\xrightarrow{F_1}\begin{bmatrix}0&1&0&0&0&0&0&0\\1&0&1&1&1&0&0&0\\0&1&0&0&0&0&0&1\\0&1&0&0&0&0&1&0\\0&1&0&0&0&1&0&0\\0&0&0&0&1&0&0&0\\0&0&0&1&0&0&0&0\\0&0&1&0&0&0&0&0\end{bmatrix}$$

,

$$I(G)=\begin{bmatrix}1&0&0&0\\1&1&1&1\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix}\xrightarrow{F_1}\begin{bmatrix}1&0&0&0&0&0&0\\1&1&1&1&0&0&0\\0&1&0&0&0&0&1\\0&0&1&0&0&1&0\\0&0&0&1&1&0&0\\0&0&0&1&0&0&0\\0&0&1&0&0&0&0\\0&1&0&0&0&0&0\end{bmatrix}$$

Where F_1 is folding

And

< Figure 6 >

Where

$$A(G)=\begin{bmatrix}0&2&0&0&0\\2&0&2&2&2\\0&2&1&0&0\\0&2&0&1&0\\0&2&0&0&1\end{bmatrix}\xrightarrow{R_1}\begin{bmatrix}0&1&0&0&0\\1&0&1&1&1\\0&1&0&0&0\\0&1&0&0&0\\0&1&0&0&0\end{bmatrix}$$

,

$$I(G)=\begin{bmatrix}2&0&0&0\\2&2&2&2\\0&2&0&0\\0&0&2&0\\0&0&0&0\end{bmatrix}\xrightarrow{R_1}\begin{bmatrix}1&0&0&0\\1&1&1&1\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix}$$

And

< Figure 7 >

Such that

$$A(G)=\begin{bmatrix}0&2&0&0&0\\2&0&2&2&2\\0&2&1&0&0\\0&2&0&1&0\\0&2&0&0&1\end{bmatrix}\xrightarrow{D_1}\begin{bmatrix}0&1&0&0&0&0&0&0&0&0&0&0&0&0\\1&0&1&0&0&1&1&0&0&1&0&0&1&1\\0&1&0&1&1&0&0&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&1&0&0&0&0&0&0&0\\0&0&0&0&0&0&1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&0&0&0&0&0&0&1\\0&1&0&0&0&0&0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0&0&0&0&0&0&0\end{bmatrix}$$

And

$$I(G)=\begin{bmatrix}2&0&0&0\\2&2&2&2\\0&2&0&0\\0&0&2&0\\0&0&0&0\end{bmatrix}\xrightarrow{D_1}\begin{bmatrix}1&0&0&0&0&0&0&0&0&0&0&0&0\\0&1&1&1&1&1&1&0&0&0&0&0&0\\0&1&0&0&0&0&0&1&1&0&0&0&0\\0&0&0&0&0&0&0&1&0&0&0&0&0\\0&0&0&0&0&0&0&0&1&0&0&0&0\\0&0&1&0&0&0&0&0&0&0&0&0&0\\0&0&0&1&0&0&0&0&0&1&1&0&0\\0&0&0&0&0&0&0&0&0&1&0&0&0\\0&0&0&0&0&0&0&0&0&0&1&0&0\\0&0&0&0&1&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&0&0&0&0&0&1\\0&0&0&0&0&0&1&0&0&0&0&0&0\\0&0&0&0&0&1&0&0&0&0&0&0&0\end{bmatrix}$$

Lemma:

Any geometric transformation on the new tree gives the original tree.

Now we will define another type of tree:

3.3 General tree

It is a tree which consists of vertices V, edges E, areas A and volumes L

Where V, E, A and L are different in dimensions, such that the connection is direct, i.e. the one dimension joined with three dimension directly, or gradually, i.e. the one dimension joined with two dimension and the two dimension joined with three dimension.

In the original graph, the vertices join between edges of the same dimension, but for the general tree the vertices joins between different dimensions.

First: Direct case:

Example:

Consider the following general tree in figure (5), where $v_0, v_1 ...., v_6$ are of 0-dimension, and $e_0, e_1 ...., e_5$ are of 1-dimension, and a_0, a_1 are of 2-dimension, and l_0, l_1 and of 3-dimension.

< Figure 8 >

Such that

$$A(G) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Where the adjacent matrix of G denoted by $A(G)$ is the n*n matrix in which the entry in row I and column J is the number of edges joining the vertices I and J.

$$I(G) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Where The incidence matrix of G denoted by $I(G)$ is the $n * m$ binary matrix where $m_{ij} = 1$ if v_i is incident with e_j and $m_{ij} = 0$otherwise,

$$M(G) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

Where the area matrix of G denoted by$M(G)$ is the $n * m$binary matrix where $m_{ij} = 1$ if v_i is incident with $a_i$and $m_{ij} = 0$ otherwise

,and

$$N(G) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Where the volume matrix of G denoted by $N(G)$is the $n * m$binary matrix where $m_{ij} = 1$ if v_i is incident with L_i and $m_{ij} = 0$ otherwise .

,

$$H(G) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

Where the edges area matrix of G denoted by $H(G)$ is the $n * m$ binary matrix where $m_{ij} = 1$ if e_i is incident with $a_i$and

$m_{ij} = 0$ otherwise .

And

$$J(G) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Where the edge volume matrix of$G$ denoted by$J(G)$is the $n * m$binary matrix where $m_{ij} = 1$ if e_i is incident with $L_i$and $m_{ij} = 0$ otherwise .

Now we will discuss the transformations for which we can obtain the usual tree from the new general tree by folding or retraction

3.4 Folding of the general tree

Let $F_i : T \longrightarrow T$, such that

$F_i(v_n v_m) = v_n v_m$, $n \neq m$, $i = 1, 2.$

< Figure 9 >

Where

$$A(G) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \overset{F_1}{\longrightarrow} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \overset{F_2}{\longrightarrow} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\ldots\ldots \xrightarrow[n\longrightarrow\infty]{Lim\ F_2} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

,

$$I(G) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \overset{F_1}{\longrightarrow} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \overset{F_2}{\longrightarrow} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \ldots\ldots$$

$$\xrightarrow[n\longrightarrow\infty]{Lim\ F_2}\begin{bmatrix} 1&0&0&0&0&0&0&0&0&0\\ 1&1&1&0&1&0&0&0&0&0\\ 0&1&0&0&0&0&1&0&0&0\\ 0&0&0&1&1&1&0&0&0&0\\ 0&0&1&0&0&0&0&1&0&0\\ 0&0&0&1&0&0&0&0&1&0\\ 0&0&0&0&0&1&0&0&0&1\\ 0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&1 \end{bmatrix}$$

,

$$M(G)=\begin{bmatrix} 0&0\\ 0&0\\ 0&0\\ 0&0\\ 0&1\\ 0&0\\ 1&0 \end{bmatrix}\xrightarrow{F_1}\begin{bmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&1&0\\ 0&0&0&1\\ 0&1&0&0\\ 0&0&0&0\\ 1&0&0&0 \end{bmatrix}\xrightarrow{F_2}\begin{bmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&1&0\\ 0&0&0&1\\ 0&1&0&0\\ 0&0&0&0\\ 1&0&0&0 \end{bmatrix}$$

,

$$N(G)=\begin{bmatrix} 0&0\\ 0&0\\ 0&1\\ 0&0\\ 0&0\\ 1&0\\ 0&0 \end{bmatrix}$$

,

$$H(G)=\begin{bmatrix} 0&0\\ 0&0\\ 0&1\\ 0&0\\ 0&0\\ 1&0 \end{bmatrix}\xrightarrow{F_1}\begin{bmatrix} 0&0&0&0\\ 0&0&1&0\\ 0&1&0&0\\ 0&0&0&1\\ 0&0&0&0\\ 1&0&0&0 \end{bmatrix}\xrightarrow{F_2}\begin{bmatrix} 0&0&0&0\\ 0&0&1&0\\ 0&1&0&0\\ 0&0&0&1\\ 0&0&0&0\\ 1&0&0&0 \end{bmatrix}$$

And

$$J(G)=\begin{bmatrix} 0&0\\ 0&1\\ 0&0\\ 1&0\\ 0&0\\ 0&0 \end{bmatrix}$$

Note that the folding must be gradually, therefore we can deduce that the

folding of the general tree gives us the original tree.

3.5 Retraction of the general tree

In this section we will discuss the retraction of the new tree by making a modification on the usual retraction

$R_1 : (G - L_i) \longrightarrow G_1, R_2 : (G_1 - a_i) \longrightarrow G_2$

Such that,

< Figure 10>

$$A(G)=\begin{bmatrix} 0&1&0&0&0&0&0\\ 1&0&1&1&1&0&0\\ 0&1&0&0&0&0&0\\ 0&1&0&0&0&1&1\\ 0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&1&0&0&0 \end{bmatrix}\xrightarrow{R_1}\begin{bmatrix} 0&1&0&0&0&0&0\\ 1&0&1&1&1&0&0\\ 0&1&0&0&0&0&0\\ 0&1&0&0&0&1&1\\ 0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&1&0&0&0 \end{bmatrix}\ldots\ldots\xrightarrow[n\longrightarrow\infty]{Lim\ R_1}\begin{bmatrix} 0&1&0&0&0&0&0\\ 1&0&1&1&1&0&0\\ 0&1&0&0&0&0&0\\ 0&1&0&0&0&1&1\\ 0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&1&0&0&0 \end{bmatrix}$$

$$\xrightarrow{R_2}\begin{bmatrix} 0&1&0&0&0&0&0\\ 1&0&1&1&1&0&0\\ 0&1&0&0&0&0&0\\ 0&1&0&0&0&1&1\\ 0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&1&0&0&0 \end{bmatrix} \ldots\ldots \xrightarrow[n\longrightarrow\infty]{Lim\ R_2} \begin{bmatrix} 0&1&0&0&0&0&0\\ 1&0&1&1&1&0&0\\ 0&1&0&0&0&0&0\\ 0&1&0&0&0&1&1\\ 0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&1&0&0&0 \end{bmatrix}$$

,

$$I(G)=\begin{bmatrix} 1&0&0&0&0&0\\ 1&1&1&0&1&0\\ 0&1&0&0&0&0\\ 0&0&0&1&1&1\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&0&1 \end{bmatrix} \xrightarrow{R_1} \begin{bmatrix} 1&0&0&0&0&0\\ 1&1&1&0&1&0\\ 0&1&0&0&0&0\\ 0&0&0&1&1&1\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&0&1 \end{bmatrix} \ldots\ldots \xrightarrow[n\longrightarrow\infty]{Lim\ R_1} \begin{bmatrix} 1&0&0&0&0&0\\ 1&1&1&0&1&0\\ 0&1&0&0&0&0\\ 0&0&0&1&1&1\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&0&1 \end{bmatrix} \xrightarrow{R_2} \begin{bmatrix} 1&0&0&0&0&0\\ 1&1&1&0&1&0\\ 0&1&0&0&0&0\\ 0&0&0&1&1&1\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&0&1 \end{bmatrix}$$

$$\ldots\ldots \xrightarrow[n\longrightarrow\infty]{Lim\ R_2} \begin{bmatrix} 1&0&0&0&0&0\\ 1&1&1&0&1&0\\ 0&1&0&0&0&0\\ 0&0&0&1&1&1\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&0&1 \end{bmatrix}$$

,

$$M(G)=\begin{bmatrix} 0&0\\ 0&0\\ 0&0\\ 0&0\\ 0&1\\ 0&0\\ 1&0 \end{bmatrix} \xrightarrow{R_1} \begin{bmatrix} 0&0\\ 0&0\\ 0&0\\ 0&0\\ 0&1\\ 0&0\\ 1&0 \end{bmatrix} \ldots\ldots \xrightarrow[n\longrightarrow\infty]{Lim\ R_1} \begin{bmatrix} 0&0\\ 0&0\\ 0&0\\ 0&0\\ 0&1\\ 0&0\\ 1&0 \end{bmatrix} \xrightarrow{R_2} \begin{bmatrix} 0&0\\ 0&0\\ 0&0\\ 0&0\\ 0&1\\ 0&0\\ 1&0 \end{bmatrix}$$

,

$$N(G)=\begin{bmatrix} 0&0\\ 0&0\\ 0&1\\ 0&0\\ 0&0\\ 1&0\\ 0&0 \end{bmatrix} \xrightarrow{R_1} \begin{bmatrix} 0&0\\ 0&0\\ 0&1\\ 0&0\\ 0&0\\ 1&0\\ 0&0 \end{bmatrix}$$

,

$$H(G)=\begin{bmatrix} 0&0\\ 0&0\\ 0&1\\ 0&0\\ 0&0\\ 1&0 \end{bmatrix} \xrightarrow{R_1} \begin{bmatrix} 0&0\\ 0&0\\ 0&1\\ 0&0\\ 0&0\\ 1&0 \end{bmatrix} \ldots\ldots \xrightarrow[n\longrightarrow\infty]{Lim\ R_1} \begin{bmatrix} 0&0\\ 0&0\\ 0&1\\ 0&0\\ 0&0\\ 1&0 \end{bmatrix} \xrightarrow{R_2} \begin{bmatrix} 0&0\\ 0&0\\ 0&1\\ 0&0\\ 0&0\\ 1&0 \end{bmatrix}$$

And

$$J(G)=\begin{bmatrix} 0&0\\ 0&1\\ 0&0\\ 1&0\\ 0&0\\ 0&0 \end{bmatrix} \xrightarrow{R_1} \begin{bmatrix} 0&0\\ 0&1\\ 0&0\\ 1&0\\ 0&0\\ 0&0 \end{bmatrix}$$

Second: Gradually case:

Example:

In this case the connection between edges are vertices and 1-dimension by 3-dimension through 2-dimension.

< Figure 11 >

By folding the graph

< Figure >

Such that

$$A(G)=\begin{bmatrix}0&1&0&0&0\\1&0&1&1&1\\0&1&0&0&0\\0&1&0&0&0\\0&1&0&0&0\end{bmatrix}\xrightarrow{F_1}\begin{bmatrix}0&1&0&0&0\\1&0&1&1&1\\0&1&0&0&0\\0&1&0&0&0\\0&1&0&0&0\end{bmatrix}\xrightarrow{F_2}\begin{bmatrix}0&1&0&0&0\\1&0&1&1&1\\0&1&0&0&0\\0&1&0&0&0\\0&1&0&0&0\end{bmatrix}\cdots\cdots\xrightarrow[n\longrightarrow\infty]{Lim\ F_2}\begin{bmatrix}0&1&0&0&0&0&0&0\\1&0&1&1&1&0&0&0\\0&1&0&0&0&0&0&0\\0&1&0&0&0&1&0&0\\0&1&0&0&0&0&1&0\\0&0&0&1&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&1&0&0&0&0&0\end{bmatrix}$$

,

$$I(G)=\begin{bmatrix}1&0&0&0\\1&1&1&1\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}\xrightarrow{F_1}\begin{bmatrix}1&0&0&0\\1&1&1&1\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}\xrightarrow{F_2}\begin{bmatrix}1&0&0&0\\1&1&1&1\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}\cdots\cdots\xrightarrow[n\longrightarrow\infty]{Lim\ F_2}\begin{bmatrix}1&0&0&0&0&0&0\\1&1&1&1&0&0&0\\0&1&0&0&1&0&0\\0&0&0&1&0&0&1\\0&0&1&0&0&1&0\\0&0&0&0&0&0&1\\0&0&0&0&0&1&0\\0&0&0&0&1&0&0\end{bmatrix}$$

,

$$M(G)=\begin{bmatrix}0&0&0\\0&0&0\\0&1&0\\1&0&0\\0&0&1\end{bmatrix}\xrightarrow{F_1}\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0\\0&0&1&1\end{bmatrix}\xrightarrow{F_2}\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0\\0&0&1&1\end{bmatrix}$$

,

$$N(G)=\begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}$$

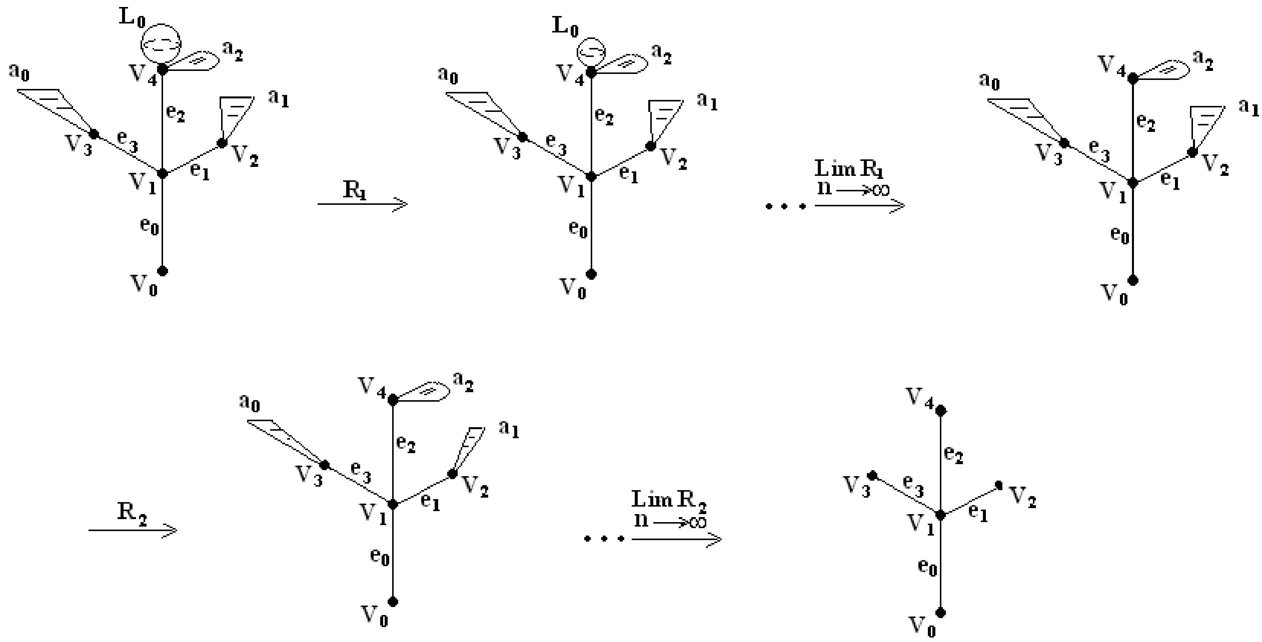
,

$$H(G)=\begin{bmatrix}0&0&0\\0&1&0\\0&0&1\\1&0&0\end{bmatrix}\xrightarrow{F_1}\begin{bmatrix}0&0&0&0\\0&1&0&0\\0&0&1&1\\1&0&0&0\end{bmatrix}\xrightarrow{F_2}\begin{bmatrix}0&0&0&0\\0&1&0&0\\0&0&1&1\\1&0&0&0\end{bmatrix}$$

And

$$J(G)=\begin{bmatrix}0\\0\\1\\0\end{bmatrix}$$

By retraction the graph in Fig. (11)



Then

$$A(G)=\begin{bmatrix}0&1&0&0&0\\1&0&1&1&1\\0&1&0&0&0\\0&1&0&0&0\\0&1&0&0&0\end{bmatrix}\overset{R_1}{\rightarrow}\begin{bmatrix}0&1&0&0&0\\1&0&1&1&1\\0&1&0&0&0\\0&1&0&0&0\\0&1&0&0&0\end{bmatrix}\ldots\ldots\xrightarrow[n\to\infty]{Lim\ R_1}\begin{bmatrix}0&1&0&0&0\\1&0&1&1&1\\0&1&0&0&0\\0&1&0&0&0\\0&1&0&0&0\end{bmatrix}\overset{R_2}{\rightarrow}\begin{bmatrix}0&1&0&0&0\\1&0&1&1&1\\0&1&0&0&0\\0&1&0&0&0\\0&1&0&0&0\end{bmatrix}$$

$$\ldots\ldots\xrightarrow[n\to\infty]{Lim\ R_2}\begin{bmatrix}0&1&0&0&0\\1&0&1&1&1\\0&1&0&0&0\\0&1&0&0&0\\0&1&0&0&0\end{bmatrix}$$

,

$$I(G)=\begin{bmatrix}1&0&0&0\\1&1&1&1\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}\overset{R_1}{\rightarrow}\begin{bmatrix}1&0&0&0\\1&1&1&1\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}\ldots\ldots$$

$$\xrightarrow[n\to\infty]{Lim\ R_1}\begin{bmatrix}1&0&0&0\\1&1&1&1\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}\overset{R_2}{\rightarrow}\begin{bmatrix}1&0&0&0\\1&1&1&1\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}\ldots\ldots$$

$$\xrightarrow[n\to\infty]{Lim\ R_2}\begin{bmatrix}1&0&0&0\\1&1&1&1\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}$$

,

$$M(G)=\begin{bmatrix}0&0&0\\0&0&0\\0&1&0\\1&0&0\\0&0&1\end{bmatrix}\overset{R_1}{\rightarrow}\begin{bmatrix}0&0&0\\0&0&0\\0&1&0\\1&0&0\\0&0&1\end{bmatrix}\ldots\ldots\xrightarrow[n\to\infty]{Lim\ R_1}\begin{bmatrix}0&0&0\\0&0&0\\0&1&0\\1&0&0\\0&0&1\end{bmatrix}\overset{R_2}{\rightarrow}\begin{bmatrix}0&0&0\\0&0&0\\0&1&0\\1&0&0\\0&0&1\end{bmatrix}$$

,

$$N(G) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \xrightarrow{R_1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

,

$$H(G) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \xrightarrow{R_1} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

And

$$J(G) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \xrightarrow{R_1} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Lemma:

The limit of folding and retraction of the general tree is the usual tree.

Application:

(1) The tree of orange can be represented as the new tree. The fruit represent volume, the leaves represent area, and the tree swing represents 1-dimension.

(2) The prickly pear tree is a kind of the new tree, as shown in the figure

< Figure 13>

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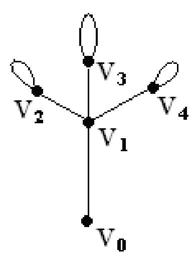


Figure 1.

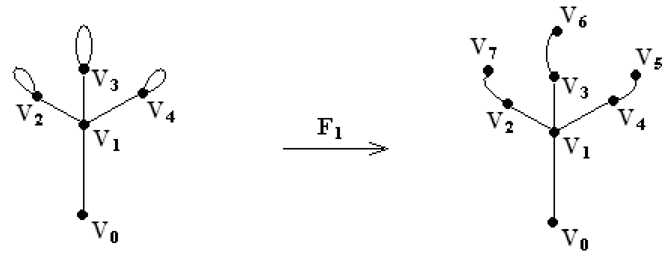


Figure 2.

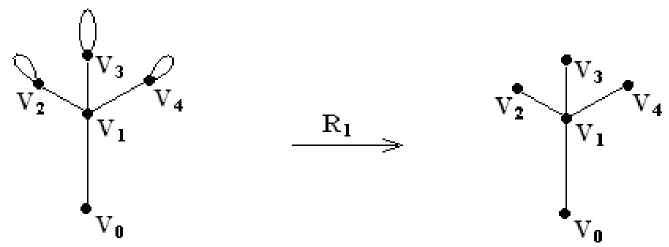
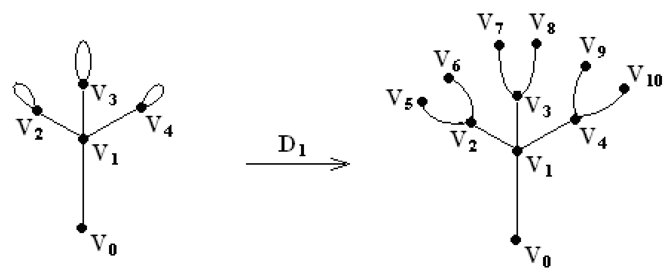


Figure 3.



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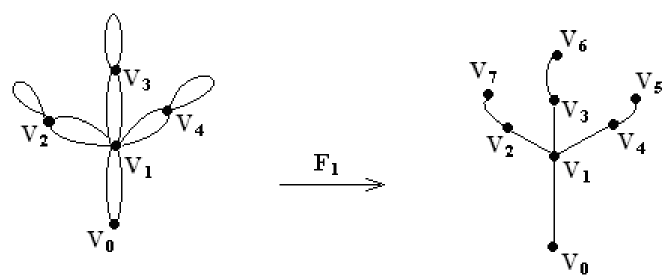

Figure 5.

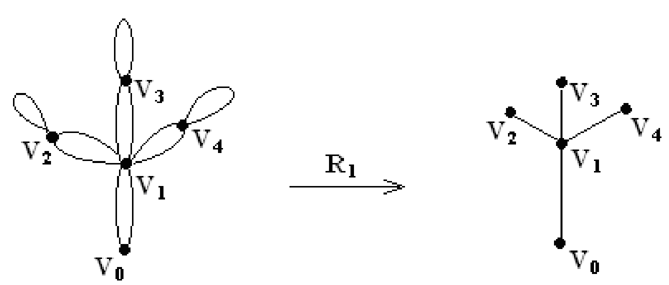

Figure 6.

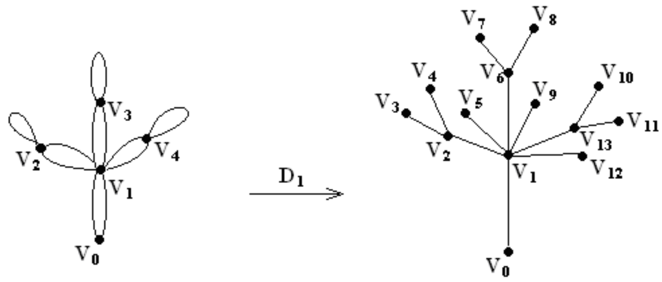

Figure 7.

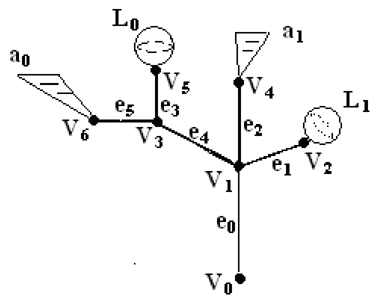

Figure 8.

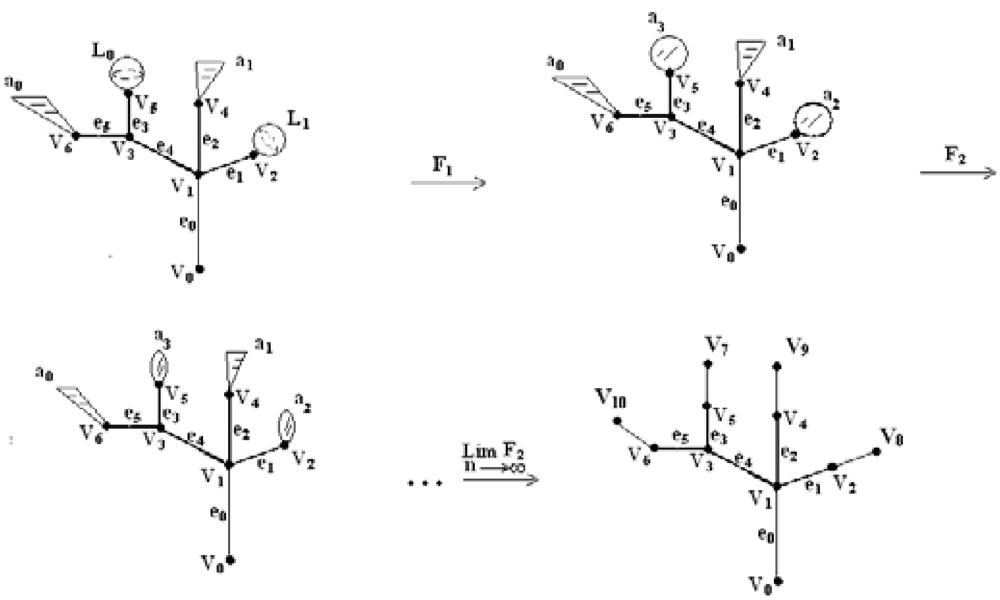

Figure 9.

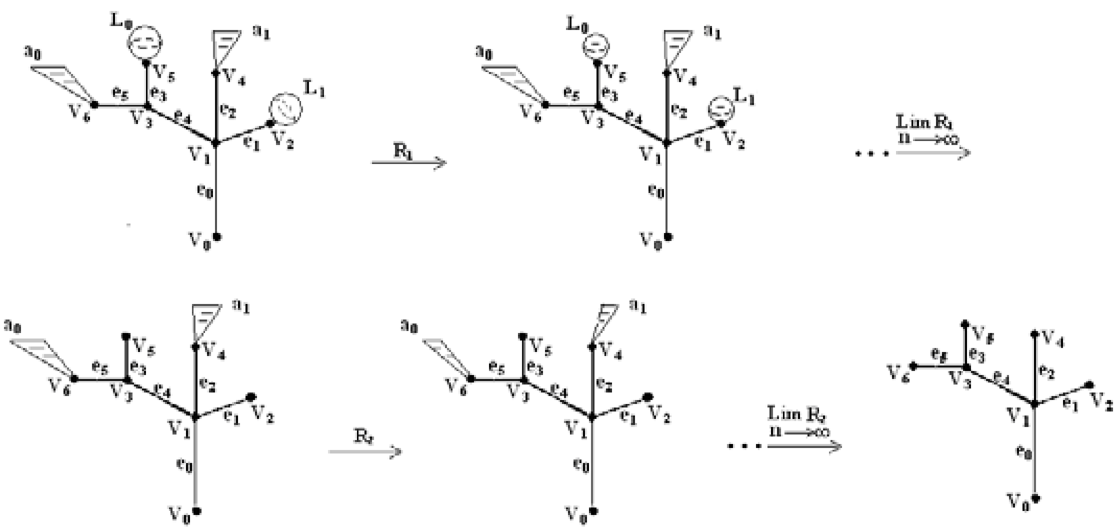

Figure 10.

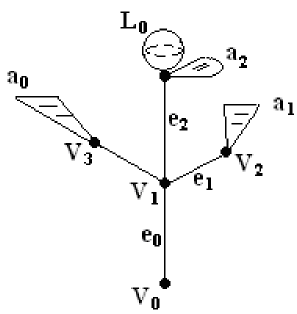

Figure 11.

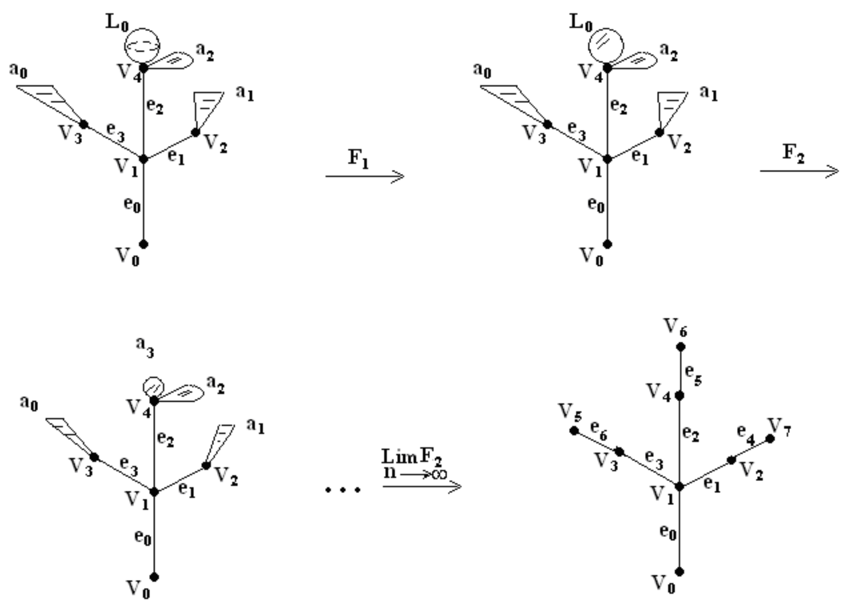

Figure 12.

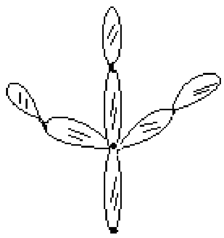

Figure 13.

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Application of Meshless Natural Neighbour Petrov-Galerkin Method in Temperature Field

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Abstract

In Meshless natura1 neighbour Petrov-Galerkin method, The natural neighbour interpolation is used as trial function and a weak form over the local polygonal sub-domains constructed by Delaunay triangular is used to obtain the discretized system of equilibrium equations, and it's a new truly meshless method. This method simplified the formation of the equilibrium equations, facilitates the imposition of essential boundary conditions and the system stiffness matrix in the present method is banded and sparse. Efforts are made to study Meshless natural neighbour Pettrov-Galerkin Method, which is extended to solve the transient heat conduction. The numerical results show that the present method is quite accurate and stable.

Keywords: Meshless local Petrov-Galerkin method, Natural neighbour interpolation, The transient heat conduction

1. Introduction

Meshless local Pettrov-Galerkin (MLPG) Method is a truly meshless method, as it does not need background meshes. For the conventional Galerkin method, the trial and the test function are chosen from the same function space, while for the MLPG method, which is actually a method of local weighted residua1, the trial and the test function are chosen from different function space. Atluri has listed six different kinds of MLPG methods named MLPG1-MLPG6 by varying the weight function. In the MLPG implementation, Moving Least Square (MLS) approximation is employed for constructing shape functions. There is an issue of imposition of essential boundary conditions as the shape function does not have Kronecker delta function property. In addition, the major drawback of MLPG is the asymmetry of the system matrices due to the use of the Petrov-Galerkin formulation. Another drawback of MLPG is that the local background integration can be very tricky due to the complexity of the integrand produced by the Petrov-Galerkin approach, especially for domains that intersect with the boundary of the problem domain. All these drawbacks limit the development of the MLPG. In recent years, more and more attention has been paid to natura1 neighbour interpolation. In this paper, an attempt is made to combine the advantages of the MLPG with the ability of easy imposition of essential boundary condition of the natural element method. A meshless method coined as natural neighbour Petrov-Galerkin method (MNNPG) is derived from the generalized meshless local Petrov-Galerkin method to solve the transient heat conduction problem. In this method, the problem domain and the boundary are discretized by the scattered nodes, and the Voronoi diagram based non-Sibsonian interpolation and is more efficient in computation of the shape functions. The 3-nodal triangular FEM shape functions are used as weight functions in each local sub-domain. The local sub-domains are constructed with Delaunay tessellations, and the Petrov-Galerkin method is used to get the discrete system equations. In included triangular regions of the sub-domain, the Gauss quadrature scheme is used to evaluate the domain integrals. Efforts are made to study Meshless natural neighbour Pettrov-Galerkin Method, which is extended to solve the transient heat conduction. The numerical results show that the present method is quite accurate and stable.

2. Natural neighbour interpolation

Natural neighbour interpolation is one kind of multivariate interpolation scheme, and Sibsonian interpolation and non-Sibsonian interpolation have been used in meshless method at present. In this paper, the approximation is constructed according to the natural neighbours of the evaluated point using non-Sibsonian interpolation, which is based on the well-known Voronoi diagram and Delaunay tessellations. The problem domain is denoted by Ω and set N is a partition of plane into regions T_i, where each region T_i is associated with a node$i$, such that any point in T_i is closer to node$i$ than to any other nodes, in mathematical form:

$$T_i = \{x \in R^2 : d(x,\ x_i) < d(x,\ x_j)\forall j \neq i\} \tag{1}$$

Where $d(x,\ x_i)$ is the distance between x and x_i.

Insert < Figure 1 > here

The important property of the Delaunay triangulation is the empty circumcircle criterion: the circumcircle of the Delaunay triangle contains no other nodes inside it. This criterion is used to find the natural neighbours of a point x (like the Gauss points), if the point x lies within the circumcircle of a Delaunay triangle. The non-Sibsonian interpolation is based on the Voronoi diagram of the evaluated point. To construct the Voronoi diagram, the natural neighbours of the evaluated point x is determined by the circumcircle criterion. The non-Sibsonian interpolation shape function of node$i$ is calculated by:

$$\phi_i = \frac{S_i(x)/h_i(x)}{\sum\limits_{j=1}^{n} S_j(x)/h_j(x)} \tag{2}$$

Where S_i is the Lebesgue measure (length in R^2) of the Voronoi boundary associated with node i, and the h_i is the distance between the evaluated point x and the node i, is the number of natural neighbours of the point x. In order to calculate S_i, the centers of the circumcircles of the triangles defined by point x, node i and another related natural neighbours of the evaluated point x should be worked out. The non-Sibsonian interpolation shape functions $\phi_i(x)$ have many properties, among which the main and crucial properties to the meshless method are:

$$0 \le \phi_i(x) \le 1,\ \phi_i(x_j) = \delta_{ij} \tag{3}$$

$$\sum_{i=1}^{n} \phi_i(x) = 1,\ x = \sum_{i=1}^{n} \phi_i(x) x_i \tag{4}$$

The Eq. (3) indicates that the shape function has Kronecker Delta function property and thus the approximation passes through the nodal values, so that the essential boundary condition can be imposed as directly as what it is done in FEM. The Eq. (4) defines a partition of unity and linear completeness, which indicates the shape function can exactly reproduce the constant and linear functions. This is important for the meshless method to get a proper convergence rate. Furthermore, the implementation of the non-Sibsonian interpolation is more efficient because the evaluation of Lebesgue measure is one dimension less than that of the Sibson interpolation. All these properties render the non-Sibsonian interpolation an attractive approximation scheme in meshless methods.

3. MLPG method for heat conduction problem

Transient heat conduction problem dependentes on time. To two-dimensional transient heat conduction problem, the temperature function is denoted by $T(x,\ y,\ t)$, according to heat conduction theory, the Transient heat conduction's Poisson equation and boundary conditions can be written as:

$$\rho c\frac{\partial T}{\partial t} - \frac{\partial}{\partial x}(k_x\frac{\partial T}{\partial x}) - \frac{\partial}{\partial y}(K_y\frac{\partial T}{\partial y}) - \rho Q = 0 \tag{5}$$

$$\begin{array}{ll} T = \bar{T} & \text{on } \Gamma_1 \\ k_x\frac{\partial T}{\partial x}n_x - k_y\frac{\partial T}{\partial y}n_y = q & \text{on } \Gamma_2 \\ k_x\frac{\partial T}{\partial x}n_x - k_y\frac{\partial T}{\partial y}n_y = h(T_a - T) & \text{on } \Gamma_3 \end{array}$$

Where ρ is Material density; c is specific heat capacity; k_x, k_y are the material conductivities along $x,\ y$ direction, respectively; Q is Internal density of heat source; $n_x,\ n_y$ are direction cosines of the unit outward normal vector. $\bar{T}$ is the temperature on Γ_1; q is the Heat current density on Γ_2; h is heat emission factor; T_a is the ambient temperature.

The problem domain Ω and its boundary are placed with scattered nodes, and the Delaunay tessellations are used to partition the whole domain into triangular regions. Each node, for example node is associated with a local sub-domain Ω_s, which is constructed by collecting all the surrounding Delaunay triangles with node being their common vertices. In each sub-domain, based on the local weighted residual method, the weak form of governing equations are satisfied, and may be written as:

$$\int_\Omega \Big[\rho c\frac{\partial T}{\partial t} - \frac{\partial}{\partial x}(k_x\frac{\partial T}{\partial x}) - \frac{\partial}{\partial y}(k_y\frac{\partial T}{\partial y}) - \rho Q\Big] w d\Omega = 0 \tag{6}$$

Where w is the weight or test function, and we use the same weight function for all the equations involved.

Using the divergence theorem, we obtain

$$\int_{\Omega_s}(k_x\frac{\partial T}{\partial x}\frac{\partial w}{\partial x}+k_y\frac{\partial T}{\partial y}\frac{\partial w}{\partial y})d\Omega+\int_{\Omega_s}w(\rho c\frac{\partial T}{\partial t}-\rho Q)d\Omega-\int_{\partial\Omega_s}w(k_x\frac{\partial T}{\partial x}n_x+k_y\frac{\partial T}{\partial y}n_y)d\Gamma=0 \quad (7)$$

Where $\partial\Omega_s$ denotes the boundary of Ω_s, and usually consists of four parts: the internal boundary L_S, which does not intersect with the global boundary. Γ_{s1}, Γ_{s2}, Γ_{s3} are the local boundary that over the global boundary, which given temperature, rate of heat flow, Convection heat transfer condition, respectively. Therefore, $\partial\Gamma_s=\Gamma_{s1}\cup\Gamma_{s2}\cup\Gamma_{s3}\cup L_s$. Considering the boundary conditions of Γ_{s2}, we can obtain:

$$\int_{\Omega_s}(k_x\frac{\partial T}{\partial x}\frac{\partial w}{\partial x}+k_y\frac{\partial T}{\partial y}\frac{\partial w}{\partial y})d\Omega+\int_{\Omega_s}w(\rho c\frac{\partial T}{\partial t}-\rho Q)d\Omega-\int_{L_s}w(k_x\frac{\partial T}{\partial x}n_x+k_y\frac{\partial T}{\partial y}n_y)d\Gamma-\int_{\Gamma_{s1}}w(k_x\frac{\partial T}{\partial x}n_x+k_y\frac{\partial T}{\partial y}n_y)d\Gamma$$

$$-\int_{\Gamma_{s2}}wqd\Gamma-\int_{\Gamma_{s3}}wh(T_a-T)d\Gamma=0 \quad (8)$$

For a sub-domain located entirely within the global domain, there is no intersection with the global boundary, the integrals over Γ_{s1}, vanish. To simplify the above equation, the 3-nodal triangular FEM shape function N_i of node i is used as weight functions. Therefore, the integrals over internal boundary L_s vanish. Furthermore, the domain integrals over Γ_{s1} have two conditions: For Fig. 2(a), easy know $N_i=0$, the integral over internal boundary Γ_{s1} vanish; For Fig. 2(b), limited by temperature condition, stiffness matrix vanishes when assemble integral items. So the Eq. (8) can be rewritten as:

$$\sum_{I=1}^{M}\int_{T_{iI}}(k_x\frac{\partial T}{\partial x}\frac{\partial w}{\partial x}+k_y\frac{\partial T}{\partial y}\frac{\partial w}{\partial y}+w\rho c\frac{\partial T}{\partial t}-w\rho Q)d\Omega-\int_{\Gamma_{s2}}wqd\Gamma-\int_{\Gamma_{s3}}wh(T_a-T)d\Gamma=0 \quad (9)$$

Where M is the total number of the Delaunay triangles which constructed local sub-domain Ω_s of node i.

< Figure 2 >

To obtain the discrete equations from Eq. (9), we use MLS to approximate the test function T as follows:

$$T(x)=\sum_{i=1}^{n}\phi_i(x)T_i \quad (10)$$

Where n is the total number of nodes for x. $\phi_i(x)$ is usually called shape function. Substitution of equation (10) into (9) for all nodes leads to the following simplified discretized system of equations:

$$\sum_{j=1}^{N}K_{ij}T_j+\frac{\partial T_j}{\partial t}C_{ij}=P_i(i=1,\ 2,\cdots,\ N) \quad (11)$$

Where

$$K_{ij}=\sum_{I=1}^{M}\int_{T_{iI}}(k_x\frac{\partial\phi_j(x)}{\partial x}\frac{\partial N_i}{\partial x}+k_y\frac{\partial\phi_j(x)}{\partial y}\frac{\partial N_i}{\partial y})d\Omega+\int_{\Gamma_{s3}}N_ih\phi_j(x)d\Gamma$$

$$C_{ij}=\sum_{I=1}^{M}\int_{\Omega_s}N_i\rho c\phi_j(x)d\Omega$$

$$P_i=\int_{\Gamma_{s2}}N_iqd\Gamma+\int_{\Gamma_{s3}}N_ihT_ad\Gamma+\int_{\Omega_s}N_i\rho Qd\Omega \quad (12)$$

Where N is the total number of the nodes in the global domain.

4. Numerical example

Consider the heat conduction problem of a rectangular plate (see Fig. 3). The length of the plate is 100. The temperature of the left boundary AB is $T=0$, the heat current of Underneath boundary BC is $q_b=0$, other boundarys are heat insulation. There is no internal heat source. The following parameters are considered: $k_x=k_y=1000$, $\rho c=1.0$, $T_0=0$, The regular nodes is discretization (9 × 9) is used in the presented work. We use Gauss integral scheme (3 point) in Delaunay triangles. From Fig. 4; Fig.5 it can be clearly seen that the MNNPG solution is in excellent agreement with the analytical solution provided by reference documentation (Liu, 2002).

<Figure 3-5>

5. Conclusions

MNNPG method proposed in this paper combines the advantages of the non-Sibsonian interpolation with the new meshless Petrov-Galerkin method. In the MNNPG, the local weak form of the equilibrium equation is used, the trail functions are constructed by non-Sibsonian interpolation and the 3-nodal triangular FEM shape functions are chosen as the test function. The global stiffness matrix obtained in MNNPG is sparse and banded, and does not need an assembly process. Natural neighbour interpolation can enforce the essential boundary condition directly. In the presented numerical examples, excellent solutions are obtained. In essence, this method is efficient, accurate and easy to implement, which reveals its potential applications to solve other problems.

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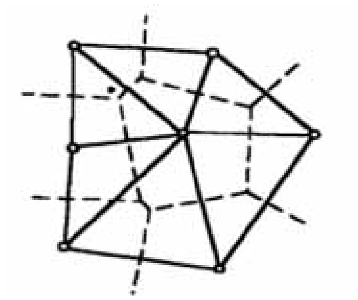

Figure 1. The Voronoi diagram and natural neighbours

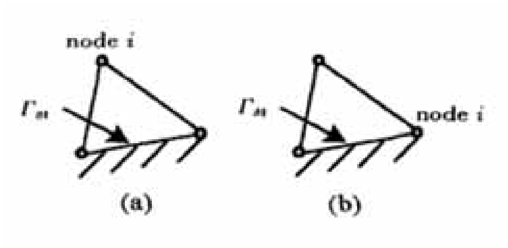

Figure 2. Essential boundary condition Γ_{s1} over sub-domain Ω_s

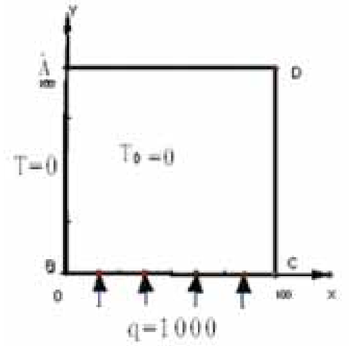

Figure 3. Heat conduction model of two-dimension

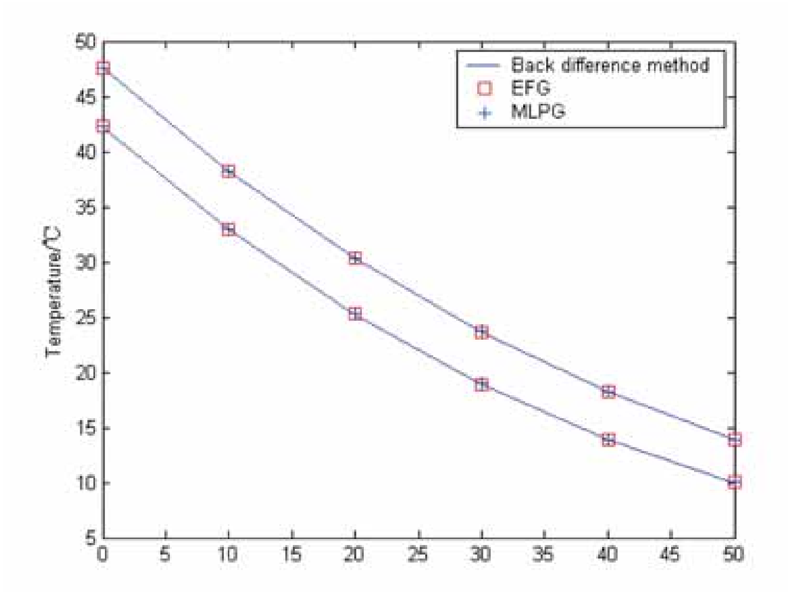

Figure 4. Comparison of the solutions obtained by using different methods along x = 80

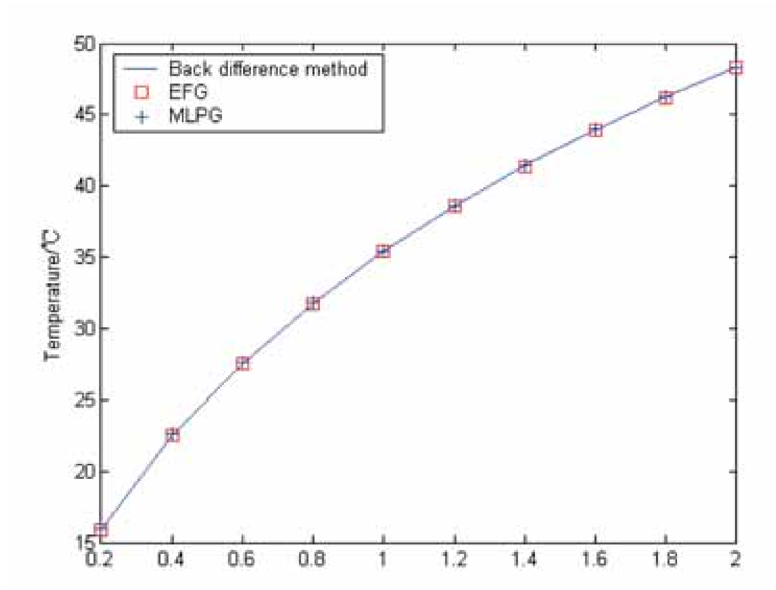

Figure 5. Comparison of the solutions obtained by using different methods at point C

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Integral Oscillation Criteria for Second-Order Linear neutral Delay Dynamic Equations on Time Scales

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Abstract

In this paper we present several sufficient conditions for oscillation of the second-order linear neutral delay dynamic equation on a time scale $\mathbb{T}$. Our results as a special case when $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{N}$ improve some well-known oscillation results for second-order neutral delay differential and difference equations.

Keywords: Oscillation, Time scales, Dynamic equation

1. Introduction

In 1988, Stefan Hilger introduced the calculus of *measure chain* in order to unify continuous and discrete analysis. Berned Aulbach, who supervised Stefan Hilger's Ph.D. thesis (Hilger, S., 1990, p18-56), points out the three main purposes of this new calculus: Unification - Extension - Discretization.

For many purposes in analysis it is sufficient to consider a special case of a measure chain, a so-called *time scale*, which simply is a closed subset of the real numbers. We denote a time scale by the symbol $\mathbb{T}$. The two most popular examples are $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$ that represent the classical theories of differential and of difference equations. Since Stefan Hilger formed the definition of derivatives and integrals on time scales, several authors has expounded on various aspects of this new theory, see the paper by (Agarwal et al., 2002, p1-26) and the references cited therein. The books on the subject of time scales, i.e., measure chain, by Bohner and Peterson (2001, 2003) summarize and organize much of time scale calculus.

In this paper, we are concerned with the oscillation of the second-order linear dynamic equation

$$(y(t) + p(t)y(t - \tau))^{\Delta\Delta} + q(t)y(t - \delta) = 0 \tag{1}$$

on a time scale $\mathbb{T}$.

Since we are interested in asymptotic behavior of solutions, we will suppose that the time scales $\mathbb{T}$ under consideration is not bounded above; i.e., it is a time scale interval of the form $[t_0, \infty)_{\mathbb{T}} = [t_0, \infty) \cap \mathbb{T}$.

Throughout this paper we assume that: τ and δ are positive constants such that the delay functions $\tau(t) := t - \tau < t$ and $\delta(t) := t - \delta < t$ satisfy $\tau(t) : \mathbb{T} \to \mathbb{T}$ and $\delta(t) : \mathbb{T} \to \mathbb{T}$ for all $t \in \mathbb{T}$,
(H1) $p(t), q(t) \in C_{rd}(\mathbb{T}, \mathbb{R}^+)$ where $C_{rd}(\mathbb{T}, \mathbb{R}^+)$ denotes the set of all function $f : \mathbb{T} \to \mathbb{R}^+$ which are right-dense continuous on $\mathbb{T}$ and $0 \le p(t) < p < 1$;

(H2) $y \in C^2_{rd}(I, \mathbb{R})$ where $I = [t_*, \infty) \subset \mathbb{T}$ for some $t_* > 0$;

(H3) $\int_0^\infty \delta(s)q(s)(1 - p(\delta(s)))\Delta s = \infty$.

By a solution of equation (1), we mean a nontrivial real value function $y(t)$ which has the properties $(y(t) + p(t)y(t - \tau)) \in$

$C_{rd}^2[t_y, \infty)$, $t_y > t_0$ and satisfying equation (1) for all $t > t_y$. Our attention is restricted to those solutions of equation (1) which exist on some half line $[t_y, \infty)$ and satisfy $sup\{|y(t)| : t > t_1\} > 0$ for any $t_1 > t_y$. A solution $y(t)$ of equation (1) is said to be oscillatory if it neither eventually positive nor eventually negative. Otherwise it is called nonoscillatory. The equation itself is called oscillatory if all its solutions are oscillatory.

We note that when $\mathbb{T} = \mathbb{R}$, we have $\sigma(t) = t$, $\mu(t) = 0$, $y^{\Delta}(t) = y'(t)$ and (1) becomes the second-order neutral delay differential equation

$$(y(t) + p(t)y(t-\tau))'' + q(t)y(t-\delta) = 0, \quad t \in \mathbb{T}. \tag{2}$$

If $\mathbb{T} = \mathbb{Z}$, we have $\sigma(t) = t + 1$, $\mu(t) = 1$, $y^{\Delta}(t) = \Delta y(t) = y(t + 1) - y(t)$ and (1) becomes the second-order neutral delay difference equation

$$\Delta^2(y(t) + p(t)y(t-\tau)) + q(t)y(t-\delta) = 0, \quad t \in \mathbb{T}. \tag{3}$$

If $\mathbb{T} = h\mathbb{Z}$, $h > 0$, we have $\sigma(t) = t + h$, $\mu(t) = h$,

$$y^{\Delta}(t) = \Delta_h y(t) = \frac{y(t+h) - y(t)}{h},$$

and (1) becomes the second-order neutral delay difference equation

$$\Delta_h^2(y(t) + p(t)y(t-\tau)) + q(t)y(t-\delta) = 0, \quad t \in \mathbb{T}.$$

If $\mathbb{T} = q^{\mathbb{N}} = \{t : t = q^n, n \in \mathbb{N}, q > 1\}$, we have $\sigma(t) = qt$, $\mu(t) = (q-1)t$,

$$y^{\Delta}(t) = \Delta_q y(t) = \frac{y(qt) - y(t)}{(q-1)t},$$

and (1) becomes the second-order q-neutral delay difference equation

$$\Delta_q^2(y(t) + p(t)y(t-\tau)) + q(t)y(t-\delta) = 0, \quad t \in \mathbb{T}.$$

Numerous oscillation criteria have been established for second-order neutral delay differential and difference equations (2), (3). See for examples [Grammatikopoulos et al., 1985, p267-274, Kubiaczyk et al., 2002, p185-212, Saker, 2003, p99-111, Sun et al., 2005, p909-918] and the references cited therein.

In this paper we improve the sufficient conditions for oscillation of the special case of nonlinear neutral delay differential equation

$$(r(t)([y(t) + p(t)y(t-\tau)]^{\Delta})^{\gamma})^{\Delta} + f(t, y(t-\delta)) = 0 \tag{4}$$

in (Saker, 2006, p123-141), (Agarwal et al., 2004, p203-217) and (Hong-Wu Wu et al., 2006, p321-331), the special case of second-order nonlinear delay dynamic equation

$$(p(t)(x^{\Delta}(t))^{\gamma})^{\Delta} + q(t)f(x(\tau(t))) = 0 \tag{5}$$

in (Zhenlai Han et al., 2007, p1-16) and the linear neutral delay differential equation

$$(y(t) + r(t)y(\tau(t)))^{\Delta\Delta} + p(t)y(\delta(t)) = 0 \tag{6}$$

in (Saker, 2007, p175-190).

Moreover, we intend to use the Riccati integral equations and the theory of integral inequalities (Kwong Man Kam, 2006, p1-18) for obtaining several oscillation criteria for (1). Hence the paper is organized as follows: In section 2, we present some preliminaries on time scales. In section 3, we establish some new sufficient conditions for oscillation of (1).

2. Some preliminaries on time scales

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. On any time scale $\mathbb{T}$, we define the forward and backward jump operators by

$$\sigma(t) := inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) := sup\{s \in \mathbb{T} : s < t\}.$$

A point $t \in \mathbb{T}$, $t > inf\mathbb{T}$ is said to be *left-dense* if $\rho(t) = t$, right-dense if $t < sup\mathbb{T}$ and $\sigma(t) = t$, *left-scattered* if $\rho(t) < t$ and *right-scattered* if $\sigma(t) > t$. The graininess function μ for a time scale $\mathbb{T}$ is defined by $\mu(t) = \sigma(t) - t$.

A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd*-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of *rd*-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R})$.

The set of functions $f : \mathbb{T} \to \mathbb{R}$ that are differentiable and whose derivative is *rd*-continuous function is denoted by $C^1_{rd} = C^1_{rd}(\mathbb{T}) = C^1_{rd}(\mathbb{T}, \mathbb{R})$.

A function $p : \mathbb{T} \to \mathbb{R}$ is called positively regressive (we write $p \in \mathfrak{R}^+$) if it is *rd*-continuous function and satisfies $1 + \mu(t)p(t) > 0$ for all $t \in \mathbb{T}$.

For a function $f : \mathbb{T} \to \mathbb{R}$ (the range $\mathbb{R}$ of f may be actually replaced by any Banach space) the (delta) derivative is defined by

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t},$$

if f is continuous at t and t is right-scattered. If t is right-dense then the derivative is defined by

$$f^{\Delta}(t) = \lim_{s\to t} \frac{f(t) - f(s)}{t - s},$$

provided this limit exists.

A function $f : [a, b] \to \mathbb{R}$ is said to be differentiable if its derivative exists, and a useful formula is

$$f^{\sigma} = f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t).$$

We will make use of the following product and quotient rules for the derivative of the product fg and the quotient f/g (where $gg^{\sigma} \neq 0$) of two differentiable functions f and g

$$(f.g)^{\Delta} = f^{\Delta} g + f^{\sigma} g^{\Delta} = f g^{\Delta} + f^{\Delta} g^{\sigma},$$

$$\left(\frac{f}{g}\right)^{\Delta} = \frac{f^{\Delta} g - f g^{\Delta}}{g g^{\sigma}}.$$

For $a, b \in \mathbb{T}$ and a differentiable function f, the Cauchy integral of f^{Δ} is defined by

$$\int_a^b f^{\Delta} \Delta t = f(b) - f(a),$$

and infinite integral is defined as

$$\int_{t_0}^{\infty} f(t)\Delta t = \lim_{b\to\infty} \int_{t_0}^{b} f(t)\Delta t.$$

An integration by parts formula reads

$$\int_a^b f(t) g^{\Delta}(t)\Delta t = [f(t)g(t)]_a^b - \int_a^b f^{\Delta}(t) g^{\sigma} \Delta t,$$

or

$$\int_a^b f^{\sigma} g^{\Delta}(t)\Delta t = [f(t)g(t)]_a^b - \int_a^b f^{\Delta}(t) g(t)\Delta t.$$

3. Main results

Before stating our main results in this paper, we start with the following lemmas.

Lemma 1 *Assume that (H3) hold and the inequality*

$$x^{\Delta\Delta}(t) + q(t)(1 - p(\delta(t))x(\delta(t)) \leq 0, \tag{7}$$

has a positive solution x on $[t_0, \infty)_{\mathbb{T}}$. Then there exists a $\mathrm{T} \in [t_0, \infty)_{\mathbb{T}}$, sufficiently large, so that $x^{\Delta}(t) \geq 0$ and $\frac{x^{\Delta}(t)}{x(t)} \leq \frac{1}{t}$ for $t \in [\mathrm{T}, \infty)_{\mathbb{T}}$.

proof. The proof is similar to the proof of Lemma 1 in (Erbe, L. ,2006, p65-78).

Lemma 2 *(Sahiner, 2005, pe1073-e1080) Suppose that the following conditions hold:*
(B1) $u \in C^2_{rd}(I, \mathbb{R})$ *where* $I = [t_*, \infty) \subset \mathbb{T}$ *for some* $t_* > 0$,
(B2) $u(t) > 0$, $u^\Delta > 0$ *and* $u^{\Delta\Delta} \leq 0$ *for* $t \geq t_*$.
Then, for each $k \in (0, 1)$*, there exists a constant* $t_k \in \mathbb{T}$, $t_k \geq t_*$*, such that*

$$u(\sigma(t)) \leq \frac{\sigma(t)}{k\delta(t)} u(\delta(t)) \quad for \quad t \geq t_k. \tag{8}$$

Lemma 3 *(Saker, 2006, p123-141) Let* $f(u) = Bu - Au^{\frac{\gamma+1}{\gamma}}$*, where* $A > 0$ *and* B *are constants,* γ *is a positive integer. Then* f *attains its maximum value on* $\mathbb{R}$ *at* $u^* = (\frac{B\gamma}{A(\gamma+1)})^\gamma$*, and*

$$\max_{u \in \mathbb{R}} f = f(u^*) = \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}} \frac{B^{\gamma+1}}{A^\gamma}.$$

Theorem 1 *Assume that (H1)-(H3) hold. Furthermore, assume that there exist positive rd-continuous* Δ*-differentiable functions* $\alpha(t)$ *and* $\beta(t)$ *with* $\beta(t) \geq t$ *such that* $\lim_{t\to\infty} \frac{\beta(t)}{\alpha(t)} = 0$, $\lim_{t\to\infty} \alpha(t) = \infty$*. If*

$$\limsup_{t\to\infty} \frac{\beta(t)}{\alpha(t)} \int_{t_1}^{t} \left(\alpha(s)q(s)(1 - p(s-\delta)) - \frac{(\alpha^\Delta(s))^2}{4\alpha(s)} \right) \Delta s \geq 0, \tag{9}$$

then every solution of Eq. (1) is oscillatory.

proof. Suppose to the contrary that $y(t)$ is a nonoscillatory solution of equation (1). Without loss of generality, we may assume that $y(t)$ is an eventually positive solution of (1) with $y(t - N) > 0$ where $N = max\{\tau, \delta\}$ for all $t > t_0$ sufficiently large. We shall consider only this case, since the substitution $z(t) = -y(t)$ transform Eq. (1) into an equation of the same form. Set

$$x(t) = y(t) + p(t)y(t - \tau). \tag{10}$$

From (10) and (1) we have

$$x^{\Delta\Delta}(t) + q(t)y(t - \delta) = 0, \tag{11}$$

for all $t > t_0$, and so $x^\Delta(t)$ is an eventually decreasing function. We first show that $x^\Delta(t)$ is eventually nonnegative. Indeed, since $q(t)$ is a positive function, the deceasing function $x^\Delta(t)$ is either eventually positive or eventually negative. Suppose there exists an integer $t_1 \geq t_0$ such that $x^\Delta(t_1) = c < 0$, then $x^\Delta(t) < x^\Delta(t_1) = c$ for $t \geq t_1$, hence $x^\Delta(t) \leq c$, which implies that

$$x(t) \leq x(t_1) + c(t - t_1) \to -\infty \quad \text{as} \quad t \to \infty,$$

which contradicts the fact that $x(t) > 0$ for all $t > t_1$. Hence $x^\Delta(t)$ is eventually nonnegative. Therefore, we see that there is some t_1 such that

$$x(t) > 0, x^\Delta(t) \geq 0, x^{\Delta\Delta} < 0, t \geq t_1. \tag{12}$$

This implies that

$$\begin{aligned} y(t) = x(t) - p(t)y(t - \tau) &= x(t) - p(t)[x(t - \tau) - p(t - \tau)y(t - 2\tau)] \\ &\geq x(t) - p(t)x(t - \tau) \geq x(t)(1 - p(t)). \end{aligned}$$

Then, for $t \geq t_1 = t_0 + \delta$ sufficiently large, we see that

$$y(t - \delta) \geq x(t - \delta)(1 - p(t - \delta)). \tag{13}$$

From (11) and (13) we obtain for $t \geq t_1$

$$x^{\Delta\Delta}(t) + q(t)(1 - p(t - \delta))x(t - \delta) \leq 0. \tag{14}$$

Then from (14), we have

$$\alpha(t)q(t)(1 - p(t - \delta)) \leq \frac{-x^{\Delta\Delta}(t)\alpha(t)}{x(t - \delta)}. \tag{15}$$

Integrating the above inequality from t_1 to t, we get

$$\int_{t_1}^{t} \alpha(s)q(s)(1-p(s-\delta))\Delta s \leq -\int_{t_1}^{t} \frac{\alpha(s)x^{\Delta\Delta}(s)}{x(s-\delta)}\Delta s,$$

hence

$$\begin{aligned}
\int_{t_1}^{t} \alpha(s)q(s)(1-p(s-\delta))\Delta s &\leq -\frac{\alpha(t)x^{\Delta}(t)}{x(t-\delta)} + \frac{\alpha(t_1)x^{\Delta}(t_1)}{x(t_1-\delta)} + \int_{t_1}^{t} x^{\Delta}(\sigma(s))(\frac{\alpha(s)}{x(s-\delta)})^{\Delta}\Delta s \\
&\leq -\frac{\alpha(t)x^{\Delta}(t)}{x(t-\delta)} + \frac{\alpha(t_1)x^{\Delta}(t_1)}{x(t_1-\delta)} + \int_{t_1}^{t} \frac{x^{\Delta}(\sigma(s))\alpha^{\Delta}(s)}{x(\sigma(s)-\delta)} \\
&- \int_{t_1}^{t} \frac{x^{\Delta}(\sigma(s))\alpha(s)x^{\Delta}(s-\delta)}{x(s-\delta)x(\sigma(s)-\delta)}\Delta s. \qquad (16)
\end{aligned}$$

In view of (12), we obtain

$$\frac{x^{\Delta}(\sigma(t))}{x(\sigma(t)-\delta)} < \frac{x^{\Delta}(t)}{x(\sigma(t)-\delta)} < \frac{x^{\Delta}(t-\delta)}{x(\sigma(t)-\delta)} < \frac{x^{\Delta}(t-\delta)}{x(t-\delta)}. \qquad (17)$$

From (16) and (17) we get

$$\begin{aligned}
\int_{t_1}^{t} \alpha(s)q(s)(1-p(s-\delta))\Delta s &\leq -\frac{\alpha(t)x^{\Delta}(t)}{x(t-\delta)} + \frac{\alpha(t_1)x^{\Delta}(t_1)}{x(t_1-\delta)} \\
&+ \int_{t_1}^{t} \frac{\alpha^{\Delta}(s)x^{\Delta}(\sigma(s))}{x(\sigma(s)-\delta)}\Delta s - \int_{t_1}^{t} \alpha(s)(\frac{x^{\Delta}(\sigma(s))}{x(\sigma(s)-\delta)})^2\Delta s \\
&\leq \frac{\alpha(t_1)x^{\Delta}(t_1)}{x(t_1-\delta)} - \frac{\alpha(t)x^{\Delta}(t)}{x(t-\delta)} + \int_{t_1}^{t} \frac{(\alpha^{\Delta}(s))^2}{4\alpha(s)}\Delta s \\
&- \int_{t_1}^{t} (\frac{\alpha^{\Delta}(s)}{2\sqrt{\alpha(s)}} - \frac{\sqrt{\alpha(s)}x^{\Delta}(\sigma(s))}{x(\sigma(s)-\delta)})^2\Delta s.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_{t_1}^{t} \alpha(s)q(s)(1-p(s-\delta))\Delta s &\leq \frac{\alpha(t_1)x^{\Delta}(t_1)}{x(t_1-\delta)} + \sqrt{\alpha(t)}(\frac{\alpha^{\Delta}(t)}{2\sqrt{\alpha(t)}} - \frac{\sqrt{\alpha(t)}x^{\Delta}(\sigma(t))}{x(\sigma(t)-\delta)}) - \frac{\alpha^{\Delta}(t)}{2} \\
&+ \int_{t_1}^{t} \frac{(\alpha^{\Delta}(s))^2}{4\alpha(s)}\Delta s - \int_{t_1}^{t} (\frac{\alpha^{\Delta}(s)}{2\sqrt{\alpha(s)}} - \frac{\sqrt{\alpha(s)}x^{\Delta}(\sigma(s))}{x(\sigma(s)-\delta)})^2\Delta s.
\end{aligned} \qquad (18)$$

Let

$$w(t) = \frac{\alpha^{\Delta}(s)}{2\sqrt{\alpha(s)}} - \frac{\sqrt{\alpha(t)}x^{\Delta}(\sigma(t))}{x(\sigma(t)-\delta)}. \qquad (19)$$

Then from (18), (19), it is easy to see that

$$\begin{aligned}
\int_{t_1}^{t} \left(\alpha(s)q(s)(1-p(s-\delta)) - \frac{(\alpha^{\Delta}(s))^2}{4\alpha(s)}\right)\Delta s &\leq \frac{\alpha(t_1)x^{\Delta}(t_1)}{x(t_1-\delta)} + \sqrt{\alpha(t)}w(t) - \frac{\alpha^{\Delta}(t)}{2} - \int_{t_1}^{t} w^2(s)\Delta s \\
&\leq \frac{\alpha(t_1)x^{\Delta}(t_1)}{x(t_1-\delta)} - \frac{\alpha(t)x^{\Delta}(\sigma(t))}{x(\sigma(s)-\delta)}. \qquad (20)
\end{aligned}$$

Then, by lemma 1, for sufficiently large t, there exists $\beta(t) \geq t$ such that $\frac{1}{\beta(t)} \leq \frac{x^{\Delta}(\sigma(t))}{x(\sigma(t))} \leq \frac{1}{t}$. Hence

$$\frac{\beta(t)}{\alpha(t)} \int_{t_1}^{t} \left(\alpha(s)q(s)(1-p(s-\delta)) - \frac{(\alpha^{\Delta}(s))^2}{4\alpha(s)}\right)\Delta s \leq \frac{\beta(t)}{\alpha(t)}\frac{\alpha(t_1)x^{\Delta}(t_1)}{x(t_1-\delta)} - 1.$$

Since $\lim_{t\to\infty} \frac{\beta(t)}{\alpha(t)} = 0$ we have

$$\frac{\beta(t)}{\alpha(t)} \int_{t_1}^{t} \left(\alpha(s)q(s)(1-p(s-\delta)) - \frac{(\alpha^{\Delta}(s))^2}{4\alpha(s)}\right)\Delta s < 0,$$

which contradicts the condition (9). The proof is complete.

Theorem 2 *Assume that (H1)-(H3) hold. Let $\alpha(t)$, $\beta(t)$ be as defined in Theorem 1 and*

$$\limsup_{t\to\infty} \frac{\beta(t)}{\alpha(t)} \int_{t_1}^{t} \left(\frac{s-\delta}{s}\alpha(s)q(s)(1-p(s-\delta)) - \frac{(\alpha^{\Delta}(s))^2}{4\alpha(s)} \right) \Delta s \geq 0 \tag{21}$$

then every solution of Eq. (1) is oscillatory.

proof. Suppose to the contrary that $y(t)$ is a nonoscillatory solution of Eq. (1) and let $t_1 \geq t_0$ be such that $y(t) \neq 0$ for all $t \geq t_1$, so without loss of generality, we may assume that $y(t)$ is an eventually positive solution of Eq. (1). From (15), we get

$$\frac{t-\delta}{t}\alpha(t)q(t)(1-p(t-\delta)) \leq \frac{-x^{\Delta\Delta}(t)\alpha(t)}{x(t-\delta)}. \tag{22}$$

We proceed as in the proof of Theorem 1 and it follows that

$$\frac{\beta(t)}{\alpha(t)} \int_{t_1}^{t} \left(\frac{s-\delta}{s}\alpha(s)q(s)(1-p(s-\delta)) - \frac{(\alpha^{\Delta}(s))^2}{4\alpha(s)} \right) \Delta s \quad < \quad 0,$$

which contradicts the condition (21). The proof is complete.

Corollary 1 *Assume that (H1)-(H3) hold and. Let $\alpha(t)$ be as defined in Theorem 1 and*

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left(\alpha(s)q(s)(1-p(s-\delta)) - \frac{(\alpha^{\Delta}(s))^2}{4\alpha(s)} \right) \Delta s = \infty, \tag{23}$$

then every solution of Eq. (1) is oscillatory.

proof. We proceed as in the proof of Theorem 1 to prove that there exists $t_1 \geq t_0$ such that (20) holds for $t \geq t_1$. From (20), it follows that

$$\int_{t_1}^{t} \left(\alpha(s)q(s)(1-p(s-\delta)) - \frac{(\alpha^{\Delta}(s))^2}{4\alpha(s)} \right) \Delta s \quad \leq \quad \frac{\alpha(t_1)x^{\Delta}(t_1)}{x(t_1-\delta)},$$

which contradicts the condition (23).

Corollary 2 *Assume that (H1)-(H3) hold. Let $\alpha(t)$ be as defined in Theorem 1 and*

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left(\frac{s-\delta}{s}\alpha(s)q(s)(1-p(s-\delta)) - \frac{(\alpha^{\Delta}(s))^2}{4\alpha(s)} \right) \Delta s = \infty, \tag{24}$$

then every solution of Eq. (1) is oscillatory.

proof. We proceed as in the proof of Theorem 1 to prove that there exists $t_1 \geq t_0$ such that (20) holds for $t \geq t_1$. From (20), it follows that

$$\int_{t_1}^{t} \left(\frac{s-\delta}{s}\alpha(s)q(s)(1-p(s-\delta)) - \frac{(\alpha^{\Delta}(s))^2}{4\alpha(s)} \right) \Delta s \quad \leq \quad \frac{\alpha(t_1)x^{\Delta}(t_1)}{x(t_1-\delta)},$$

which contradicts the condition (24).

Theorem 3 *Assume that (H1)-(H3) hold. Let $\alpha(t)$ be as defined in Theorem 1 such that for some positive constant $k \in (0,1)$,*

$$\limsup_{t\to\infty} \frac{\beta(t)}{\alpha(t)} \int_{t_1}^{t} \left(k(\frac{s-\delta}{\sigma(s)})\alpha(s)q(s)(1-p(s-\delta)) - \frac{(\alpha^{\Delta}(s))^2}{4\alpha(s)} \right) \Delta s \geq 0, \tag{25}$$

then every solution of Eq. (1) is oscillatory.

proof. Suppose that Eq. (1) has a nonoscillatory solution $y(t)$. We may assume without loss of generality that $y(t) > 0$ for all $t > t_0$. We will consider only this case, since the proof when $y(t)$ is eventually negative is similar. In view of Lemma 2, for each positive constant $k \in (0,1)$, there exists a $t_1 = max\{t_k, t_0\}$ such that

$$x(t) \leq x(\sigma(t)) \leq \frac{\sigma(t)}{k(t-\delta)}x(t-\delta) \leq \frac{\sigma(t)}{k(t-\delta)}x(t) \quad \text{for} \quad t \geq t_1. \tag{26}$$

From (15) and from (26), we get

$$\frac{k(t-\delta)}{\sigma(t)}\alpha(t)q(t)(1-p(t-\delta)) \le \frac{-x^{\Delta\Delta}(t)\alpha(t)}{x(t-\delta)}. \tag{27}$$

We proceed as in the proof of Theorem 1, so we get

$$\begin{aligned}\int_{t_1}^{t}\left(\frac{k(t-\delta)}{\sigma(t)}\alpha(s)q(s)(1-p(s-\delta)) - \frac{(\alpha^{\Delta}(s))^2}{4\alpha(s)}\right)\Delta s \quad &\le \quad \frac{\alpha(t_1)x^{\Delta}(t_1)}{x(t_1-\delta)} + \sqrt{\alpha(t)}w(t) - \frac{\alpha^{\Delta}(t)}{2} - \int_{t_1}^{t} w^2(s)\Delta s \\ &\le \quad \frac{\alpha(t_1)x^{\Delta}(t_1)}{x(t_1-\delta)} - \frac{\alpha(t)x^{\Delta}(\sigma(t))}{x(\sigma(s)-\delta)}. \end{aligned} \tag{28}$$

From lemma 1, for sufficiently large t, there exists $\beta(t) \ge t$ such that $\frac{1}{\beta(t)} \le \frac{x^{\Delta}(\sigma(t))}{x(\sigma(t))} \le \frac{1}{t}$. Hence

$$\frac{\beta(t)}{\alpha(t)}\int_{t_1}^{t}\left(\frac{k(t-\delta)}{\sigma(t)}\alpha(s)q(s)(1-p(s-\delta)) - \frac{(\alpha^{\Delta}(s))^2}{4\alpha(s)}\right)\Delta s \quad \le \quad \frac{\beta(t)}{\alpha(t)}\frac{\alpha(t_1)x^{\Delta}(t_1)}{x(t_1-\delta)} - 1.$$

Since $\lim_{t\to\infty}\frac{\beta(t)}{\alpha(t)} = 0$ we have

$$\frac{\beta(t)}{\alpha(t)}\int_{t_1}^{t}\left(\frac{k(t-\delta)}{\sigma(t)}\alpha(s)q(s)(1-p(s-\delta)) - \frac{(\alpha^{\Delta}(s))^2}{4\alpha(s)}\right)\Delta s \quad < \quad 0,$$

which contradicts the condition (25). The proof is complete.

Corollary 3 *Assume that (H1)-(H3) hold. Let $\alpha(t)$ be as defined in Theorem 1 such that for some positive constant $k \in (0,1)$,*

$$\limsup_{t\to\infty}\int_{t_1}^{t}\left(k(\frac{s-\delta}{\sigma(s)})\alpha(s)q(s)(1-p(s-\delta)) - \frac{(\alpha^{\Delta}(s))^2}{4\alpha(s)}\right)\Delta s = \infty, \tag{29}$$

then every solution of Eq. (1) is oscillatory.

Assume that the condition (23) fails, and

$$R(t) = \int_{t}^{\infty}\left(\alpha(s)q(s)(1-p(s-\delta)) - \frac{(\alpha^{\Delta}(s))^2}{4\alpha(s)}\right)\Delta s. \tag{30}$$

In this case we have the following result.

Theorem 4 *Assume that (H1)-(H3) hold and there exists a positive rd-continuous Δ-differentiable functions $\alpha(t)$ such that* $\lim_{t\to\infty}\alpha(t) = \infty$,

$$\limsup_{t\to\infty}\frac{t-\delta}{\alpha(t)}R(t) > 1, \tag{31}$$

then every solution of Eq. (1) *is oscillatory.*

proof. Assume that Eq. (1) has a positive solution $y(t)$ for all $t \ge t_1$. Then from condition (31) we have,

$$\frac{t_1-\delta}{\alpha(t_1)}R(t_1) > 1. \tag{32}$$

From lemma 1, for sufficiently large t, we have

$$\frac{tx^{\Delta}(t)}{x(t)} \le 1. \tag{33}$$

Then from (32) and (33) we get

$$\frac{t_1-\delta}{\alpha(t_1)}R(t_1) > \frac{(t_1-\delta)x^{\Delta}(t_1-\delta)}{x(t_1-\delta)} > \frac{(t_1-\delta)x^{\Delta}(t_1)}{x(t_1-\delta)}.$$

Let

$$N(t) = \frac{\alpha(t)x^{\Delta}(t)}{x(t-\delta)} > 0. \tag{34}$$

For that,

$$R(t_1) > N(t_1) \tag{35}$$

From (20), (34) and (35) we get

$$\frac{\alpha(t)N(\sigma(t))}{\alpha(\sigma(t))} < \int_{t_1}^{\infty}\left(\alpha(s)q(s)(1-p(s-\delta)) - \frac{(\alpha^{\Delta}(s))^2}{4\alpha(s)}\right)\Delta s - \int_{t_1}^{t}\left(\alpha(s)q(s)(1-p(s-\delta)) - \frac{(\alpha^{\Delta}(s))^2}{4\alpha(s)}\right)\Delta s. \tag{36}$$

From (36), for sufficiently large t, we have

$$\frac{\alpha(t)N(\sigma(t))}{\alpha(\sigma(t))} < 0.$$

which is a contradiction. This complete the proof.

Remark 1 *From Theorem 1 and Theorem 2 we can obtain different conditions for oscillation of Eq.* (1) *by choosing* $\alpha(t) = t$.

Corollary 4 *Assume that (H1)-(H3) hold. Let* $\beta(t)$ *as defined in Theorem 1 such that* $\lim_{t\to\infty}\frac{\beta(t)}{t} = 0$. *If*

$$\limsup_{t\to\infty}\frac{\beta(t)}{t}\int_{t_1}^{t}\left(sq(s)(1-p(s-\delta)) - \frac{1}{4s}\right)\Delta s > 0,$$

then every solution of Eq. (1) *is oscillatory.*

Corollary 5 *Assume that (H1)-(H3) hold. Let* $\beta(t)$ *as defined in Theorem 1 such that* $\lim_{t\to\infty}\frac{\beta(t)}{t} = 0$. *If*

$$\limsup_{t\to\infty}\frac{\beta(t)}{t}\int_{t_1}^{t}\left((s-\delta)q(s)(1-p(s-\delta)) - \frac{1}{4s}\right)\Delta s > 0,$$

then every solution of Eq. (1) *is oscillatory.*

Corollary 6 *Assume that (H1)-(H3) hold. If*

$$\limsup_{t\to\infty}\int_{t_1}^{t}\left(sq(s)(1-p(s-\delta)) - \frac{1}{4s}\right)\Delta s = \infty,$$

then every solution of Eq. (1) *is oscillatory.*

Corollary 7 *Assume that (H1)-(H3) hold and*

$$\limsup_{t\to\infty}\int_{t_1}^{t}\left((s-\delta)q(s)(1-p(s-\delta)) - \frac{1}{4s}\right)\Delta s = \infty,$$

then every solution of Eq. (1) *is oscillatory.*

The following theorem gives Philos-type oscillation criteria for Eq. (1).
First, let us introduce now the class of functions $\Re$ which will be extensively used in the sequel.
Let $\mathbb{D}_0 \equiv \{(t,s) \in \mathbb{T}^2 : t > s \geq t_0\}$ and $\mathbb{D} \equiv \{(t,s) \in \mathbb{T}^2 : t \geq s \geq t_0\}$. The function $H \in C_{rd}(\mathbb{D}, \mathbb{R})$ is said to belongs to the class $\Re$ if

(i) $H(t,t) = 0$, $t \geq t_0$, $H(t,s) > 0$ on $\mathbb{D}_0$,

(ii) H has a continuous Δ-partial derivative $H^{\Delta_s}(t,s)$ on $\mathbb{D}_0$ with respect to the second variable. (H is *rd*-continuous function if H is *rd*-continuous function in t and s).

Theorem 5 *Assume that (H1)-(H3) hold. Furthermore, assume that there exist a positive rd-continuous Δ-differentiable functions* $\alpha(t)$ *and* $\beta(t)$ *with* $\beta(t) \geq t$ *such that* $\lim_{t\to\infty}\frac{\beta(t)}{\alpha(t)} = 0$, $\lim_{t\to\infty}\alpha(t) = \infty$ *and*

$$\limsup_{t\to\infty}\frac{1}{H(t,t_1)}\frac{\beta(t)}{\alpha(t)}\int_{t_1}^{t}\left(\alpha(s)q(s)(1-p(s-\delta))H(t,s) - \frac{B^2(t,s)\alpha^{\sigma^2}}{4\alpha(s)H(t,s)}\right)\Delta s > 0,$$

where

$$B(t,s) = \frac{\alpha^{\Delta}(s)H(t,s)}{\alpha^{\sigma}} + H^{\Delta_s}(t,s), \tag{37}$$

then every solution of Eq. (1) *is oscillatory.*

proof. Suppose to the contrary that $y(t)$ is a nonoscillatory solution of Eq. (1) and let $t \geq t_1$ be such that $y(t) \neq 0$ for all $t \geq t_1$, so without loss of generality, we may assume that $y(t)$ is an eventually positive solution of Eq. (1) with $y(t-N) > 0$ where $N = \max\{\tau, \delta\}$ for all $t \geq t_1$ sufficiently large. We proceed as in the proof of Theorem 1. From (15) we get

$$\int_{t_1}^{t} \alpha(s)q(s)(1-p(s-\delta))H(t,s)\Delta s \leq -\int_{t_1}^{t} \frac{\alpha(s)H(t,s)x^{\Delta\Delta}(s)}{x(s-\delta)}\Delta s$$

and then

$$\begin{aligned}\int_{t_1}^{t} \alpha(s)q(s)(1-p(s-\delta))H(t,s)\Delta s \leq\ & \frac{\alpha(t_1)H(t,t_1)x^{\Delta}(t_1)}{x(t_1-\delta)} - \frac{\alpha(t)H(t,t)x^{\Delta}(t)}{x(t-\delta)} + \int_{t_1}^{t} \frac{x^{\Delta}(\sigma(s))\alpha^{\Delta}(s)H(t,s)}{x(\sigma(s)-\delta)}\Delta s \\ &+ \int_{t_1}^{t} \frac{x^{\Delta}(\sigma(s))\alpha^{\sigma}(s)H^{\Delta_s}(t,s)}{x(\sigma(s)-\delta)}\Delta s - \int_{t_1}^{t} \frac{\alpha(s)x^{\Delta}(\sigma(s))H(t,s)x^{\Delta}(s-\delta)}{x(s-\delta)x(\sigma(s)-\delta)}\Delta s.\end{aligned} \tag{38}$$

Then by using (17) we get

$$\begin{aligned}\int_{t_1}^{t} \alpha(s)q(s)(1-p(s-\delta))H(t,s)\Delta s \leq\ & \frac{\alpha(t_1)H(t,t_1)x^{\Delta}(t_1)}{x(t_1-\delta)} + \int_{t_1}^{t} (H^{\Delta_s}(t,s) + \frac{\alpha^{\Delta}(s)H(t,s)}{\alpha^{\sigma}})\frac{x^{\Delta}(\sigma(s))\alpha^{\sigma}}{x(\sigma(s)-\delta)}\Delta s \\ &- \int_{t_1}^{t} \alpha(s)H(t,s)(\frac{x^{\Delta}(\sigma(s))}{x(\sigma(s)-\delta)})^2\Delta s,\end{aligned} \tag{39}$$

where $H(t,t) = 0$. Therefore by using Lemma 3, with

$$\gamma = 1,\ B = (H^{\Delta_s}(t,s) + \frac{\alpha^{\Delta}(s)H(t,s)}{\alpha^{\sigma}}),\ A = \frac{\alpha(s)H(t,s)}{\alpha^{\sigma^2}} \quad \text{and} \quad u = \frac{x^{\Delta}(\sigma(s))\alpha^{\sigma}}{x(\sigma(s)-\delta)}, \tag{40}$$

we get that

$$\int_{t_1}^{t} \left(\alpha(s)q(s)(1-p(s-\delta))H(t,s) - \frac{B^2\alpha^{\sigma^2}}{4\alpha(s)H(t,s)}\right)\Delta s \leq \frac{\alpha(t_1)H(t,t_1)x^{\Delta}(t_1)}{x(t_1-\delta)}. \tag{41}$$

So

$$\frac{\beta(t)}{\alpha(t)H(t,t_1)}\int_{t_1}^{t} \left(\alpha(s)q(s)(1-p(s-\delta))H(t,s) - \frac{B^2\alpha^{\sigma^2}}{4\alpha(s)H(t,s)}\right)\Delta s \leq \frac{\beta(t)}{\alpha(t)}\frac{\alpha(t_1)x^{\Delta}(t_1)}{x(t_1-\delta)}.$$

Since $\lim_{t\to\infty}\frac{\beta(t)}{\alpha(t)} = 0$ we have

$$\frac{\beta(t)}{\alpha(t)H(t,t_1)}\int_{t_1}^{t} \left(\alpha(s)q(s)(1-p(s-\delta))H(t,s) - \frac{B^2\alpha^{\sigma^2}}{4\alpha(s)H(t,s)}\right)\Delta s \leq 0,$$

which contradicts the condition (37). Then every solution of Eq. (1) oscillates.

Corollary 8 *Assume that (H1)-(H3) hold. Let $\alpha(t)$, $B(t,s)$ be as defined in Theorem 5 and*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_1)}\int_{t_1}^{t} \left(\alpha(s)q(s)(1-p(s-\delta))H(t,s) - \frac{B^2(t,s)\alpha^{\sigma^2}}{4\alpha(s)H(t,s)}\right)\Delta s = \infty, \tag{42}$$

then every solution of Eq. (1) is oscillatory.

proof. By proceeding as in the proof of Theorem 5 and from (41) we get

$$\frac{1}{H(t,t_1)}\int_{t_1}^{t} \left(\alpha(s)q(s)(1-p(s-\delta))H(t,s) - \frac{B^2\alpha^{\sigma^2}}{4\alpha(s)H(t,s)}\right)\Delta s \leq \frac{\alpha(t_1)x^{\Delta}(t_1)}{x(t_1-\delta)},$$

which contradicts the condition (42). Then every solution of Eq. (1) oscillates.

Theorem 6 *Assume that (H1)-(H3) hold. Let $\alpha(t)$, $\beta(t)$, $B(t,s)$ be as defined in Theorem 5 and*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_1)}\frac{\beta(t)}{\alpha(t)}\int_{t_1}^{t} \left(\frac{s-\delta}{s}\alpha(s)q(s)(1-p(s-\delta))H(t,s) - \frac{B^2(t,s)\alpha^{\sigma^2}}{4\alpha(s)H(t,s)}\right)\Delta s > 0, \tag{43}$$

then every solution of Eq. (1) is oscillatory.

proof. By proceeding as in Theorem 5 we get

$$\int_{t_1}^{t}\left(\frac{s-\delta}{s}\alpha(s)q(s)(1-p(s-\delta))H(t,s)-\frac{B^2\alpha^{\sigma^2}}{4\alpha(s)H(t,s)}\right)\Delta s \le \frac{\alpha(t_1)H(t,t_1)x^{\Delta}(t_1)}{x(t_1-\delta)}$$

and then,

$$\frac{\beta(t)}{\alpha(t)H(t,t_1)}\int_{t_1}^{t}\left(\frac{s-\delta}{s}\alpha(s)q(s)(1-p(s-\delta))H(t,s)-\frac{B^2\alpha^{\sigma^2}}{4\alpha(s)H(t,s)}\right)\Delta s \le \frac{\beta(t)}{\alpha(t)}\frac{\alpha(t_1)x^{\Delta}(t_1)}{x(t_1-\delta)}.$$

Since $\lim_{t\to\infty}\frac{\beta(t)}{\alpha(t)}=0$ we have

$$\frac{\beta(t)}{\alpha(t)H(t,t_1)}\int_{t_1}^{t}\left(\frac{s-\delta}{s}\alpha(s)q(s)(1-p(s-\delta))H(t,s)-\frac{B^2\alpha^{\sigma^2}}{4\alpha(s)H(t,s)}\right)\Delta s \quad\le\quad 0$$

which contradicts the condition (43). Then every solution of Eq. (1) oscillates.

Remark 2 *With an appropriate choice of the functions $H \in C_{rd}(\mathbb{D},\mathbb{R})$ and $h \in C_{rd}(\mathbb{D}_0,\mathbb{R})$. We can take $H(t,s) = (t-s)^m, (t,s)\in\mathbb{D}$ with $m>1$. It is clear that H belongs to the class $\mathfrak{R}$.*
Now, we claim that

$$((t-s)^m)^{\Delta_s} \le -m(t-\sigma(s))^{m-1}. \tag{44}$$

proof.

We consider the following two case:
Case 1: If $\mu(t)=0$ then $((t-s)^m)^{\Delta_s} = -m(t-\sigma(s))^{m-1}$.
Case 2: If $\mu(t)\neq 0$ then we have

$$\begin{aligned}((t-s)^m)^{\Delta_s} &= \frac{1}{\mu(s)}[(t-\sigma(s))^m-(t-s)^m]\\ &= \frac{-1}{\sigma(s)-s}[(t-s)^m-(t-\sigma(s))^m].\end{aligned} \tag{45}$$

Using Hardy et al. inequality (Hardy, 1952)

$$x^m-y^m \ge my^{m-1}(x-y) \qquad \text{for all} \quad x\ge y>0 \quad \text{and} \quad m\ge 1 \tag{46}$$

we have

$$[(t-s)^m-(t-\sigma(s))^m] \ge m(t-\sigma(s))^{m-1}(\sigma(s)-s). \tag{47}$$

Then from (45) and (47), we have

$$((t-s)^m)^{\Delta_s} \le -m(t-\sigma(s))^{m-1}. \tag{48}$$

and this proves (44).

From the above claim and Theorem 5, we have the following Kamenev-type oscillation criteria for Eq. (1).

Corollary 9 *Assume that (H1)-(H3) hold. Let $\alpha(t)$, $\beta(t)$ be as defined in Theorem 5 and*

$$\limsup_{t\to\infty}\frac{1}{t^m}\frac{\beta(t)}{\alpha(t)}\int_{t_0}^{t}\left(\alpha(s)q(s)(1-p(s-\delta))(t-s)^m-\frac{C^2(t,s)\alpha^{\sigma^2}}{4\alpha(s)(t-s)^m}\right)\Delta s>0,$$

where

$$C(t,s)=\frac{\alpha^{\Delta}(s)(t-s)^m}{\alpha^{\sigma}}-m(t-\sigma(s))^{m-1},$$

then every solution of Eq. (1) *is oscillatory.*

Corollary 10 *Assume that (H1)-(H3) hold. Let $\alpha(t)$ be as defined in Theorem 5 and $C(t,s)$ be as defined in Corollary 9*

$$\limsup_{t\to\infty}\frac{1}{t^m}\int_{t_0}^{t}\left(\alpha(s)q(s)(1-p(s-\delta))(t-s)^m-\frac{C^2(t,s)\alpha^{\sigma^2}}{4\alpha(s)(t-s)^m}\right)\Delta s=\infty,$$

then every solution of Eq. (1) *is oscillatory.*

Theorem 7 *Assume that (H1)-(H3) hold. Let $\alpha(t)$, $\beta(t)$ be as defined in Theorem 5 and*

$$\limsup_{t\to\infty} \frac{\beta(t)}{\alpha(t)} \int_{t_1}^{t} q(s)(1 - p(s-\delta))\Delta s > 0, \tag{49}$$

then every solution of Eq. (1) *oscillates.*

proof. Suppose to the contrary that Eq. (1) has a nonoscillatory solution $y(t)$. We may assume that there exists $t_1 \geq t_0$ such that $y(t) > 0$ for all $t \geq t_1$.
We proceed in this theorem as in Theorem 5 and from (14) we get

$$\begin{aligned} x^{\Delta}(t_1) - x^{\Delta}(t) &\geq \int_{t_1}^{t} q(s)(1 - p(s-\delta))x(s-\delta)\Delta s \\ x^{\Delta}(t_1) &\geq x(t_1 - \delta) \int_{t_1}^{t} q(s)(1 - p(s-\delta))\Delta s. \end{aligned}$$

For that,

$$\frac{x^{\Delta}(t_1)}{x(t_1 - \delta)} \geq \int_{t_1}^{t} q(s)(1 - p(s-\delta))\Delta s.$$

Since $\lim_{t\to\infty} \frac{\beta(t)}{\alpha(t)} = 0$ we have

$$\frac{\beta(t)}{\alpha(t)} \int_{t_1}^{t} q(s)(1 - p(s-\delta))\Delta s \leq 0, \tag{50}$$

which is contradicts (49), and consequently, Eq. (1) has no eventually positive solution. Similarly, by using the same technique we can prove that Eq. (1) has no eventually negative solution. Thus Eq. (1) is oscillatory.

Corollary 11 *Assume that (H1) hold. If*

$$\limsup_{t\to\infty} \int_{t_1}^{t} q(s)(1 - p(s-\delta))\Delta s = \infty,$$

then every solution of Eq. (1) *is oscillatory.*

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The Minimal Description of Formal Concept Analysis

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Abstract

In rough sets, there were some of results of minimal description formed by covering, this paper gives some remarks about them.Using the idea of minimal description in formal concept analysis, discusses the corresponding relation between the covering of objects set and the attributes set.

Keywords: Rough set, Formal concept analysis, Minimal description, Covering, Partition

In 1982, R.Wille (1999) proposed a new model to represent the formal concepts associated to a context (G, M, I), named formal concept analysis based on the formal context, which is a pair consisting of a set of objects (the extension) and a set of attributes (the intension) such that the intension consists of exactly those attributes that the objects in the extension have in common, and the extension contains exactly those objects that share all attributes in the intension. The extension and the intension of a concept uniquely determine each other. The main goal is to reveal the hierarchical structure of formal concepts and to investigate the dependencies among attributes. It provides a theoretical framework for the discovery and design of concept hierarchies from relational information systems. It serves as a basis for conceptual data analysis and knowledge processing. The family of all formal concepts is a complete lattices, which is an effective method for several real-world applications in data analysis, such as object-oriented databases, inheritance lattices, mining for association rules, generating frequent sets (F.M.Baltasar, 1998; J.S. Deogun, 2004; So Kuznetsov, 2002; R.Godin, 1995) etc, one of the important challenges in data handling is generating or navigating the concept lattice of binary relation.

The theory of rough sets as a tool for processing uncertain and incomplete information, proposed by Z.Pawlak (1991) in 1982, in which the lower approximation and upper approximation of an arbitrary subset of universe U are the basic operators.

The concepts of the lower and upper approximations in rough sets theory are fundamental to the examination of granularity in knowledge, and are an effective way of studying imprecision, vagueness, and uncertainty, and rough set has been successed using all kinds of artificial intelligence fields, such as: data mining , machine learning. There are many authors studying rough sets and formal concept analysis, the readers can see (Zhang wenxiu, 2006; A.Burusco, 1994; A.Burusco, 2000; R. Belohavek, 1995). In (A.Skowron, 1992; Z.Bonikowski, 1998; William Zhu, 2003; J.A.Pomykala, 1987), the concept of covering of a universe was presented to construct the upper and the lower approximations of an arbitrary set. In (Z.Bonikowski, 1998), the authors studied the reduct of covering of generalized rough sets; In (Z.Bonikowski, 1998; Z.Bonikowski, 2003), the authors mainly studied the structure of cover, authors proposed a representative approximation spaces, that is, discussed the point x of universe U ($i.e. x \in U$) by covering, they defined the description of x and the minimal description of x; In (Xu WeiHua, 2007), the authors considered some new concepts and main results in generalized rough sets induced by a covering; In (Qiu Weigen, 2006), the author discussed fined covering fuzzy generalized rough sets; In (Eric C.C.Tsang, 2008), the authors defined the induced cover and the intersection of a family of covers by using the induced cover in covering generalized rough sets.

The rest of the paper is organized as following, in section 1, we give some basic knowledge of concept lattices and rough sets. In section 2, give some remarks of the minimal description which depict by covering in covering approximation space. In section 3 , we use minimal description to objects set and attributes set of formal context, and discuss the corresponding relation of the covering of objects set and the covering of attributes set, and conclusions are given in section 4.

1. Preliminaries

Definition 1.1 (B.Ganter, 1999) Let (G, M, I) be a formal context, where $G = \{a_1, \cdots, a_m\}$ is an objects set, $M =$

$\{x_1, \cdots, x_n\}$ is an attributes set, a binary relation $I \subseteq G \times M$. If object $g \in G$ has attribute $m \in M$ under relation I, denote gIm or $(g, m) \in I$. Typically, 1 means $(g, m) \in I$, and 0 means $(g, m) \notin I$.(In the following , we will use this denotation).

Definition 1.2 (B.Ganter, 1999) In formal context $K = (G, M, I)$, $\forall A \subseteq G, \forall B \subseteq M$, define:

$A' = \{m \in M \mid gIm, \forall g \in A\}$, i.e., the set of all attributes shared by all objects from A.

$B' = \{g \in G \mid gIm, \forall m \in M\}$, i.e., the set of all objects sharing all attributes from B.

Definition 1.3 (B.Ganter, 1999) The formal concept of formal context $K = (G, M, I)$ is the set pair (A, B), where $A \subseteq G, B \subseteq M$, and satisfying $A' = B, B' = A$. A, B is called the extent and intent of concept (A, B), respectively, and using $\mathcal{B}(G, M, I)$ denote the set of all concepts of $K = (G, M, I)$.

Definition 1.4 (Z.Bonikowski, 1998) Let U be a finite nonempty set of objects, called the **universe**, let C be a covering of U, i.e. C is a family of nonempty subsets of U whose union is U, i.e. $U = \bigcup_{X \in C} X$. The ordered pair $\mathcal{A} = (U, C)$ is called an **approximation space**.

Remark 1.1 It should be noticed that the definition of covering, the join of members of C need not be an empty set, that is, $\exists X_i, X_j \in C,\ s.t.\ X_i \cap X_j \neq \emptyset$. In other word, the covering of U can not be a partition of U, conversely, a partition of U is also a covering of U.

Definition 1.5 (Z.Bonikowski, 1998) Let $x \in U$, the family $\{X \in C \mid x \in X\}$ is called the **description of** x.

Definition 1.6 (Z.Bonikowski, 1998) Let $x \in U$, the family $Md(x) = \{X \in C \mid x \in X \wedge \forall Y \in C(x \in Y \wedge Y \subseteq X \Rightarrow X = Y)\}$ is called the **minimal description of** x.

Definition 1.7 (Z.Bonikowski, 1998) An object $x \in U$ such that $Md(x) = \{X\}$ is called **representative element of** X.

Definition 1.8 (Xu WeiHu, 2007) Let U be the universe, C be a subsets family of U, if $\bigcup C = U$, then C is the covering of U, the ordered pair (U, C) is called a **covering approximation space**. $\forall X, Y \in C$, $\exists Z_1, \cdots, Z_k \in C$, such that, $X \cap Y = Z_1 \cup \cdots \cup Z_k$, then called C is a fined covering of U, the ordered pair $(U, C)$is called a **fined covering approximation space**.

Remark 1.2 It is important to note that in the definition of fined covering, C includes the empty set, that is, $\emptyset \in C$.

Definition 1.9 (Qiu Weigen, 2006) Suppose U is a finite universe, and $C = \{C_1, \cdots, C_n\}$ is a cover of U. For every $x \in U$, let $C_x = \cap\{C_j \mid C_j \in C, x \in C_j\}$, then $Cov(C) = \{C_x \mid x \in U\}$ is also a cover of U, we call it the **induced cover of** C.

Remark 1.3 For all $C_x, C_y \in C, C_x \neq C_y, C_x \cap C_y$ can not be empty, for example:
$U = \{1, \cdots, 8\}, C = \{\{1, 3, 4\}, \{2, 5, 7\}, \{1, 2, 8\}, \{3, 4, 6, 7\}, \{2, 6, 7\}, \{1, 7, 8\}, \{2, 5, 7, 8\}, \{4, 5, 7\}\}$, in here, $C_3 = \{3, 4\}, C_4 = \{4\}, C_3 \cap C_4 \neq \emptyset$, so C_x is a covering of U, not a partition of U, however, for all $C_x \in Cov(C)$, if $x \neq y, C_x \cap C_y = \emptyset$, then $Cov(C)$ is a partition of U; if $C_x \cap C_y \neq \emptyset$, and $C_x \subseteq C_y$, or $C_y \subseteq C_x$, then $Md(x)$ is a singleton set, and is a special covering of U.

Definition 1.10 (Qiu Weigen, 2006) Suppose U is a finite universe, and $\Delta = \{C_1, \cdots, C_m\}$ is a family of cover of U. For every $x \in U$, let $\Delta_x = \cap\{C_{ix} \mid C_{ix} \in Cov(C_i), x \in C_{ix}\}$, then $Cov(\Delta) = \{\Delta_x \mid x \in U\}$ is also a cover of U, we call it the **induced cover of** Δ.

Remark 14 For all $x \in U$, if $x \in Cov(\Delta)$, then must exist C_i, such that $x \in Cov(C_i)$, that is , $Cov(\Delta) \subseteq Cov(C_i)$. In other words, $Cov(\Delta)$ is the smaller minimum than $Cov(C_i)$.

2. Some remarks about the existed results

Remark 2.1 In the definition of covering, $X_i \neq X_j \in C$, it can exist $X_i \cap X_j \neq \emptyset, \forall x \in U$, at least exist one $X \in C,\ s.t.\ x \in X$, so, the minimal description of x must exist, if x belongs to the two or more than two subsets which do not include each other, then the elements of $Md(x)$ more than 1, that is $\mid Md(x) \mid \geq 2$, for example:

Example 1 Let universe $U = \{1, 2, \cdots, 5\}, C = \{\{1, 3\}, \{1, 2, 3\}, \{2, 3, 4\}, \{3, 5\}, \{1, 3, 5\}, \{4, 5\}\}$.

Obviously, C is the covering of U. Using $Dp(x)$ denote the description of x, then $Dp(1) = \{\{1, 3\}, \{1, 2, 3\}, \{1, 3, 5\}\}$, $Dp(2) = \{\{1, 2, 3\}, \{2, 3, 4\}\}$, $Dp(3) = \{\{1, 2, 3\}, \{2, 3, 4\}, \{3, 5\}, \{1, 3, 5\}\}$, $Dp(4) = \{\{4, 5\}, \{2, 3, 4\}\}$, $Dp(5) = \{\{1, 3, 5\}, \{3, 5\}, \{4, 5\}\}$. And $Md(1) = \{\{1, 3\}, \{1, 3, 5\}\}$, $Md(2) = \{\{1, 2, 3\}, \{2, 3, 4\}\}$, $Md(3) = \{\{1, 2, 3\}, \{2, 3, 4\}, \{3, 5\}\}$, $Md(4) = \{\{4, 5\}, \{2, 3, 4\}\}$, $Md(5) = \{\{3, 5\}, \{4, 5\}\}$.

Proportion 2.1 Let C be nonempty family of U, $C = \{X_i \mid \emptyset \neq X_i \subseteq U, i = 1, \cdots, k\}$, if $U = \bigcup_{i=1}^{k} X_i$, and $\forall x \in U$, exist unique minimal description of x, that is, $\exists! X \in \{X \in C \mid x \in X\}$, such that $Md(x) = \{X\}$, then $\forall i \neq j, X_i, X_j \in C, X_i \cap X_j = \emptyset$, or $X_i \subseteq X_j$. Furthermore, if $x, y \in X$, then $Md(x) = Md(y)$.

Proof If covering C is the partition of the universe U, then $\forall x \in U$, at least one $X_i \in C,\ s.t.\ x \in X_i$, if x only belongs to the unique X_i, then $\forall i \neq j, X_i, X_j \in C, X_i \cap X_j = \emptyset$, if x belongs to more than one X_i, then $\forall i \neq j, X_i, X_j \in C, X_i \subseteq X_j$, otherwise, $Md(x) = \{X_i, X_j \mid X_i, X_j \in C\}$, it is contradict with the minimal description of x.

Remark 2.2 In (Z.Bonikowski, 1998), the authors defined fined covering, where $\forall X, Y \in C$, $\exists Z_1, \cdots, Z_k \in C$, such that $X \cap Y = Z_1 \cup \cdots \cup Z_k$, however, $X \cap Y$ can be empty, if $X \cap Y \neq \emptyset$, although there exists in theory, it is difficult to achieve in practice, even in (Z.Bonikowski, 1998), the example that the authors gave did not satisfy $\forall X, Y \in C, X \cap Y \neq \emptyset$, for example: $X_2 = \{a, c\}, X_5 = \{e, b\}, X_2 \cap X_5 = \emptyset$, so, it must permit $X \cap Y = \emptyset$, that is, in the definition of covering includes empty set, in addition, the fined covering formed by non-empty subsets class almost is singleton set, I don't think it has too much meaning.

Remark 2.3 From the definition of the representative element, we can declare, if the every element x of the universe U is a representative element, that is $\forall x \in U$, $\exists X \in \{X \in C \mid x \in X\}$, such that, $Md(x) = \{X\}$, $\forall X, Y \in C$, then it must have $X \cap Y = \emptyset$ or $X \subseteq Y$, if $\forall X, Y, X \cap Y = \emptyset$, then the covering C is formed the partition of U, it is identical with the result of proportion 2.1.

3. The minimal description of formal context

The formal context (G, M, I) is studying the concept pair (A, B) which extent and intent are objects subset $A \subseteq G$ and attributes subset $B \subseteq M$, respectively, so, we can consider the covering formed by their subset family, and the minimal description of elements.

Example 2 The following table 1 is formal context (G, M, I), where $G = \{1, 2, \cdots, 8\}, M = \{a, b, \cdots, i\}$.

Given $C_G = \{X_1, X_2, X_3, X_4, X_5\}$, where $X_1 = \{1, 3\}, X_2 = \{1, 2, 4\}, X_3 = \{3, 5, 7\}, X_4 = \{6, 8\}, X_5 = \{6, 7, 8\}$. Obviously, $\bigcup_{i=1}^{5} X_i = G$, and: $Dp(1) = Md(1) = \{X_1, X_2\}$, $Dp(2) = Md(2) = \{X_2\}$, $Dp(3) = Md(3) = \{X_1, X_3\}$, $Dp(4) = Md(4) = \{X_2\}, Dp(5) = Md(5) = \{X_3\}, Dp(6) = Dp(8) = \{X_4, X_5\}, Md(6) = Md(8) = \{X_4\}, Dp(7) = Md(7) = \{X_3, X_5\}$.

$C_M = \{Y_1, Y_2, Y_3, Y_4, Y_5\}$, where $Y_1 = \{a, f, g, h\}, Y_2 = \{a, c, d\}, Y_3 = \{b, e, i\}, Y_4 = \{f, g\}$. Obviously, $\bigcup_{i=1}^{4} Y_i = M$,and: $Dp(a) = Md(a) = \{Y_1, Y_2\}$, $Dp(b) = Md(b) = Dp(e) = Md(e) = Dp(i) = Md(i) = \{Y_3\}$, $Dp(c) = Md(c) = Dp(d) = Md(d) = \{Y_2\}$, $Dp(f) = Dp(g) = \{Y_4, Y_1\}, Md(f) = Md(g) = \{Y_4\}, Dp(h) = Md(h) = \{Y_1\}$.

We all know, the all objects in G share all attributes of M, conversely, all attributes in M common to the all objects of G, so, we have:

Theorem 3.1 The covering C_G is formed by the subset of objects set G, every objects subset which possesses maximal attributes set is corresponding a covering of the attributes set.

A nature question is whether a pair of the objects subset and its corresponding attributes subset is formed a concept? The answer is negative, for example:

$X_1 = \{1, 3\} \subseteq G$, its corresponding the maximal attributes subset is $\{a, b, c, g, h\} \subseteq M$, but $(\{1, 3\}, \{a, b, c, g, h\})$ is not a concept, because $\{a, b, c, g, h\}' = \{3\}$. In here, we consider objects subset corresponding the common attributes, that is, the intent of the objects subset, it also does not form the concept. Such as, $X_1 = \{1, 3\}, \{1, 3\}' = \{a, b, g\}$, but $\{a, b, g\}' = \{1, 2, 3\}$, hence, $(\{1, 3\}, \{a, b, g\})$ is not a concept.

Remark 3.1 If replace "objects subset and its corresponding the maximal attributes subset" for "objects subset and its corresponding the common attributes subset" in theorem 3.1, it can occur the attributes subsets can not formed the covering of the attributes set. For example, consider example 2, given the covering of object set, $X_1' = \{a, b, g\}, X_2' = \{a, g\}, X_3' = \{a\}, X_4' = \{a, c, d, f\}, X_5' = \{a, c, d\}$, $\bigcup_{i=1}^{5} X_i' = \{a, b, c, d, f, g\} \neq M$, the intents of object subsets are not formed the covering of the attribute set.

It is similar to the objects set, the covering of the attributes set has similar results.

Theorem 3.2 The covering C_M is formed by the subset of attributes set M, every attributes subset which shares maximal objects set is corresponding a covering of the objects set.

A similar question is whether a pair of the attributes subset and its corresponding objects subset is formed a concept? Furthermore, attributes subset corresponding the common objects, that is, the extent of the attributes subset whether is formed a concept? The answer is also negative, for example: in example 2, let $Y = \{a, c, g\} \subseteq M$, and its corresponding objects subset is $\{1, 3, 4\} \subseteq G$, but $(\{1, 3, 4\}, \{a, c, g\})$ is not a concept, because $\{1, 3, 4\}' = \{a, b, g, h\}$.

Remark 3.2 If replace "attributes subset and its corresponding the maximal objects subset" for "attributes subset and its corresponding the sharing common objects subset" in theorem 3.2, it can occur the objects subsets are not formed the covering of the objects set. For example:

Example 3 The following table 2 is a formal context, where $G = \{1, 2, \cdots, 5\}, M = \{a, b, \cdots, i\}$, the figure 1 is Hasse figure of its concept lattice.

Given the covering of attributes subset $C_M = \{Y_1, Y_2, Y_3, Y_4\}$, where $Y_1 = \{a, c, f, h\}, Y_2 = \{d, g, i\}, Y_3 = \{b, d, e, f, g\}, Y_4 = \{b, f, h\}$, $Y_1' = \{1\}, Y_2' = \{3\}, Y_3' = \emptyset, Y_4' = \{4\}$, $\bigcup_{i=1}^{4} Y_i' = \{1, 3, 4\} \neq G$, that is the extent of attributes subset is not formed the covering of the objects set .

A natural problem, what conditions can make the binary ordered pair of the objects subset and its corresponding attributes subset formed a concept?

In fact, this is not difficult, we only need choose the covering subset from the point of Hasse figure of formal context, which can assure objects subset and its corresponding common attributes subset or attributes subset and its corresponding sharing objects subset formed a concept, that is, the extent of concept is the objects subset, and the intent is the attributes subset. For example, consider the Hasse figure (figure 1), let $C_M = \{\{a, c\}, \{g, i\}, \{c, f, h\}, \{b, e, g\}\}$, and $\{a, c\}' = \{1, 2\}, \{g, i\}' = \{2, 3\}, \{c, f, h\}' = \{1, 4\}, \{b, e, g\}' = \{5\}$, from Hasse figure, we can see, $(\{1, 2\}, \{a, c\})(\{2, 3\}, \{g, i\}), (\{1, 4\}, \{c, f, h\}), (\{5\}, \{b, e, g\})$ are all concepts. It is similar to the objects set. However, there is a flaw, like as the remark 3.1, 3.2, it will cause that the objects subset or the attributes subset can not formed the covering of objects set and attributes set. Such as, in example 3, $C_M = \{\{1, 2, 3\}, \{1, 2, 4\}, \{2, 3, 5\}, \{4, 5\}\}$, and $\{1, 2, 3\}' = \{a\}, \{1, 2, 4\}' = \{c\}, \{2, 3, 5\}' = \{g\}, \{4, 5\}' = \{b\}$, from Hasse figure, we can see, $(\{1, 2, 3\}, \{a\}), (\{1, 2, 4\}, \{c\}), (\{2, 3, 5\}, \{g\}), (\{4, 5\}, \{b\})$ are all concepts, but $\{a\}\cup\{c\}\cup\{g\}\cup\{b\} = \{a, b, c, g\} \neq M$.

In order to overcome this drawback, we can take covering subsets from the bottom upward of Hasse figure .

Remark 3.3 In (Qiu Weigen, 2006), the object x of C_x shares the common attribute that can't be formed a covering of attributes set, and if consider the maximum attributes set, then it must be a covering of attributes set, for the covering of attributes set has similar results.

4. Conclusion

In this paper, we further discuss the minimal description of rough sets, in order to understand some the former results, we give some remarks, at the same time, we use the minimal description under covering to the formal concepts of formal context, we study the covering that formed by the concept pair of objects subset and attributes subset, discuss their corresponding relation, and illustrate them by examples.

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Table 1.

	a	b	c	d	e	f	g	h	i
1	1	1	0	0	0	0	1	0	0
2	1	1	0	0	0	0	1	1	0
3	1	1	1	0	0	0	1	1	0
4	1	0	1	0	0	0	1	1	1
5	1	1	0	1	0	1	0	0	0
6	1	1	1	1	0	1	0	0	0
7	1	0	1	1	1	0	0	0	0
8	1	0	1	1	0	1	0	0	0

Table 2.

	a	b	c	d	e	f	g	h	i
1	1	0	1	0	0	1	0	1	0
2	1	0	1	0	0	0	1	0	1
3	1	0	0	1	0	0	1	0	1
4	0	1	1	0	0	1	0	1	0
5	0	1	0	0	1	0	1	0	0

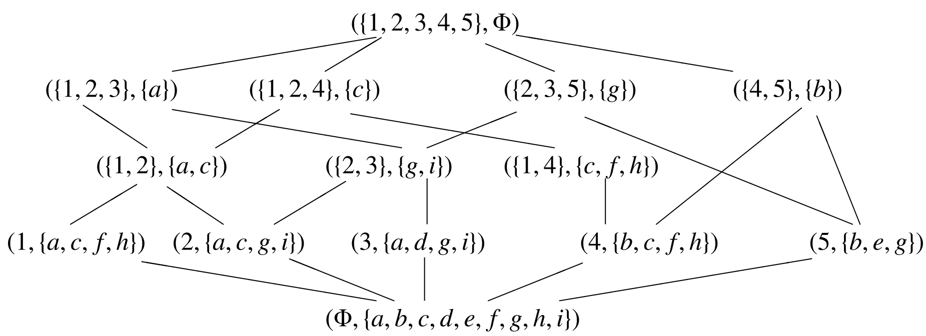

Figure 1

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A Model to Study Effect of Rapid Buffers and Na^+ on Ca^{2+} Oscillations in Neuron Cell

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Abstract

Ca^{2+} plays a vital role in muscle mechanics, cardiac electrophysiology, secretion, hair cells, and adaptation in photoreceptors. It is a vital second messenger used in signal transduction. Calcium controls cell movement, cell differentiation, ciliary beating. Many cells exhibit oscillations in intracellular [Ca^{2+}] in response to agonist such as hormones and neurotransmitters. Many cells use oscillations in calcium concentration to transmit messages (Sneyd J. *et al*, 2006, p. 151-163). In this paper, an attempt has been made to develop a model to study calcium oscillations in neuron cells. This model incorporates the effect of variable Na^+ influx, sodium-calcium exchange (NCX) protein, Sarcolemmal Calcium ATPase (SL) pump, Sarco-Endoplasmic Reticulum CaATPase (SERCA) pump, sodium and calcium channels, and IP_3R channel. The proposed mathematical model leads to a system of partial differential equations which has been solved numerically using Forward Time Centered Space (FTCS) approach. The numerical results have been used to study the relationships among different types of parameters such as buffer concentration, disassociation rate, calcium permeability, etc.

Keywords: Signal transduction, Calcium oscillations, Na^+ influx, NCX protein, SL pump, SERCA pump, IP_3R channel, FTCS approach

1. Introduction

Calcium oscillations are known to play a key role in a number of mechanisms like the secretion in the pituitary and parotid glands, the contraction of smooth muscle, and cardiac inotrophy and induction of arrhythmias (Jafri M.S. *et al*, 1992, p. 235-246). These oscillations are supposed to contain frequency encoded signals that help in using Calcium as a second messenger while avoiding its high intracellular concentrations (Keener J. *et al*, 1998, p. 53-56). A number of investigators have reported the oscillatory behaviour of calcium due to intracellular concentration of inositol 1, 4, 5-trisphosphate (IP_3). In the process of signal transduction, intracellular calcium acts like a switch and decides whether a particular signal needs to be further propagated or not.There are mainly two types of receptors, Rynodine Receptors (RyRs) or Inositol Triphosphate Receptors (IP_3Rs) that are located at the membrane of endoplasmic reticulum (in neurons) or sarcoplasmic reticulum (in myocytes) which cause an efflux of Ca^{2+} in the cytosol. The release of Ca^{2+} through IP_3Rs is as a result of some agonist or neurotransmitter binding to its receptor which can cause via G protein link to phospholipase C (PLC), the cleavage of phosphotidylinositol (4,5)-bisphosphate (PIP_2) to inositol triphosphate (IP_3) and diacylglycerol (DAG). This released IP_3 is free to diffuse through the cytosol and binds with IP_3R leading to the subsequent opening of these receptors and release of Ca^{2+} from the intracellular stores. Calcium oscillation can be classified into mainly two types: 1) that is induced by changing membrane potential as in the case of an action potential and the associated periodic entry of Ca^{2+} through voltage-gated Ca^{2+} channels,2) that occur in the presence of voltage clamp. The latter part can be further categorized based on the fact that the oscillatory Ca^{2+} flux is from RyRs or IP_3Rs. The period of IP_3-dependent oscillations ranges from a few seconds to a few minutes. There is a great deal of evidence that in many cell types, these oscillations occur at constant $[IP_3]$ and are therefore not driven by oscillations in $[IP_3]$. It is also observed that as $[IP_3]$ increases, the steady state $[Ca^{2+}]$ increases, the oscillation frequency increases, and the amplitude of the oscillations remains approximately constant. Calcium oscillations usually occur only when $[IP_3]$ is greater than some critical value and disappear again when $[IP_3]$ gets too large. Thus, there is an intermediate range of IP_3 concentrations that generate Ca^{2+} oscillations (Keener J. *et al*, 1998, p. 53-56).In this paper, we have also studied oscillations induced in calcium due to change in IP_3 concentration. Here a mathematical model is proposed which incorporates nearly all important and necessary biophysical parameters like buffers, L-type calcium channel, calcium pump, sodium calcium exchanger (NCX), and calcium leak. Further, we assume that buffers exhibit rapid buffering.

Another factor which might have significant effect on calcium oscillation and which has not been given much importance is the sodium ion concentration. We have incorporated it through the NCX protein. We have considered an exchange ratio of 4:1 (Fujioka Y., 2000, p.611-623) with respect to sodium and calcium ions respectively. To make the model more realistic we have included the ER. The proposed mathematical model leads to a system of partial differential equations. We have used finite difference approach for the simulation of the proposed model for which a program has been developed in MATLAB and run on a Pentium IV Dual Core 1.00 GB RAM, 1.73GHz processor to obtain the numerical result. The time taken per simulation is 240 seconds for time, t = 30 seconds.

2. Mathematical Model

Our mathematical model assumes the following reaction-diffusion kinetics (Wagner J. *et al*, 1994, p.447-456) (Smith G.D., 1996, p.3064-3072),

$$[Ca^{2+}] + [B_m] \underset{k^-}{\overset{k^+}{\rightleftarrows}} [CaB_m] \tag{1}$$

where, $[B_m]$ and $[CaB_m]$ are free and bound buffers respectively. Using Ficks law of diffusion and law of mass action and assuming rapid buffering approximation, we have the following partial differential equation (Smith G.D., 1996, p.3064-3072),

$$\frac{\partial[Ca^{2+}]}{\partial t} = \beta\left(D_{Ca} + \gamma_m D_{CaBm}\right)\nabla^2\left[Ca^{2+}\right] - \frac{2\beta\gamma_m D_{CaBm}}{K_m + [Ca^{2+}]}\nabla\left[Ca^{2+}\right].\nabla\left[Ca^{2+}\right] \tag{2}$$

$$\begin{aligned} &where, \\ &\beta = (1 + \gamma_s + \gamma_m)^{-1} \; and \\ &\gamma_m = \frac{K_m\,[B_m]_T}{(K_m + [Ca^{2+}])^2} \end{aligned}$$

D_{Ca}, and D_{CaB_m} are the diffusion coefficients of free calcium, and calcium bound buffer respectively, and K_m is disassociation rate constant. For stationary buffers, $D_{CaBm} = 0$.

Our proposed mathematical model also contains the following parameters, to study the effect of rapid buffer, Na^+ ions and ER over Ca^{2+} oscillations,

2.1 Ion channels

The Ca^{2+} and Na^+ channels have been modeled using the Goldman-Hodgkin-Katz (GHK) current equation (Keener J. *et al*, 1998, p. 53-56):

$$I_s = P_s z_s^2 \frac{F^2 V_m}{RT} \frac{[S]_i - [S]_o \exp\left(-\frac{z_s F V_m}{RT}\right)}{\left(1 - \exp\left(-\frac{z_s F V_m}{RT}\right)\right)} \tag{3}$$

where $[S]_i$,$[S]_o$, are the intracellular and extracellular ion concentration (Molar),respectively .P_S is the permeability (m/s) of S ion, z_s is valence of S ion. F is Faradays constant (C/moles).V_m is membrane potential (Volts). R is Real gas constant (J/K moles) and T is Absolute temperature (Kelvin). Equation (3) is converted into molar/second by using the following equation

$$\sigma_s = \frac{-I_s}{z_s F V_{cyt}} \tag{4}$$

The negative sign in equation (4) is taken because by convention inward current is taken to be negative. The GHK equation is derived from the constant field approximation which assumes that the electric field in the membrane is constant, and thus decoupled from the effects of charges moving through the membrane.

2.2 Na^+ / Ca^{2+} Exchange (NCX) Protein

The NCX protein is essential for excitation-contraction coupling in cardiac myocytes (Fujioka Y., 2000, p.611-623). It helps in the extrusion of cytosolic calcium in neurons and hence regulates neurotransmitter release, (Blaustein M.P., 1999, p. 763-854). The pump is assumed to be electrogenic in nature as one calcium leaves the cytosol for intake of four sodium ions. In our model we have taken an exchange ratio of 4:1 with respect to sodium and calcium ions respectively (Fujioka Y., 2000, p.611-623). The amount of energy required to extrude an ion against its concentration gradient is given by:

$$\Delta_s = z_s F V_m + RT \log\left(\frac{S_i}{S_o}\right) \tag{5}$$

So using $\triangle Ca^{2+} = 4\triangle Na^+$, we have,

$$\sigma_{NCX} = Ca_o \left(\frac{Na_i}{Na_o}\right)^4 \exp\left(\frac{2FVm}{RT}\right) \tag{6}$$

$$\overline{\sigma}_{NCX} = Na_o \left(\frac{Ca_i}{Ca_o}\right)^{1/4} \exp\left(-\frac{FVm}{2RT}\right) \tag{7}$$

2.3 Sarcolemmal Calcium ATPase pump (SL CaATPase pump)

It is a P-type ATPase which is also known as Plasma Membrane Calcium ATPase pump (PMCA). Energy obtained from ATP is used to extrude calcium ions out of the cytosol. The kinetics of the pump follows MichaelisMenten kinetics (Nelson D.L., 2005) (Blackwell K.T., 2005, p.1-27). So the net efflux of calcium ions out of the cytosol is given by:

$$\sigma_{SLPump} = \frac{V_{SLPump}}{1 + \left(\frac{K_{SLPump}}{Ca_i}\right)^H} \tag{8}$$

where, V_{SLpump} is the maximum pump capacity, K_{SLpump} is half of the maximum pump capacity at steady state and H is the Hills coefficient.

2.4 Sarco Endoplasmic Reticulum CaATPase (SERCA) pump

The SERCA pump uses the chemical energy produced from the conversion of adenosine triphosphate (ATP) into adenosine diphosphate (ADP) to transport calcium ions across the membrane from the cytosol to the ER, against its concentration gradient, (Sneyd J. *et al*, 2006, p. 151-163). It binds calcium on the cytosolic side and releases it on the ER side. The SERCA pump has been modelled by a simple Hills equation as follows:

$$\sigma_{SERCApump} = \frac{v_{SERCA}}{1 + \left(\frac{k_{SERCA}}{Ca_i}\right)^H} \tag{9}$$

where, v_{SERCA} is the maximal pump rate, k_{SERCA} is half of the maximum pump capacity at steady state and H is the Hills coefficient.

2.5 Endoplasmic Reticulum leak and IP_3R Flux

There is a certain amount of leak from the ER to the cytosol along the concentration gradient and an IP_3 flux given by the following term (Jafri S. *et al*, 1995, p.2139-2153):

$$c_1(leakcons + ipflux((X_{110})_i^j)^3)((u_{er})_i^j - u_i^j) \tag{10}$$

Where, c_1 is the ratio of volume of ER to cytosol, leakcons is the calcium leak rate constant, ipflux is the maximum IP_3 receptor flux, X_{110} fraction of open channels; u_{er} and u are the calcium concentrations in the ER and cytosol respectively.

Variation of channel states, that is, whether closed or opened is given as follows

$$\frac{dX_{100}}{dt} = -a_2[Ca^{2+}]X_{100} - a_5[Ca^{2+}]X_{100} + b_5X_{110} \tag{11}$$

$$\frac{dX_{110}}{dt} = -a_2[Ca^{2+}]X_{110} + b_2c_2[IP_3] + a_5[Ca^{2+}]X_{100} - b_5X_{110} \tag{12}$$

Where, X_{100} and X_{110} represent the fraction of closed channels and open channels respectively.a_2 is inhibitory receptor binding constant,a_5 is activation receptor binding constant, b_2 is inhibitory receptor disassociation constant, and b_5 is activation receptor disassociation constant.

2.6 ER Calcium Concentration

Calcium in the ER is assumed to be under rapid buffering approximation (Jafri S. *et al*, 1995, p.2139-2153), and is as given below,

$$\frac{\partial[Ca^{2+}]_{ER}}{\partial t} = \left(\frac{1}{c_1}\right)\beta_{ER}\left(\begin{array}{l} \left(\left(D^{ER}_{Ca} + \gamma^{ER}_m D^{ER}_{CaBm}\right)\nabla^2[Ca^{2+}]_{ER} - \frac{2\gamma^{ER}_m D^{ER}_{CaBm}}{K^{ER}_m+[Ca^{2+}]_{ER}}\nabla[Ca^{2+}]_{ER}.\nabla[Ca^{2+}]_{ER}\right).c_1 \\ -c_1\left(v_2 + v_1 X^3_{110}\right)\left([Ca^{2+}]_{ER} - [Ca^{2+}]\right) + \frac{v_3[Ca^{2+}]^2}{k_3^2+[Ca^{2+}]^2} \end{array}\right) \tag{13}$$

$$where,\ \ \beta_{ER} = \left(1 + \frac{K^{ER}_s\,[B_s]_{ER}}{(K^{ER}_s + [Ca^{2+}]_{ER})^2} + \frac{K^{ER}_m\,[B_m]_{ER}}{(K^{ER}_m + [Ca^{2+}]_{ER})^2}\right)^{-1}\ and$$

$$\gamma^{ER}_m = \frac{K^{ER}_m\,[B_m]_{ER}}{(K^{ER}_m + [Ca^{2+}]_{ER})^2}$$

2.7 $[IP_3]$ Variation

The IP_3 concentration varies with time according to the equation given below, (Keizer J. *et al*, 1992, p. 649-660)

$$\frac{d[IP_3]}{dt} = v_3 f(t) + \frac{v_6[Ca^{2+}]_i}{k_6 + [Ca^{2+}]_i} - v_7[IP_3] \tag{14}$$

where, f(t)=0 or 1 (defines pulses of IP_3), v_3 is external IP_3 input rate, v_6 is maximum calcium dependent IP_3 input rate, v_7 is IP_3 decay rate constant and k_6 is activation constant, calcium dependent IP_3 input. The first term in above equation denotes IP_3 pulse, second term denotes calcium induced IP_3 production and last term denotes IP_3 degradation.

Combining equations (1-14) we get the proposed mathematical model as given below,

$$\frac{\partial[Ca^{2+}]}{\partial t} = \beta\left(D_{Ca} + \gamma_m D_{CaBm}\right)\nabla^2\left[Ca^{2+}\right] - \frac{2\beta\gamma_m D_{CaBm}}{Km + [Ca^{2+}]}\nabla\left[Ca^{2+}\right].\nabla\left[Ca^{2+}\right]$$
$$\sigma_{NCX} - \sigma_{SLPump} - \sigma_{SERCA} + \sigma_{ERleak} \tag{15}$$

$$\frac{\partial[Na^{+}]}{\partial t} = \beta_{sod}\left(-\sigma_{Na} + \sigma_{NCX} - \sigma_{leak}\right) \tag{16}$$

Along with the initial-boundary conditions,

A. Initial condition:

$$[Ca^{2+}]_{t=0} = 0.1\mu Mand[Na^{+}]_{t=0} = 12mM \tag{17}$$

B. Boundary conditions:

$$\lim_{r\to 0}\left(-2\pi r^2\beta\left(D_{Ca} + \gamma_m D_{CaBm}\right)\frac{d[Ca^{2+}]}{dr}\right) = \beta\sigma_{Ca} \tag{18}$$

$$\lim_{r\to\infty}[Ca^{2+}] = 0.1\mu M \tag{19}$$

Our problem is to solve equation (15) and (16) coupled with equation (17-19). For our convenience we are writing 'u' in lieu of $[Ca^{2+}]$ and 'v' in lieu of $[Na^{+}]$. Applying finite difference method (Forward Time Centered Space) on equation (15 16), we get

$$\frac{u_i^{j+1} - u_i^j}{k} = \beta_i^j\left(\begin{array}{l} \left(\left(D_{eff}\right)_i^j\ \frac{(u^j_{i+1}-2u^j_i+u^j_{i-1})}{(h^2)}\ \right) + \frac{(u^j_{i+1}-u^j_{i-1})}{h}\left(\frac{D_{eff}{}^j_i}{r_i} - \frac{((\gamma_m{}^j_i D_m)(u^j_{i+1}-u^j_{i-1}))}{(2h(K_m+u^j_i))}\ \right) \\ +c_1(leakcons + ipflux((X_{110})^j_i)^3)((u_{er})^j_i - u^j_i) - \frac{sprate*(u^j_i)^2}{((u^j_i)^2+spdissrat^2)} \\ -\frac{1}{(1+\mathrm{e}^{(\mathrm{v}^j_i-130)/14})}u_{out}\left(\frac{v^j_i}{v_{out}}\right)^4 e^{2\varepsilon} - \frac{V_{SLpump}}{1+\left(\frac{K_{SLpump}}{u^j_i}\right)^{1.6}} \end{array}\right)$$

And

$$\frac{v_i^{j+1}-v_i^j}{k}=\beta_{sod_i^j}\left(\frac{1}{\left(1+\ \mathrm{e}^{(\mathrm{v}_i^j-130)/14)}\right)}v_{out}\left(\frac{u_i^j}{u_{out}}\right)^{(1/4)}e^{(-\varepsilon/2)}+\frac{\mathrm{k_{Cain}}}{(\mathrm{k_{Cain}}+\mathrm{u}_i^j)}\frac{((3*10^5)P_{Na}\ \varepsilon)}{(1-e^{\varepsilon})}(v_i^j e^{\varepsilon}-v_{out})-\mathrm{P_L}(\mathrm{v_{out}}-\mathrm{v}_i^j)\right) \tag{20}$$

where, $\varepsilon = FV_m/RT$ is a dimensionless quantity, 'h' represents spatial step and 'k' represents time step, 'i' and 'j' represents the index of space and time respectively. Since, the above expression is not valid at the mouth of the channel; therefore the approximation at the mouth of the channel is given by

$$u_1^{j+1}=k.\beta_1^j\left(\begin{array}{l}\left(\left(D_{eff}\right)_1^j\frac{(2u_2^j-2u_1^j)}{(h^2)}\right)+c_1(leakcons+ipflux((X_{110})_1^j)^3)((u_{er})_1^j-u_1^j)\\-\frac{sprate*(u_1^j)^2}{((u_1^j)^2+spdissrat^2)}-\frac{1}{(1+\mathrm{e}^{(\mathrm{v}_1^j-130)/14})}u_{out}\left(\frac{v_1^j}{v_{out}}\right)^4e^{2\varepsilon}\\-\frac{V_{SLpump}}{1+\left(\frac{K_{SLpump}}{u_1^j}\right)^{1.6}}+sigma\end{array}\right)+u_1^j \tag{21}$$

And

$$v_1^{j+1}=k.\beta_{sod_1^j}\left(\frac{1}{\left(1+e^{(v_1^j-130)/14)}\right)}v_{out}\left(\frac{u_1^j}{u_{out}}\right)^{(1/4)}e^{(-\varepsilon/2)}+\frac{\mathrm{k}_{Cain}}{(k_{Cain}+u_1^j)}\frac{((3*10^5)\ P_{Na}\varepsilon)}{(1-e^{\varepsilon})}(v_1^j e^{\varepsilon}-v_{out})-P_L(v_{\mathrm{out}}-v_1^j)\right)+v_1^j \tag{22}$$

3. Result and Discussion

In this section the results and discussion of the obtained numerical solution is given. The values of all the biophysical parameters used are as given in table 1. Figures (1-6) are the plots obtained for $[IP_3]$ = 0.6μM whereas figures (7-12) are obtained for variable IP_3 concentration.

Mobile Buffers (Simulation Time =48s)

Figure 1 shows the oscillations in calcium concentration against time with variable mobile buffer concentration for a constant $[IP_3]$ concentration of 0.6μM. This figure shows that it is not necessary to have oscillations in $[IP_3]$ to observe oscillations in calcium concentration, (Keizer J. *et al*, 1992, p. 649-660). As the buffer concentration increases the free calcium concentration decreases but in the time interval t= [8-15]s and t= [27-30]s it is vice-versa. The reason for this is that as the calcium concentration increases the NCX protein activates which causes larger extrusion of calcium ions from the cytosol. Hence the calcium concentration inside the cytosol decreases.

Figure 2 shows the variation with time and buffer disassociation rate for two mobile endogenous buffers, calmodulin (K_m=2μM) and (K_m=3μM). As the disassociation rate increases more calcium ions become free and so the calcium concentration increases. However, in the time duration [7-15]s and [25-30]s it is vice-versa, i.e., lower disassociation rate has higher calcium concentration. This is because as the calcium ions increases the NCX protein activates which extrudes 1 calcium ion for four sodium ions entering the cytosol. Also the frequency has increased on increasing the buffer disassociation rate. This is due to the faster kinetics involved with higher disassociation rate. Also there is latency in the oscillation, that is, delay in time required to attain maximum value. When the disassociation rate is increased the calcium achieves the peak value faster.

Stationary buffers (Simulation Time =48s)

Figure 3 shows that as the stationary buffer concentration increases the calcium concentration decreases as more number of ions get bound to the buffer. However, in the intervals t= [7-15]s and [26-30]s the trend reverses. This is due to the fact that initially only the Sarcolemmal pump is working but as the calcium concentration increases the NCX protein gets activated which extrudes larger amount of calcium ions from the cytosol and hence the concentration decreases. The stationary buffers are immobile and hence they affect only the temporal variation of calcium and do not affect the spatial profile. The Calcium achieves the peak value faster when the buffer concentration is lower. This explains the latency observed.

Figure 4 shows variations in calcium concentration with time and buffer disassociation rate. Initially calcium concentration increases with increase in disassociation rate. This is so because with increases in disassociation rate more calcium ions become free. However in the time intervals [8-15]s and [26-30]s calcium decreases with increase in disassociation rate. This is because as the calcium ions reach higher concentrations the NCX protein gets activated which extrudes more calcium ions and hence the calcium ion concentration inside the cytosol decreases. As the buffer disassociation rate increases the frequency of oscillation also increases. This is because higher disassociation rate implies faster Calcium kinetics, that is, the binding and disassociation increases. We also observe latency in the oscillations. As the disassociation rate is increased the peak value is achieved faster.

Figure 5 shows the variations of calcium concentration with time and the leak rate constant of calcium from the ER into the cytosol are as shown in the above figure. As the calcium leak from the ER into the cytosol is increased the amount of free calcium in the cytosol increases. However, in intervals [5-12]s and [24-30]s the opposite is observed because with increase in calcium ions the NCX protein extrudes larger amount of calcium ions and hence the net calcium ion concentration in the cytosol decreases.

Figure 6 shows the sodium concentration variation with time. For the sodium concentration we have incorporated (1) sigmoidal deactivation of the NCX protein (2) calcium dependent inactivation of the L-type sodium channel and (3) a leak term which balances the sodium entering the cytosol via sodium channel and NCX protein. The sodium decreases up to 4s and thereafter increases. It decreases first due to deactivation of the NCX protein and calcium dependent inactivation of the sodium channel. After that it increases as the NCX protein is no longer deactivated and the sodium channel is also no longer inactivated.

Variable IP_3

Figure 7 shows a comparison of calcium concentration and IP_3 concentration on time scale. It is observed that the peaks of calcium and IP_3 occur at about the same time. This is because IP_3 oscillations are believed to induce calcium oscillations. By incorporating variable IP_3 it is observed that the frequency of calcium oscillation increases. Since earlier we observed only two peaks in 30 s time interval but now we observe 3 peaks. This has also been proved by Kusters *et al*, (Kusters *et al*, 2005, p. 3741-3756).Calcium is released from the ER when the IP_3 binds to the IP_3R and is refilled back into the ER by the SERCA pump. This increases and decreases the calcium in cytosol more rapidly when we consider a varying IP_3 which is dependent on the fraction of open channels. Also the calcium maximum concentration increases from 0.37 μM to 0.7 μM. This is because of increased released of calcium from the ER. In the earlier case we had taken the IP_3 value to be 0.6 μM whereas the maximum IP_3 value here is 1.15 μM. IP_3 is regulated by the fraction of open channels in the ER which regulates the amount of calcium released from the ER.

Figure 8 compares cytoplasmic and ER calcium concentration on time scale. Both show oscillations with respect to time. Calcium concentration is high in the ER compared to the cytoplasm. There is a leak of calcium from the ER to the cytoplasm and calcium uptake by ER through the SERCA pump. It is thus observed that when the calcium concentration in the ER is minimum it has a maximum in the cytoplasm. This has also been proved by Kusters *et al*, (Kusters *et al*, 2005, p. 3741-3756).It is also observed that the calcium concentration in the ER is continuously decreasing. This is so because the calcium extrusion from the cytosol via the NCX protein and Sarcolemmal pump is large and hence the amount of calcium being refilled into the ER by the SERCA pump is decreasing.

Figure 9(a) depicts variation of cytosolic calcium variation with ER buffer concentration and time. The effects of changing buffer concentration are negligible near the source due to rapid buffering. As we move away from the source the effect of other parameters becomes visible. So the effects are more visible at the second peak compared to the first peak. It is observed that as the ER buffer concentration increases the cytosolic calcium increases. This is because as ER buffer concentration increases more calcium inside the ER gets bound to the buffers so concentration gradient decreases and hence lesser calcium ions leak from the ER to cytosol. The pumping by the SERCA pump depends on the cytosolic calcium concentration which pumps lesser amount of calcium from the cytosol and hence the cytosolic calcium concentration increases. Also ER buffer concentration has little effect on cytoplasmic calcium oscillations. This is in agreement with the findings of Kusters *et al* (Kusters *et al*, 2005, p. 3741-3756).

Figure 9(b) depicts variation of cytosolic calcium variation with cytosolic buffer concentration and time. The effects of changing buffer concentration are less near the source due to rapid buffering. So the effects are more visible at the second peak compared to the first peak. It is observed that as the cytosolic buffer concentration increases the cytosolic calcium decreases. This is because as buffer concentration increases more calcium in the cytosol gets bound to the buffers. Hence the cytosolic calcium concentration decreases.

Figure 10(a) depicts variation of ER calcium concentration with ER buffer concentration and time. The effects of changing buffer concentration are more evident at the second peak compared to the first peak due to rapid buffering in the ER. It is observed that as the ER buffer concentration increases the ER calcium increases. This is because as ER buffer concentration increases more calcium inside the ER gets bound to the buffers so there is lesser concentration gradient so lesser calcium ions leak from the ER. Hence the ER calcium concentration increases.

Figure 10(b) depicts variation of ER calcium concentration with cytoplasmic buffer concentration and time. As the cytoplasmic buffer increases more calcium ions inside the cytosol get bound. The refilling by SERCA pump is proportional to the cytosolic calcium so lesser cytosolic calcium implies that there is lesser refilling of the ER and hence the ER calcium concentration decreases. Also as calcium concentration in the cytoplasm decreasing there is more concentration gradient and hence more calcium leaks from the ER and hence the ER calcium concentration decreases.

Figure 11(a) depicts variation of cytosolic calcium variation with ER buffer disassociation rate and time. The effects of changing buffer disassociation rate are negligible at first spike due to rapid buffering. As we move away from the source the effect of disassociation rate becomes more evident. It is observed that as the ER buffer disassociation rate increases the cytosolic calcium increases. This is because as ER buffer disassociation rate increases more calcium inside the ER becomes free so there is a larger leak of calcium ions from the ER into the cytosol. Thus calcium ion concentration inside the cytosol increases. ER buffer disassociation rate has negligible effect on cytosolic calcium oscillations as also shown by Kusters *et al* (Kusters *et al*, 2005, p. 3741-3756).

Figure 11(b) depicts variation of cytosolic calcium variation with cytoplasmic buffer disassociation rate and time. The results are evident, that is, as the disassociation rate increases the calcium concentration in the cytosol also increases.

Figure 12(a) depicts variation of ER calcium concentration with ER buffer disassociation rate and time. The effects of changing buffer disassociation rate are more evident at second spike due to rapid buffering. It is observed that as the ER buffer disassociation rate increases the ER calcium increases, this is in agreement with the physiological facts. As ER buffer disassociation rate increases more calcium inside the ER gets free. This is because disassociation rate is the ratio of disassociation to association rate, so larger disassociation rate implies that disassociation is greater than association so more calcium ions are released from the buffer.

Figure 12(b) depicts variation of ER calcium concentration with cytosolic buffer disassociation rate and time. As the cytosolic buffer disassociation rate increases the calcium concentration inside the cytosol increases and hence a greater amount of refilling of calcium into the ER by the SERCA pump. Hence the calcium concentration inside the ER increases. Also the concentration gradient across the ER decreases and hence there is less amount of calcium leaking from the ER into the cytosol so the ER calcium concentration increases.

4. Conclusion

Here we have studied the effects of sodium ions on calcium oscillations which have not been given much importance till date. We have used an exchange ratio of 4:1 with respect to sodium and calcium ions. We have tried to make the model more realistic by incorporating the different parameters as mentioned above. Some of the results obtained above are in agreement with the work done by previous researchers. The new results obtained are also in agreement with the physiological facts. The numerical solutions obtained above can be used to study the relationships among different types of parameters in the normal and abnormal conditions which can be useful to biomedical scientists for developing new protocols for the diagnosis and treatment of various neurological diseases.

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Table 1. Value of biophysical parameters used

Symbol	Parameter	Value	Reference
P_{ca}	Calcium permeability	$5.4\text{x}10^{-6}$ metre/sec	In this paper
P_{Na}	Sodium permeability	$6.07\text{x}10^{-10}$ metre/sec	(Tewari S. et al, 2008, p. 205-210)
F	Faraday's constant	96487coulomb/moles	known fact
R	Real gas constant	8.314 J/K mole	known fact
T	Absolute temperature	310°K	in this paper
z_{Ca}	Calcium valence	2	known fact
z_{Na}	Sodium valence	1	known fact
V_m	Resting Membrane potential	-70 mV	(Fain G.L., 1999)
D_{Ca}	Diffusion coefficient	250 μm^2/s	(Tewari S. et al, 2008, p. 205-210)
$(D_{Ca})^{ER}$	Diffusion coefficient	250 μm^2/s	In this paper
D_m	Mobile buffer diffusion coefficient	75 μm^2/s	(Jafri S. et al, 1995, p.2139-2153)
$(D_m)^{ER}$	Mobile buffer diffusion coefficient in ER	75 μm^2/s	(Jafri S. et al, 1995, p.2139-2153)
K_m	Mobile buffer disassociation rate	2 µM	(Smith G.D. et al, 1996, p.2527-2539)
K_m^{ER}	Mobile buffer disassociation rate	2 µM	In this paper
K_s	Stationary buffer disassociation rate	10 µM	In this paper
K_s^{ER}	Stationary buffer disassociation rate	10 µM	(Jafri S. et al, 1995, p.2139-2153)
K_{msod}	Sodium mobile buffer disassociation rate	10 mM	(Shannon T.R. et al, 2004, p. 3351-3371)
B_m	Mobile buffer concentration	75 µM	(Jafri S. et al, 1995, p.2139-2153)
B_m^{ER}	Mobile buffer concentration	250 µM	(Jafri S. et al, 1995, p.2139-2153)
B_s	stationary buffer concentration	225 µM	(Jafri S. et al, 1995, p.2139-2153)
B_s^{ER}	stationary buffer concentration	1 mM	In this paper
B_{msod}	Sodium mobile buffer concentration	5.35 mM	(Shannon T.R. et al, 2004, p. 3351-3371)
Ca_{out}	Extracellular calcium concentration	1.8 mM	(Shannon T.R. et al, 2004, p. 3351-3371)
Na_{out}	Extracellular sodium concentration	145 mM	(Shannon T.R. et al, 2004, p. 3351-3371)
Ca_i	Cytosolic calcium concentration	0.1 µM	(Shannon T.R. et al, 2004, p. 3351-3371)
Na_i	Cytosolic sodium concentration	12 mM	(Shannon T.R. et al, 2004, p. 3351-3371)
V_{SLPump}	Maximum pump capacity of PMCA	0.9 µM/sec	In this paper
K_{SLPump}	Half maximal pump capacity	0.1 µM	(Jafri S. et al, 1995, p.2139-2153)
H	Hill's coefficient	1.6	(Shannon T.R. et al, 2004, p. 3351-3371)
leakcons	Leak constant from ER	2.0 sec^{-1}	(Jafri S. et al, 1995, p.2139-2153)
ipflux	Maximum IP_3 flux rate	300 sec^{-1}	(Jafri S. et al, 1995, p.2139-2153)
volrat	Ratio of ER to cytoplasmic volume	0.185	(Jafri S. et al, 1995, p.2139-2153)
irbc	Inhibitory receptor binding constant	0.2 $\mu M^{-1}sec^{-1}$	(Jafri S. et al, 1995, p.2139-2153)
arbc	Activation receptor binding constant	20 $\mu M^{-1}sec^{-1}$	(Jafri S. et al, 1995, p.2139-2153)
irdc	Inhibitory receptor dissociation constant	0.2 sec^{-1}	(Jafri S. et al, 1995, p.2139-2153)
ardc	Activation receptor dissociation constant	$20\text{x}82.3\text{x}10^{-3}$ sec^{-1}	(Jafri S. et al, 1995, p.2139-2153)
scapumprate	SERCA maximal pump rate	45 µM/sec	(Jafri S. et al, 1995, p.2139-2153)
scadissrat	SERCA disassociation constant	0.1 µM	(Jafri S. et al, 1995, p.2139-2153)
Kcain	Calcium dependent inactivation	10 µM	(Kusters et al, 2005, p. 3741-3756)
ipin	External IP_3 input rate	2 µM/sec	(Keizer J. et al, 1992, p. 649-660)
ft	IP_3 pulse	1	In this paper
v_6	Maximum Ca^{2+} dependent IP_3 input rate	0.710 µM/sec	(Keizer J. et al, 1992, p. 649-660)
v_7	IP_3 decay rate constant	2.0 s^{-1}	(Keizer J. et al, 1992, p. 649-660)
k_6	Activation const.Ca^{2+} dependent IP_3 input	1.1 µM	(Keizer J. et al, 1992, p. 649-660)

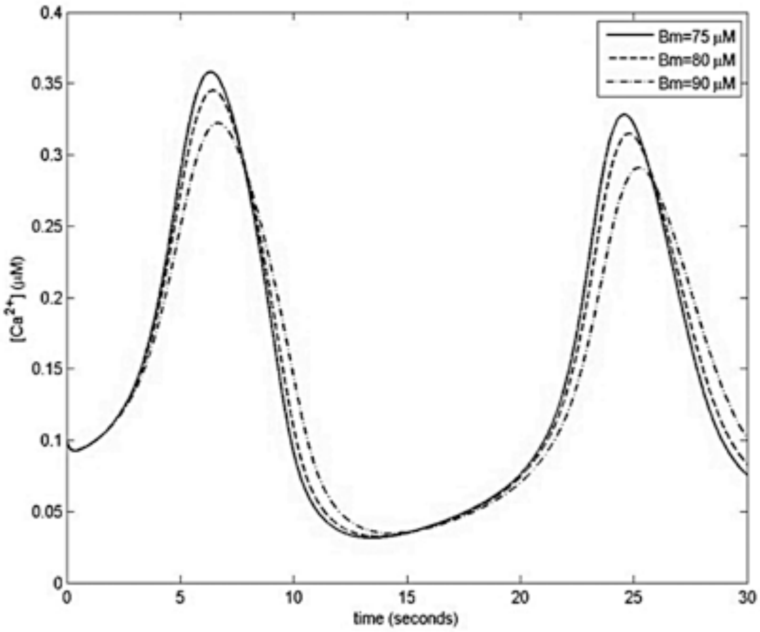

Figure 1. Variation of calcium with time and cytoplasmic buffer concentration

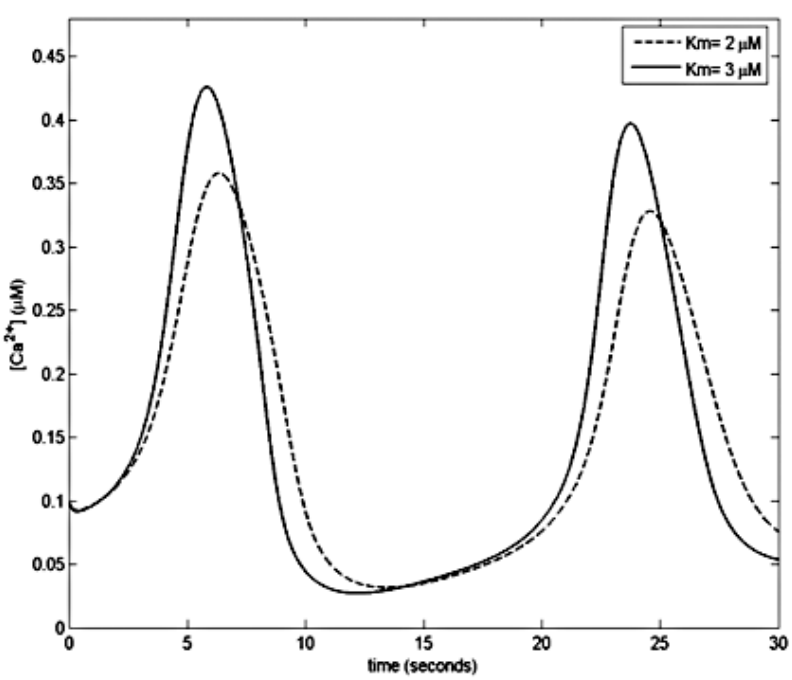

Figure 2. Variation of calcium with time and buffer dissociation rate

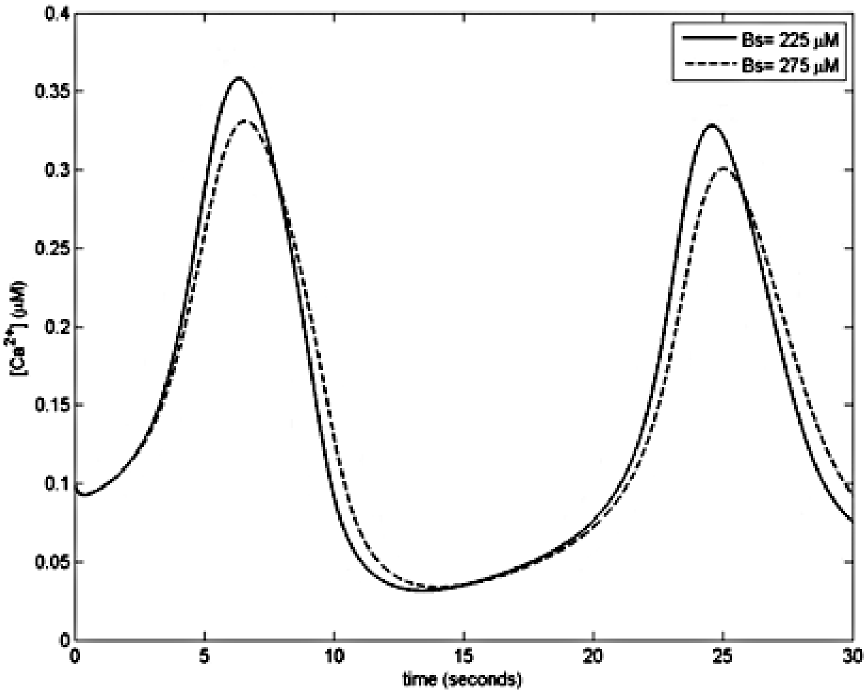

Figure 3. Calcium oscillation with time and stationary buffer concentration

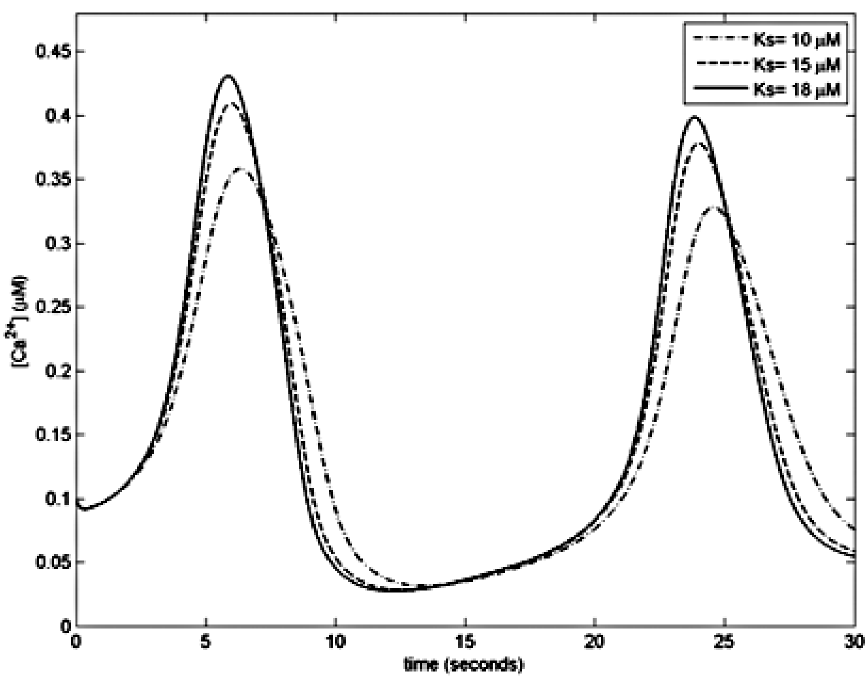

Figure 4. Calcium variation with time and stationary buffer disassociation rate

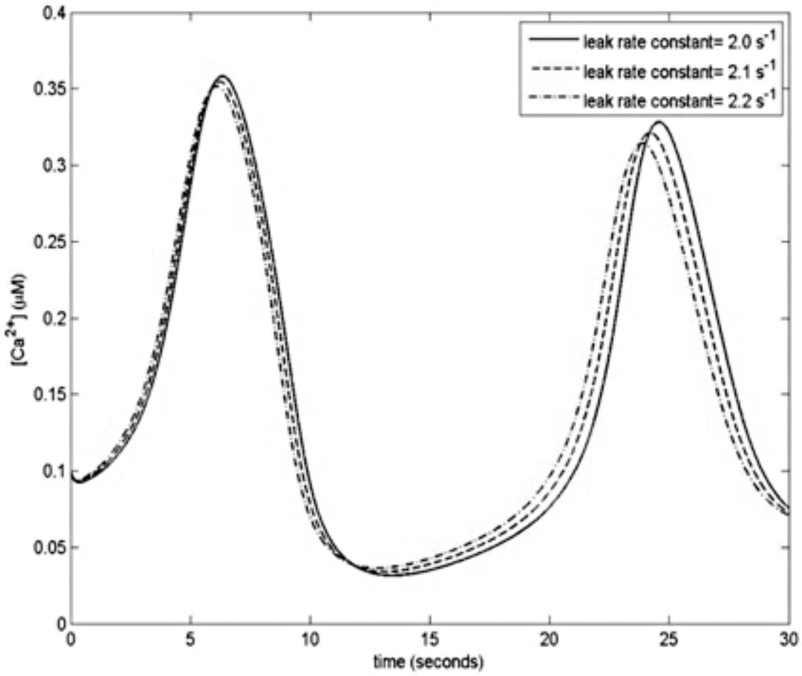

Figure 5. Calcium variation with leak rate constant

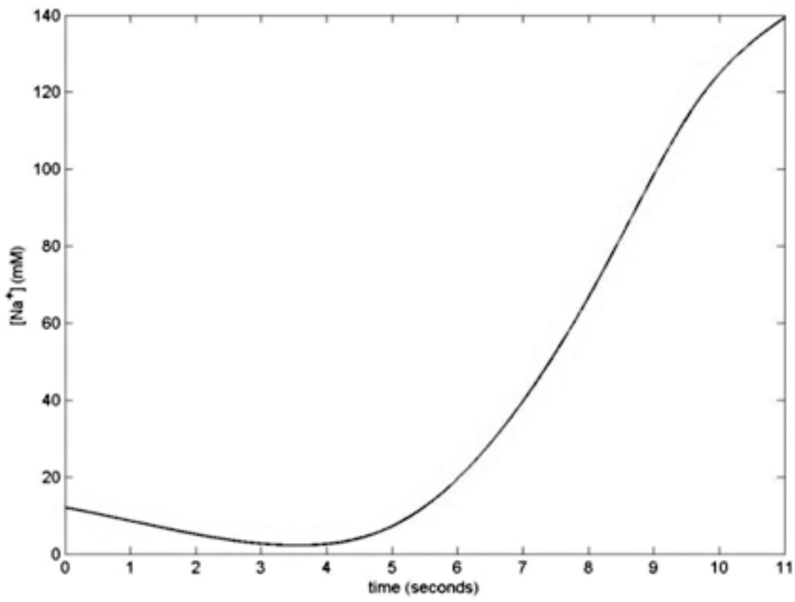

Figure 6. Sodium concentration variations with time

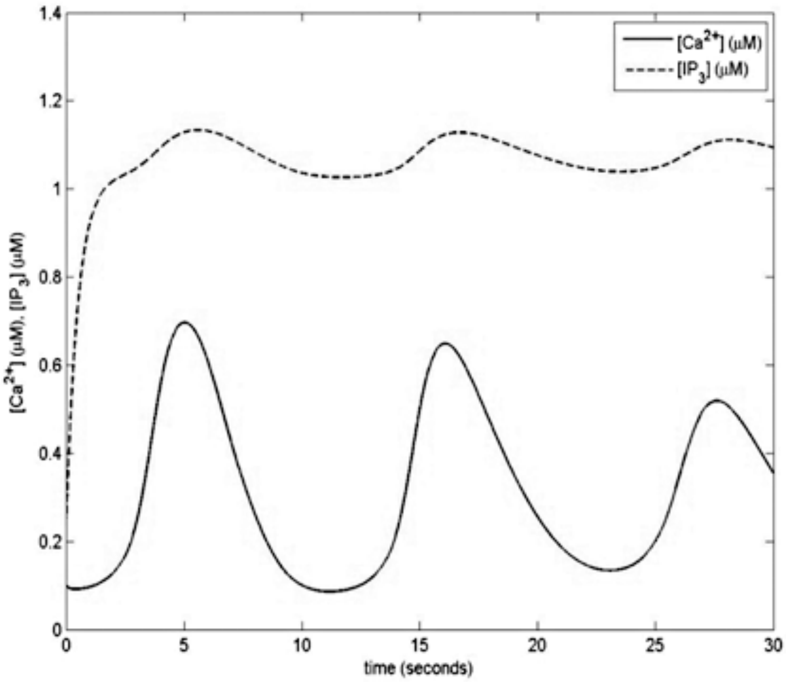

Figure 7. Variations in calcium and inositol triphosphate with time

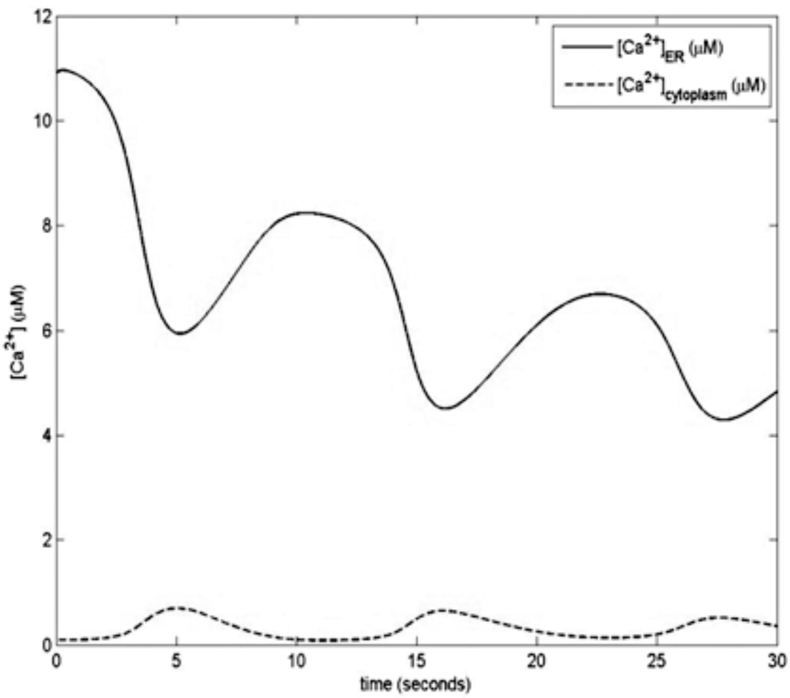

Figure 8. Variations in cytoplasmic and er calcium with time

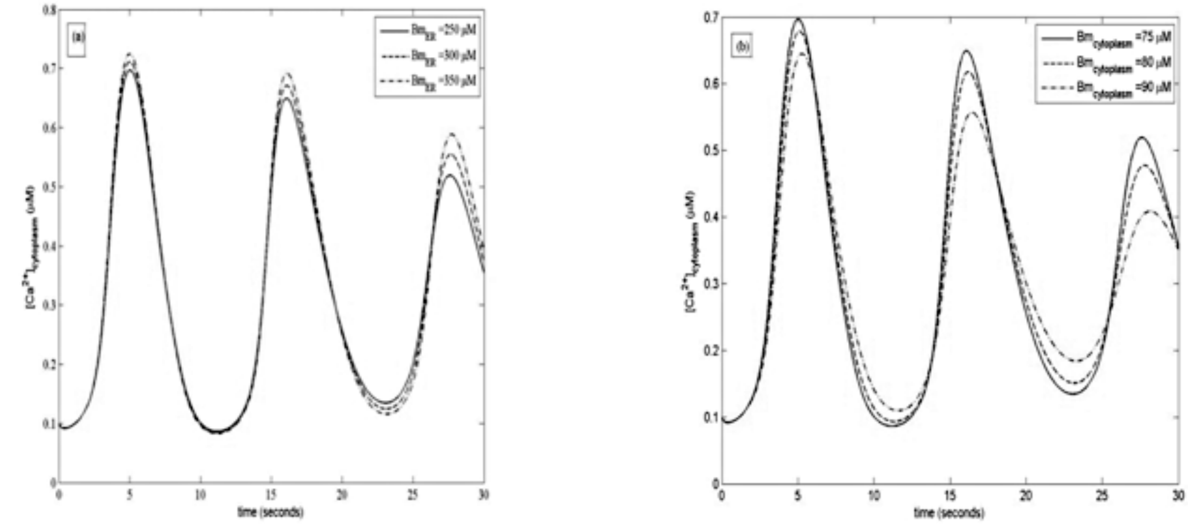

Figure 9. Variation of cytosolic calcium with time and (a) er and (b) cytoplasmic buffer concentrations

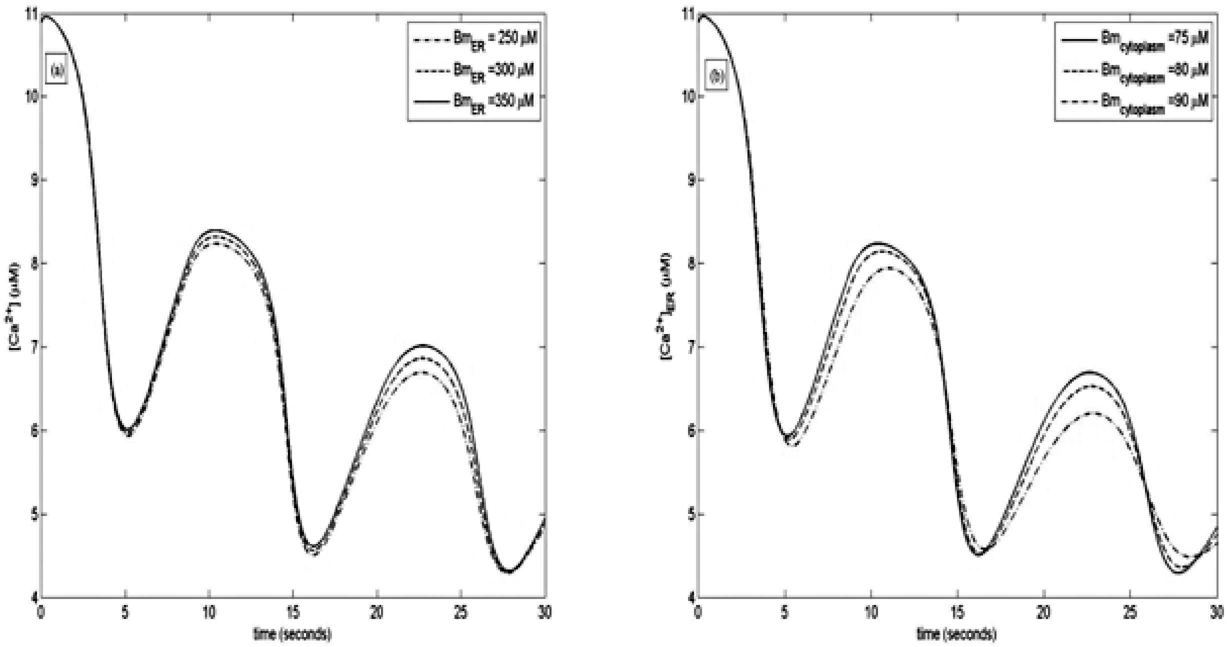

Figure 10. Plot of er calcium variation with (a) er and (b) cytoplasmic buffer concentration and time

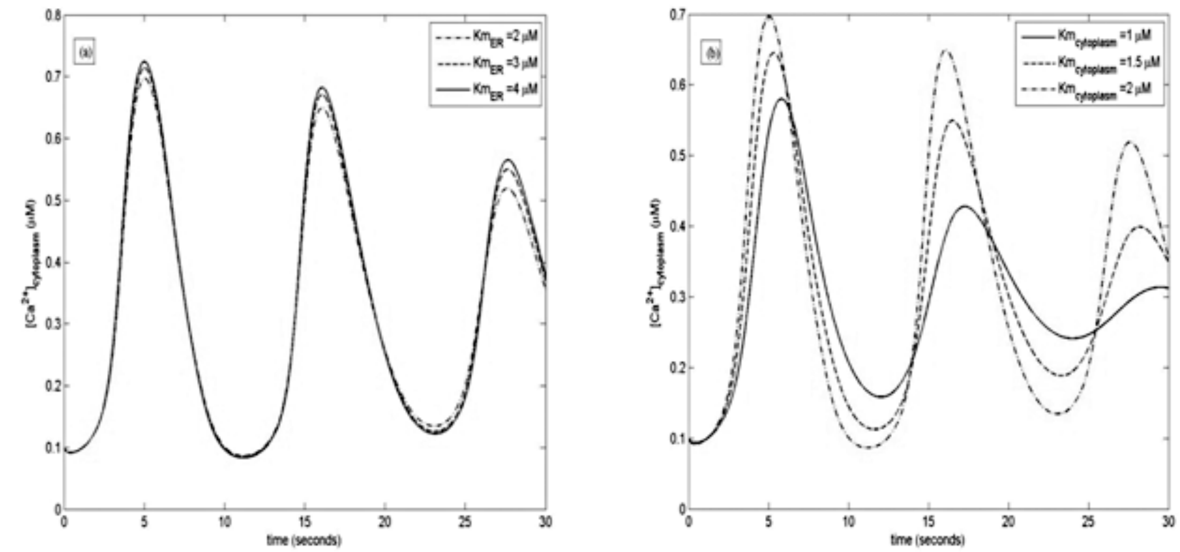

Figure 11. Cytosolic calcium variation with (a) er and (b) cytoplasmic buffer disassociation rate and time

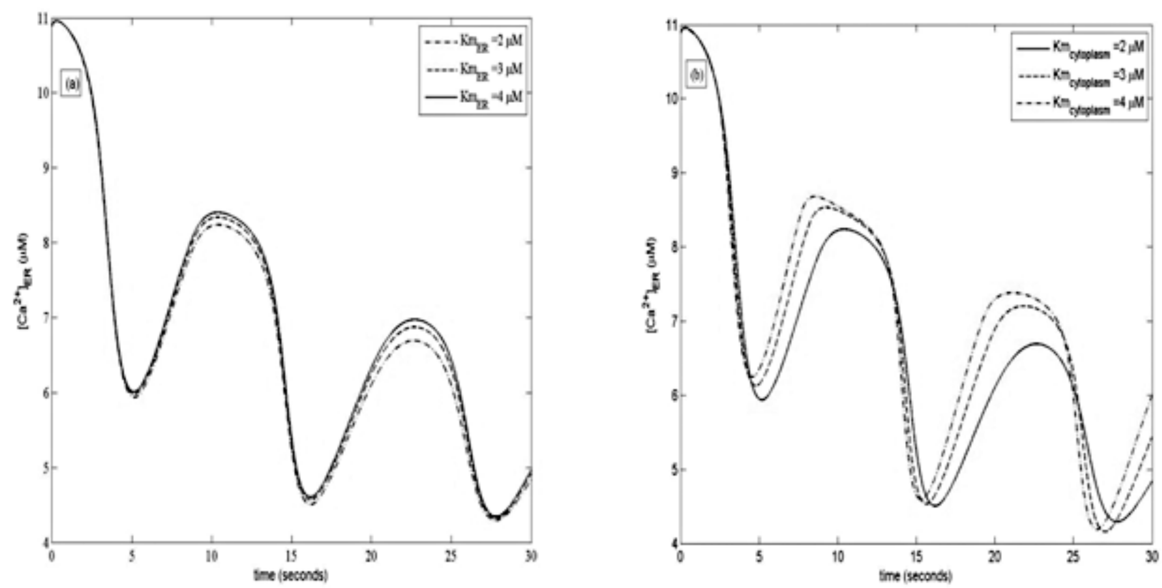

Figure 12. Plot of er calcium variation with (a) er and (b) cytoplasmic buffer disassociation rate and time

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Teaching Effect Assessment Method of Basis Courses in Engineering Institutions Based on Homogeneous Markov Chain

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Abstract

Using Markov chain model and by the changes of state transition about system, this paper describes the dynamic characteristics of teaching method of the basis course in the engineering institutions, which reflects the management of institutions and effect of teaching and learning.

Keywords: Homogeneous Markov chain, Public courses, Teaching assessment, Transition matrix

1. Introduction

Real, comprehensive, accurate assessment of the project based on teaching effectiveness of institutions can promote the improvement of teaching quality and has important role in high-quality talent. Teaching quality assessment is a complex dynamic system engineering, and teaching method of assessment should be determined by the nature and characteristics of teaching. A suitable teaching assessment method and model suit for all schools does not exist; Teacher's participation is the basic condition for implementing any method of teaching assessment, and also is the security for achieving goal of teaching evaluation ; The goal of any teaching assessment method or model use is only one, i.e. improve the teaching, improve large-area fully quality of teaching and train talents.

Many colleges have carried teaching assessment. Practices have proved that classroom teaching assessment is an important mean for feedback the teaching information, which can help teachers and students to effectively monitor the classroom instruction and classroom learning process, improve the quality of teaching and promote self-improvement of teachers. However, for some reasons, in practice the current traditional method based solely on student's average performance, variance and teacher evaluating students's learning attitude to assess the quality of teaching is one-sided and inaccurate. Because the test scores of students depend on many factors, where the basis difference of students is a very important factor. Many colleges exist many problems of teaching assessment, which have a certain negative effect on teaching and learning activities.

Homogeneous Markov chain analysis is a statistical method based on probability and using mathematical models to analyze the number development and changes in the process of object relations. Homogeneous Markov chain is widely used in the forecast stock prices, business unit forecast of human resource flows, the RMB exchange rate forecast, as well as a variety of market forecast. Zhenhua Ma, in literature (Zhenhua Ma, 2002) , researched Markov chain theory and applications. Xiangyang Cheng, in literature (Xiangyang Cheng, 2007), using the properties of homogeneous Markov chain, analysis the models of structure of school talents to carry out modeling analysis.

2. Principle of Homogeneous Markov Chain

In a random process, if the probability of a state transition from one state to another only has anything to do with the current state, but has nothing with the state before this moment, which is known as Markov process. Markov chain is a Markov process for discrete state and time , referred to Markov chain. According to the composition of Markov chain, the process has the following three characteristics:

(1) Discreteness of process. The development of system can be separated into limited states in time.

(2) Stochastic process. The system from one state to another state is a random, and transition is valued by the probability of original history conditions.

(3) Process without aftereffect. The transition probability of system only is with the current state, and has nothing to do with the previous state. That is, the ith result of certain elements of a system only influenced by the $(i-1)$th result in the transition, and has nothing to do with other results.

A random process $\{X_n, n = 0, 1, 2\cdots\}$ is a set of random variables and X_n can take various different values, which is known as the state. If the transition probability of a random process $\{X_n, n = 0, 1, 2\cdots\}$ from one state to another state only is with the current state and has nothing with the state before this moment. That is, if the conditional probability distribution of X_{n+1} in the process $\{X_n, n = 0, 1, 2\cdots\}$ is only dependent on the value of X_n, and independent of the prevenient value, whose process is called Markov process. Markov chain is a Markov process about discrete time and discrete state. If a Markov chain transfers from the state i in u moment to state j in $t+u$ moment, its probability has nothing to do with the starting time u, which is known as the homogeneous Markov chain. If denoting transition probability from state i to state j by P_{ij}, $P_{ij} = P\{X_{n+1} = j|X_n = i\}, i, j = 0, 1, 2\cdots$ and transition probability matrix is P. So that a homogeneous Markov chain is completely determined by a transition probability matrix P and the probability distribution in zero time $x = 0, 1, 2\cdots$. By the properties of homogeneous Markov chain, A_i in the ith state and A_{i+1} in the $(i+1)$th state have relationships: $A_{i+1} = A_iP$.

In the quantitative indicators of teaching effectiveness for basic courses in engineering institutions, homogeneous Markov chain takes the ratio of each grade student number than the total number as state variable, where those grades are excellent (hyper-90 points), good (80-89 points), medium (70-79 points), passed (60-69 points) and failed (under 59 points) groups in a certain test, denoted by vector $P(t) = (X_1(t), X_2(t), X_3(t), X_4(t), X_5(t))$, where t is moment, $t \in N$. Because homogeneous Markov chain has nothing to do with the state before moment t(without aftereffect), we can study the change rule of state vector when t changes, which can effect physical education and quality evaluation. Suppose in the first examination, excellent, good, medium, passed and failed students of n students in a class are $n_i(i = 1, 2, 3, 4, 5)$, the state vector $P(1) = (n_1/n, n_2/n, n_3/n, n_4/n, n_5/n)$ is called initial vector. To study the teaching quality , we continue to analysis level changes of the above students in the next examination. If in the second test, the number of original n_1 excellent students maintained still excellent is n_{11} and students transforming to "good", "medium", "passed", "failed" students are $n_{12}, n_{13}, n_{14}, n_{15}$. So the examination transition condition for the first excellent students is

$$P_1 = (\frac{n_{11}}{n_1}, \frac{n_{12}}{n_1}, \frac{n_{13}}{n_1}, \frac{n_{14}}{n_1}, \frac{n_{15}}{n_1}).$$

Similarly, the rest examination transition condition for the rest grade students are

$$P_2 = (\frac{n_{21}}{n_2}, \frac{n_{22}}{n_2}, \frac{n_{23}}{n_2}, \frac{n_{24}}{n_2}, \frac{n_{25}}{n_2}),$$

$$P_3 = (\frac{n_{31}}{n_3}, \frac{n_{32}}{n_3}, \frac{n_{33}}{n_3}, \frac{n_{34}}{n_3}, \frac{n_{35}}{n_3}),$$

$$P_4 = (\frac{n_{41}}{n_4}, \frac{n_{42}}{n_4}, \frac{n_{43}}{n_4}, \frac{n_{44}}{n_4}, \frac{n_{45}}{n_4}).$$

Where $n_{ij}(i, j = 1, 2, 3, 4, 5)$ denote the number from state i to state j. The transition case is expressed as a matrix:

$$P = \begin{bmatrix} \frac{n_{11}}{n_1} & \frac{n_{12}}{n_1} & \frac{n_{13}}{n_1} & \frac{n_{14}}{n_1} & \frac{n_{15}}{n_1} \\ \frac{n_{21}}{n_2} & \frac{n_{22}}{n_2} & \frac{n_{23}}{n_2} & \frac{n_{24}}{n_2} & \frac{n_{25}}{n_2} \\ \frac{n_{31}}{n_3} & \frac{n_{32}}{n_3} & \frac{n_{33}}{n_3} & \frac{n_{34}}{n_3} & \frac{n_{35}}{n_3} \\ \frac{n_{41}}{n_4} & \frac{n_{42}}{n_4} & \frac{n_{43}}{n_4} & \frac{n_{44}}{n_4} & \frac{n_{45}}{n_4} \\ \frac{n_{51}}{n_5} & \frac{n_{52}}{n_54} & \frac{n_{53}}{n_5} & \frac{n_{54}}{n_5} & \frac{n_{55}}{n_5} \end{bmatrix} = (p_{i,j}).$$

P is a transition probability matrix. Student achievements according to state transition probability matrix of Homogeneous Markov chain is bound to more stable. $X = (x_1, x_2, x_3, x_4, x_5)$ has non-impact, and influencing the student achievement is the teaching and learning process from the first examination to the second examination, as well as the change condition P according to this process. Through the transition probability matrix of student's learning state, we can predict ultimately the steady state of a class of students. For teachers, it can be used to evaluate and predict the quality of teaching class. If giving different scores to different levels in the end, we can get a comprehensive evaluation result of teaching and learning performance.

3. Application Examples

Select two "Advanced Mathematics" courses of seven classes to assess the results. The examination transition vectors of all students computed by contrasting two scores are as follows:

$$P_1 = (0 \quad 1.0000 \quad 0 \quad 0 \quad 0),$$

$$P_2 = (0.1000 \quad 0.1000 \quad 0.8000 \quad 0 \quad 0)$$

$$P_3 = (0.0385 \quad 0.3077 \quad 0.2692 \quad 0.2692 \quad 0.1154)$$

$$P_4 = (0 \quad 0 \quad 0.2308 \quad 0.4615 \quad 0.3077)$$

$$P_5 = (0 \quad 0.0192 \quad 0.0769 \quad 0.1731 \quad 0.7308)$$

Then the transition matrix is as follows:

$$P = \begin{bmatrix} 0 & 1.0000 & 0 & 0 & 0 \\ 0.1000 & 0.1000 & 0.8000 & 0 & 0 \\ 0.0385 & 0.3077 & 0.2692 & 0.2692 & 0.1154 \\ 0 & 0 & 0.2308 & 0.4615 & 0.3077 \\ 0 & 0.0192 & 0.0769 & 0.1731 & 0.7308 \end{bmatrix}.$$

The limit vector of this transition matrix is

$$x = (0.0206 \quad 0.1136 \quad 0.2413 \quad 0.2432 \quad 0.3813).$$

Analysis results for the examination of every learning class can be seen in Table 1, Figure 1 and Table 2.

From the examples, we can see that the ranking changes using homogeneous Markov chain assessment instead of traditional methods. We may know that the second class ranking is unchanged and it ranks first now, which explains that the second class has not only good basis , but also the largest progress. The third, forth and seventh classes improve ranking, which is known that it has a relatively larger progress. The first, fifth and sixth is opposite.

4. Conclusions

Homogeneous Markov chain approach can be sensitive to reflect the true effect of teaching. Traditional teaching approach is based on student's scores of a particular distortion, only taking into account achievements of students, which results the assessment. The analysis result of homogeneous Markov chain is only concerned with the transition matrix, and has nothing to do with one examination result of students, which is assessed according to the transition state of two examinations. The good or bad effect of teaching is reflected by the rise or decline of students scores, which reflects superiority and objectivity of Markov chain theory. Researching the teaching method is the basic subject, and also is the eternal subject. Only pursuing high-quality effect and high standard teaching effect for teachers can truly embody the diathesis education and the forever value of quality education, find out assessment method suit for basis courses in colleges, and train high-quality talents.

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Table 1. Examination Analysis

	the first examination		the second examination	
teaching class	average score	ranking	average score	ranking
one	74.08	3	75.63	2
two	79.63	1	82.45	1
three	78.87	2	73.80	5
four	72.44	5	75.13	3
five	73.53	4	74.40	4
six	69.10	6	65.25	6
seven	63.51	7	60.92	7

Table 2. Comprehensive Evaluation

teaching class	limiting vector	score	ranking
one	(0.1404 0.4496 0.2817 0.0514 0.0769)	78.33	4
two	(0.3351 0.4759 0.1457 0.0320 0.0112)	85.63	1
three	(0.2811 0.3855 0.2239 0.0849 0.0246)	82.52	3
four	(0.2810 0.4277 0.1896 0.0638 0.0378)	82.55	2
five	(0.1882 0.2426 0.2210 0.2098 0.1384)	72.86	5
six	(0.0111 0.0785 0.1587 0.3239 0.4278)	53.52	7
seven	(0.0206 0.1136 0.2413 0.2432 0.3813)	56.96	6

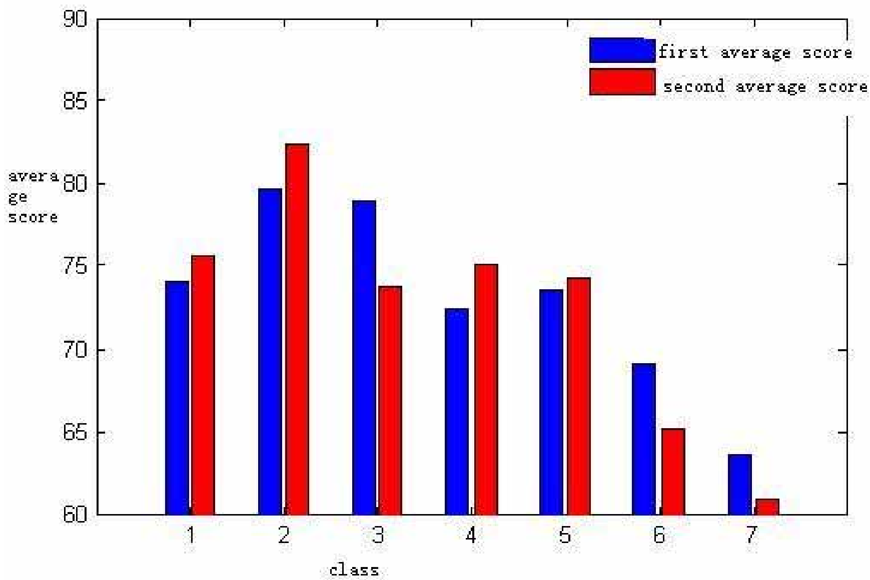

Figure 1. Average scores comparison of classes

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Fuzzy Ideals and Fuzzy Filters of Ordered Ternary Semigroups

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Abstract

The notion of ternary semigroups was introduced by Lehmer in 1932 and that of fuzzy sets by Zadeh in 1965. Any semigroup can be reduced to a ternary semigroup but a ternary semigroup does not necessarily reduce to a semigroup. A partially ordered semigroup T is called an ordered ternary semigroup if for all $x_1, x_2, x_3, x_4 \in T, x_1 \leq x_2$ implies $x_1x_3x_4 \leq x_2x_3x_4, x_3x_1x_4 \leq x_3x_2x_4$ and $x_3x_4x_1 \leq x_3x_4x_2$. In this paper, we study fuzzy ternary subsemigroups (left ideals, right ideals, lateral ideals, ideals) and fuzzy left filters (right filters, lateral filters, filters) of ordered ternary semigroups.

Keywords: Fuzzy ideals, Prime fuzzy ideals, Filters, Fuzzy filters, Ordered ternary semigroups

1. Introduction and Preliminaries

In 1932, Lehmer gave the definition of ternary semigroups (Lehmer, 1932). A nonempty set T is called a *ternary semigroup* if there exists a ternary operation $T \times T \times T \to T$, written as $(x_1, x_2, x_3) \mapsto x_1x_2x_3$ satisfying the following identity for any $x_1, x_2, x_3, x_4, x_5 \in T$,

$$[[x_1x_2x_3]x_4x_5] = [x_1[x_2x_3x_4]x_5] = [x_1x_2[x_3x_4x_5]].$$

Any semigroup can be reduced to a ternary semigroup. However, Banach showed that a ternary semigroup does not necessarily reduce to a semigroup by this example.

Example 1.1 $T = \{-i, 0, i\}$ is a ternary semigroup while T is not a semigroup under the multiplication over complex numbers.

The next example is also a ternary semigroup but not a semigroup.

Example 1.2 $\mathbb{Z}^-$ is a ternary semigroup while $\mathbb{Z}^-$ is not a semigroup under the multiplication over integers.

However, Los showed that every ternary semigroup can be embedded in a semigroup (Los, 1955).

The algebraic structures of ternary semigroups were studied by some authors, for example, Sioson studied ideals in ternary semigroups (Sioson, 1965), Santiago studied regular ternary semigroups (Santiago, 1990), Dixit and Dewan studied quasi-ideals and bi-ideals in ternary semigroups (Dixit and Dewan, 1995), Kar and Maity studied congruences of ternary semigroups (Kar and Maity, 2007) and Iampan studied minimal and maximal lateral ideals of ternary semigroups (Iampan, 2007).

A partially ordered semigroup T is called an *ordered ternary semigroup* if for all $x_1, x_2, x_3, x_4 \in T, x_1 \leq x_2$ implies $x_1x_3x_4 \leq x_2x_3x_4, x_3x_1x_4 \leq x_3x_2x_4$ and $x_3x_4x_1 \leq x_3x_4x_2$.

Example 1.3 $(\mathbb{Z}^-, \cdot, \leq)$ is a ordered ternary semigroup.

Let T be an ordered ternary semigroup. For nonempty subsets A, B and C of T, let $ABC := \{abc \mid a \in A, b \in B \text{ and } c \in C\}$. For a nonempty subset A of T, we note $(A] := \{t \in T \mid t \leq h \text{ for some } h \in A\}$. A nonempty subset S of T is called a *ternary subsemigroup* of T if $(S] \subseteq S$ and $SSS \subseteq S$. A nonempty subset A of T is called a *left ideal* of T if $(A] \subseteq A$ and $TTA \subseteq A$, a *right ideal* of T if $(A] \subseteq A$ and $ATT \subseteq T$, and a *lateral ideal* of T if $(A] \subseteq A$ and $TAT \subseteq A$. If A is a left, right and lateral ideal of T, A is called an *ideal* of T.

The notion of fuzzy sets was introduced by Zadeh (Zadeh, 1965). Several researchs were conducted on the generalizations of the notion of fuzzy sets. Fuzzy semigroups have been first considered by Kuroki (Kuroki, 1981, 1991 and 1993) and fuzzy ordered semigroups by Kehayopulu and Tsingelis (Kehayopulu, 1990, Kehayopulu and Tsingelis, 1999 and 2002, Kehayopulu, Xie and Tsingelis, 2001). In 2008, Shabir and Khan studied fuzzy filters in ordered semigroups (Shabir and Khan, 2008).

Let T be an ordered ternary semigroup. A function f from T to the unit interval $[0, 1]$ is called a *fuzzy subset* of T. The ordered ternary semigroup T itself is a fuzzy subset of T such that $T(x) = 1$ for all $x \in T$, denoted also by T. If $A \subseteq T$, the *characteristic function* f_A of A is a fuzzy subset of T defined as follows:

$$f_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

Let T be an ordered ternary semigroup and f a fuzzy subset of T. The fuzzy subset f' defined by $f'(x) = 1 - f(x)$ for all $x \in T$ is called the *complement* of f in T.

The aim of this paper is to study fuzzy ternary subsemigroups (left ideals, right ideals, lateral ideals, ideals) and fuzzy left filters (right filters, lateral filters, filters) of ordered ternary semigroups that are studied analogously to the concept of fuzzy ideals and fuzzy filters in ordered semigroups.

2. Main Results

Now we define fuzzy ternary subsemigroups, fuzzy left ideals, fuzzy right ideals, fuzzy lateral ideals and fuzzy ideals of ordered ternary semigroups. Let T be a ternary semigroup. A fuzzy subset f of T is called

a *fuzzy ternary subsemigroup* of T if (1) $x \leq y$ implies $f(x) \geq f(y)$ and (2) $f(xyz) \geq \min\{f(x), f(y), f(z)\}$ for all $x, y, z \in T$,

a *fuzzy left ideal* of T if (1) $x \leq y$ implies $f(x) \geq f(y)$ and (2) $f(xyz) \geq f(z)$ for all $x, y, z \in T$,

a *fuzzy right ideal* of T if (1) $x \leq y$ implies $f(x) \geq f(y)$ and (2) $f(xyz) \geq f(x)$ for all $x, y, z \in T$,

a *fuzzy lateral ideal* of T if (1) $x \leq y$ implies $f(x) \geq f(y)$ and (2) $f(xyz) \geq f(y)$ for all $x, y, z \in T$ and

a *fuzzy ideal* of T if (1) $x \leq y$ implies $f(x) \geq f(y)$ and (2) $f(xyz) \geq \max\{f(x), f(y), f(z)\}$ for all $x, y, z \in T$.

Lemma 2.1 Let T be an ordered ternary semigroup and A a nonempty subset of T. Then $(A] \subseteq A$ if and only if $x \leq y$ implies $f_A(x) \geq f_A(y)$.

Proof Assume that $(A] \subseteq A$. Let $x, y \in T$ such that $x \leq y$.

Case 1: $y \notin A$. Then $f_A(y) = 0 \leq f_A(x)$.

Case 2: $y \in A$. Since $(A] \subseteq A, x \in A$. So $f_A(x) = 1 \geq f_A(y)$.

Conversely, let $x \in (A]$. Then there exists $y \in A$ such that $x \leq y$. So $f_A(x) \geq f_A(y) = 1$. This implies $x \in A$. □

Now we characterize ternary subsemigroups (left ideals, right ideals, lateral ideals, ideals) of ordered ternary semigroups in terms of fuzzy ternary subsemigroups (fuzzy left ideals, fuzzy right ideals, fuzzy lateral ideals, fuzzy ideals).

Theorem 2.1 Let T be an ordered ternary semigroup and A a nonempty subset of T. The following statements are true.

(1) A is a ternary subsemigroup of T if and only if f_A is a fuzzy ternary subsemigroup of T.

(2) A is a left ideal (right ideal, lateral ideal, ideal) of T if and only if f_A is a fuzzy left ideal (fuzzy right ideal, fuzzy lateral ideal, fuzzy ideal) of T.

Proof (1) Assume that A is a ternary subsemigroup of T. By Lemma 2.1, $x \leq y$ implies $f_A(x) \geq f_A(y)$. Next, let $x, y, z \in T$.

Case 1 : $x, y, z \in A$. Since A is a ternary subsemigroup of T, $xyz \in A$. Therefore $f_A(xyz) = 1 \geq \min\{f(x), f(y), f(z)\}$.

Case 2 : $x \notin A$ or $y \notin A$ or $z \notin A$. Thus $f_A(x) = 0$ or $f_A(y) = 0$ or $f_A(z) = 0$. Hence $\min\{f_A(x), f_A(y), f_A(z)\} = 0 \leq f_A(xyz)$.

Conversely, assume that f_A is a fuzzy ternary subsemigroup of T. By Lemma 2.1, $(A] \subseteq A$. Next, let $x, y, z \in A$. So $f_A(x) = f_A(y) = f_A(z) = 1$. Since f_A is a fuzzy ternary subsemigroup of T, $f_A(xyz) \geq \min\{f_A(x), f_A(y), f_A(z)\} = 1$. Then $xyz \in A$.

(2) Assume that A is a left ideal of T. By Lemma 2.1, we have that $x \leq y$ implies $f_A(x) \geq f_A(y)$. Next, let $x, y, z \in T$.

Case 1 : $z \in A$. Since A is a left ideal of T, $xyz \in A$. Then $f_A(xyz) = 1$. Therefore $f_A(xyz) \geq f_A(z)$.

Case 2 : $z \notin A$. So $f_A(z) = 0$. Hence $f_A(xyz) \geq f_A(z)$.

Conversely, assume that f_A is a fuzzy left ideal of T. By Lemma 2.1, $(A] \subseteq A$. Next, let $x, y \in T$ and $z \in A$. Since f_A is a fuzzy left ideal of T and $z \in A$, $f_A(xyz) \geq f_A(z) = 1$. So $xyz \in A$.

The other parts of (2) can be proved in similarly way. □

Now we define left filters, right filters, lateral filters, filters, fuzzy left filters, fuzzy right filters, fuzzy lateral filters and fuzzy filters of ordered ternary semigroups.

Let T be an ordered ternary semigroup. A nonempty subset F of T is called

a *left filter* of T if (1) $F^3 \subseteq F$, (2) for all $x, y \in T, x \leq y$ and $x \in F$ imply $y \in F$ and (3) for all $x, y, z \in T, xyz \in F$ implies $z \in F$,

a *right filter* of T if (1) $F^3 \subseteq F$, (2) for all $x, y \in T, x \leq y$ and $x \in F$ imply $y \in F$ and (3) for all $x, y, z \in T, xyz \in F$ implies $y \in F$,

a *lateral filter* of T if (1) $F^3 \subseteq F$, (2) for all $x, y \in T, x \leq y$ and $x \in F$ imply $y \in F$ and (3) for all $x, y, z \in T, xyz \in F$ implies $x \in F$ and

a *filter* of T if (1) $F^3 \subseteq F$, (2) for all $x, y \in T, x \leq y$ and $x \in F$ imply $y \in F$ and (3) for all $x, y, z \in T, xyz \in F$ implies $x, y, z \in F$.

A fuzzy subset f of T is called

a *fuzzy left filter* of T if for all $x, y, z \in T$ (1) $x \leq y$ implies $f(x) \leq f(y)$, (2) $f(xyz) \geq \min\{f(x), f(y), f(z)\}$ and (3)$f(xyz) \leq f(z)$,

a *fuzzy right filter* of T if for all $x, y, z \in T$ (1) $x \leq y$ implies $f(x) \leq f(y)$, (2) $f(xyz) \geq \min\{f(x), f(y), f(z)\}$ and (3)$f(xyz) \leq f(x)$,

a *fuzzy lateral filter* of T if for all $x, y, z \in T$ (1) $x \leq y$ implies $f(x) \leq f(y)$, (2) $f(xyz) \geq \min\{f(x), f(y), f(z)\}$ and (3)$f(xyz) \leq f(y)$ and

a *fuzzy filter* of T if for all $x, y, z \in T$ (1) $x \leq y$ implies $f(x) \leq f(y)$, (2) $f(xyz) = \min\{f(x), f(y), f(z)\}$.

We also characterize left filters (right filters, lateral filters, filters) of ordered ternary semigroups in terms of fuzzy left filters (fuzzy right filters, fuzzy lateral filters, fuzzy filters).

Theorem 2.2 Let F be a nonempty subset of an ordered ternary semigroup T. Then F is a left filter (right filter, lateral filter, filter) of T if and only if the characteristic function f_F of F is a fuzzy left filter (right filter, lateral filter, filter) of T.

Proof Assume that F is a left filter of T. Let $x, y \in T$ such that $x \leq y$.

Case 1: $x \notin F$. Then $f_F(x) = 0$. Then $f_F(x) \leq f_F(y)$.

Case 2: $x \in F$. Since $x \leq y$ and F is a left filter of $T, y \in F$. Thus $f_F(y) = 1$. Hence $f_F(x) \leq f_F(y)$.

Next, let $x, y, z \in T$.

Case 1: $x, y, z \in F$. Then $xyz \in F$. Hence $f_F(xyz) = 1$. Therefore $f_F(xyz) \geq \min\{f_F(x), f_F(y), f_F(z)\}$.

Case 2: $x \notin F$ or $y \notin F$ or $z \notin F$. So $f_F(x) = 0$ or $f_F(y) = 0$ or $f_F(z) = 0$. This implies $f_F(xyz) \geq \min\{f_F(x), f_F(y), f_F(z)\}$.

Finally, let $x, y, z \in T$.

Case 1: $xyz \in F$. Since F is a left filter of T and $xyz \in F, z \in F$. So $f_F(z) = 1$. Therefore $f_F(xyz) \leq f_F(z)$.

Case 2: $xyz \notin F$. Then $f_F(xyz) = 0$. Therefore $f_F(xyz) \leq f_F(z)$.

Conversely, assume f_F is a fuzzy left filter of T. Let $x, y, z \in F$. Then $f_F(x) = f_F(y) = f_F(z) = 1$. Thus $f_F(xyz) \geq \min\{f_F(x), f_F(y), f_F(z)\} = 1$. Hence $xyz \in F$. Next, let $x, y \in T$. Assume $x \leq y$ and $x \in F$. Then $f_F(x) \leq f_F(y)$ and $f_F(x) = 1$. Thus $f_F(y) = 1$, this implies $y \in F$. Finally, let $x, y, z \in T$ such that $xyz \in F$. So $f_F(xyz) = 1$. Since f_F is a fuzzy left filter of T, then $f_F(z) \geq f_F(xyz)$. This implies $f_F(z) = 1$. So $z \in F$.

The other parts can be proved in similarly way. □

Let T be an ordered ternary semigroup. A nonempty subset S of T is called a *prime subset* of T if for all $x, y, z \in T, xyz \in S$ implies $x \in S$ or $y \in S$ or $z \in S$. A ternary subsemigroup S of T is called a *prime ternary subsemigroup* of T if S is a prime subset of T. *Prime left ideals, prime right ideals, prime lateral ideals* and *prime ideals* of T are defined analogously. A fuzzy subset f of T is called a *prime fuzzy subset* of T if $f(xyz) \leq \max\{f(x), f(y), f(z)\}$ for all $x, y, z \in T$. A fuzzy ternary subsemigroup f of T is called a *prime fuzzy ternary subsemigroup* of T if f is a prime fuzzy subset of T. *Prime fuzzy left ideals, prime fuzzy right ideals, prime fuzzy lateral ideals* and *prime fuzzy ideals* of T are defined analogously.

Theorem 2.3 Let T be an ordered ternary semigroup and A a nonempty subset of T. The following statements are true.

(1) A is a prime subset of T if and only if f_A is a prime fuzzy subset of T.

(2) A is a prime ternary subsemigroup (prime left ideal, prime right ideal, prime lateral ideal, prime ideal) of T if and only if f_A is a prime fuzzy ternary subsemigroup (prime fuzzy left ideal, prime fuzzy right ideal, prime fuzzy lateral ideal, prime fuzzy ideal) of T.

Proof (1) Let A be a prime subset of T and $x, y, z \in T$.

Case 1 : $xyz \in A$. Since A is a prime subset of T, $x \in A$ or $y \in A$ or $z \in A$. So $\max\{f_A(x), f_A(y), f_A(z)\} = 1 \geq f_A(xyz)$.

Case 2 : $xyz \notin A$. So $f_A(xyz) = 0 \leq \max\{f_A(x), f_A(y), f_A(z)\}$.

Conversely, let $x, y, z \in T$ such that $xyz \in A$. Thus $f_A(xyz) = 1$. Since f_A is prime, $\max\{f_A(x), f_A(y), f_A(z)\} = 1$. Then $f_A(x) = 1$ or $f_A(y) = 1$ or $f_A(z) = 1$. Hence $x \in A$ or $y \in A$ or $z \in A$.

(2) follows from (1) and Theorem 2.1. □

Let f be a fuzzy subset of an ordered ternary semigroup T. For any $t \in [0, 1]$, the set

$$f_t = \{x \in T \mid f(x) \geq t\} \text{ and } f_t^s = \{x \in T \mid f(x) > t\}$$

are called a *t-levelset* and a *t-strong levelset* of f, respectively.

Theorem 2.4 Let f be a fuzzy subset of an ordered ternary semigroup T. The following statements are true.

(1) f is a fuzzy ternary subsemigroup of T if and only if for all $t \in [0, 1]$, if $f_t \neq \emptyset$, then f_t is a ternary subsemigroup of T.

(2) f is a fuzzy left ideal (fuzzy right ideal, fuzzy lateral ideal, fuzzy ideal) of T if and only if for all $t \in [0, 1]$, if $f_t \neq \emptyset$, then f_t is a left ideal (right ideal, lateral ideal, ideal) of T.

Proof (1) Assume that f is a fuzzy ternary subsemigroup of T. Let $t \in [0, 1]$ such that $f_t \neq \emptyset$. Let $x \in (f_t]$. Then there exists $y \in f_t$ such that $x \leq y$. Thus $f(x) \geq f(y) \geq t$. Hence $x \in f_t$. Next, let $x, y, z \in f_t$. Then $f(x), f(y), f(z) \geq t$. Thus $\min\{f(x), f(y), f(z)\} \geq t$. Since f is a fuzzy ternary subsemigroup of T, $f(xyz) \geq t$. Hence $xyz \in f_t$.

Conversely, assume for all $t \in [0, 1]$, if $f_t \neq \emptyset$, then f_t is a ternary subsemigroup of T. Let $x, y \in T$ such that $x \leq y$. Choose $t = f(y)$. Thus $y \in f_t$. This implies $x \in f_t$. Then $f(x) \geq t = f(y)$. Next, let $x, y, z \in T$. Choose $t = \min\{f(x), f(y), f(z)\}$. Then $f(x), f(y), f(z) \geq t$. Thus $x, y, z \in f_t$. Since f_t is a ternary subsemigroup of T, $xyz \in f_t$. Therefore $f(xyz) \geq t = \min\{f(x), f(y), f(z)\}$.

(2) Assume that f is a fuzzy left ideal of T. Let $t \in [0, 1]$. Suppose that $f_t \neq \emptyset$. Let $x \in (f_t]$. Then there exists $y \in f_t$ such that $x \leq y$. Thus $f(x) \geq f(y) \geq t$. Next, let $x, y, z \in T$ and $z \in f_t$. So $f(xyz) \geq f(z) \geq t$. Therefore $xyz \in f_t$.

Conversely, assume for all $t \in [0, 1]$, if $f_t \neq \emptyset$, then f_t is a left ideal of T. Let $x, y \in T$ such that $x \leq y$. Choose $t = f(y)$. Thus $y \in f_t$. This implies $x \in f_t$. Then $f(x) \geq t = f(y)$. Next, let $x, y, z \in T$. Choose $t = f(z)$. Thus $z \in f_t$, this implies $f_t \neq \emptyset$. By assumption, we have f_t is a left ideal of T. So $xyz \in f_t$. Therefore $f(xyz) \geq t$. So $f(xyz) \geq f(z)$.

The other parts of (2) can be proved in a similar way. □

Theorem 2.5 Let f be a fuzzy subset of an ordered ternary semigroup T. The following statements are true.

(1) f is a prime fuzzy subset of T if and only if for all $t \in [0, 1]$, if $f_t \neq \emptyset$, then f_t is a prime subset of T.

(2) f is a prime fuzzy ternary subsemigroup (prime fuzzy left ideal, prime fuzzy right ideal, prime fuzzy lateral ideal, prime fuzzy ideal) of T if and only if for all $t \in [0, 1]$, if $f_t \neq \emptyset$, then f_t is a prime ternary subsemigroup (prime left ideal, prime right ideal, prime lateral ideal, prime ideal) of T.

Proof (1) Assume that f is a prime fuzzy subset of T. Let $t \in [0, 1]$. Suppose that $f_t \neq \emptyset$. Let $x, y, z \in T$ such that $xyz \in f_t$. Thus $f(xyz) \geq t$. Since f is prime, $f(x) \geq t$ or $f(y) \geq t$ or $f(z) \geq t$. Hence $x \in f_t$ or $y \in f_t$ or $z \in f_t$.

Conversely, let $x, y, z \in T$. Choose $t = f(xyz)$. Thus $xyz \in f_t$. Since f_t is prime, $x \in f_t$ or $y \in f_t$ or $z \in f_t$. Then $f(x) \geq t$ or $f(y) \geq t$ or $f(z) \geq t$. Therefore $\max\{f(x), f(y), f(z)\} \geq t = f(xyz)$.

(2) follows from (1) and Theorem 2.4. □

Theorem 2.6 Let f be a fuzzy subset of an ordered ternary semigroup T. Then f is a fuzzy ternary subsemigroup (fuzzy left ideal, fuzzy right ideal, fuzzy lateral ideal, fuzzy ideal) of T if and only if for all $t \in [0, 1]$, if $f_t^s \neq \emptyset$, then f_t^s is a ternary subsemigroup (left ideal, right ideal, lateral ideal, ideal) of T.

Proof The proof of this theorem is similar to the proof of Theorem 2.4. □

Theorem 2.7 Let f be a fuzzy subset of an ordered ternary semigroup T. Then f is a prime fuzzy subset (prime fuzzy ternary subsemigroup, prime fuzzy left ideal, prime fuzzy right ideal, prime fuzzy lateral ideal, prime fuzzy ideal) of T if and only if for all $t \in [0, 1]$, if $f_t^s \neq \emptyset$, then f_t^s is a prime subset (prime ternary subsemigroup, prime left ideal, prime right ideal, prime lateral ideal, prime ideal) of T.

Proof The proof of this theorem is similar to the proof of Theorem 2.5. □

Lemma 2.2 Let f be a fuzzy subset of an ordered ternary semigroup T. The following statements are equivalent.

(1) $f'(xyz) \leq \max\{f'(x), f'(y), f'(z)\}$ for all $x, y, z \in T$.

(2) $f(xyz) \geq \min\{f(x), f(y), f(z)\}$ for all $x, y, z \in T$.

Proof Straightforward. □

Theorem 2.8 Let f be a fuzzy subset of an ordered ternary semigroup T. Then f is a fuzzy left filter (fuzzy right filter, fuzzy lateral filter, fuzzy filter) of T if and only if the complement f' of f is a prime fuzzy left ideal (prime fuzzy right ideal, prime fuzzy lateral ideal, prime fuzzy ideal) of T.

Proof Assume f is a fuzzy left filter of T. Let $x, y \in T$ such that $x \leq y$. Since f is a fuzzy left filter of $T, f(x) \leq f(y)$. This implies $f'(x) \geq f'(y)$. Next, let $x, y, z \in T$. Since f is a fuzzy left filter of $T, f(xyz) \leq f(z)$. Thus $f'(xyz) \geq f'(z)$. Finally, let $x, y, z \in T$. Since f is a fuzzy left filter of $T, f(xyz) \geq \min\{f(x), f(y), f(z)\}$. By Lemma 2.2, $f'(xyz) \leq \max\{f'(x), f'(y), f'(z)\}$.

Conversely, assume f' is a prime fuzzy left ideal of T. Let $x, y \in T$ such that $x \leq y$. Since f' is a fuzzy left ideal of $T, f'(x) \geq f'(y)$. Therefore $f(x) \leq f(y)$. Next, let $x, y, z \in T$. Since f' is prime, $f'(xyz) \leq \max\{f'(x), f'(y), f'(z)\}$. By Lemma 2.2, we have $f(xyz) \geq \min\{f(x), f(y), f(z)\}$. Finally, let $x, y, z \in T$. Since f' is a fuzzy left ideal of $T, f'(xyz) \geq f'(z)$. Hence $f(xyz) \leq f(z)$.

The other parts can be proved in similarly way. □

Corollary 2.1 Let F be a nonempty subset of an ordered ternary semigroup T. Then F is a left filter (right filter, lateral filter, filter) of T if and only if the complement f'_F of f_F is a prime fuzzy left ideal (fuzzy right filter, fuzzy lateral filter, fuzzy filter) of T.

Proof It follows by Theorem 2.2 and Theorem 2.8. □

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The M/M/1 Queue with Single Working Vacation Serving at a Slower Rate during the Start-up Period

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Abstract

We consider the M/M/1 queue with single working vacation serving at a slower rate during the start-up period. In order to save energy and reduce waste, the server works at a slower rate rather than completely stops during a vacation and start-up period. Using quasi birth and death process and matrix-geometric solution method, we obtain the distribution of the nember of customers in the system, the average nember of the customers and the average sojourn time of a customer in the stationary state.

Keywords: Start-up period, State transition rate matrix, Quasi birth and death process, Matrix-geometric solution, Single working vacation

1. Introduction

Over the past two decades, the vacation queues have been investigated extensively. In a classical vacation queue, a server may completely stop service or do some additional work during a vacation. The vacation queues have been extended to computer networks, communications systems, as well as production management, inventory management and other fields(Doshi B T, 1990).

Recently, a class of semi-vacation policies has been introduced by Servi and Finn. Such a vacation is called a working vacation (WV). The server works at a slower rate rather than completely stops during a vacation. Servi and Finn(Servi, L. D., 2002) studied an M/M/1 queue with multiple working vacations, and obtained the of the number of customers in the system and the of waiting time distribution. Later, Liu, Xu and Tian(2002) gave simple explicit expressions of distribution for the stationary queue length and waiting time which have intuitionistic probability sense. Kin, Choi and Chae, Wu and Takagi(2006) generalized the work of to an M/G/1 queue with multiple working vacations(Wu, D., 2006), Baba investigated a GI/M/1 queue multiple working vacations. Recently, Tian ,Zhao and Wang(2008) study an M/M/1 queue with single working vacation. According to Tian, Zhao and Wang's research£this paper introduced a start-up period. In addition, there is a slow rate of service during the start-up period.

In this paper, we study an M/M/1 queue with single working vacation serving at a slower rate in during the start-up period. Firstly, the system is in a closed state, when a customer arrives, leading to a start-up period. After the start-up period, the system becomes a normal service state. Until there are no customers in the queue, it changes into the working vacation state. When the working vacation ends, if there are customers in the queue, the system becomes a normal service state: if there are no customers in the queue, the system is closed. Until a customer arrives, a new cycle begins. In this model, when the number of customers in the system is relatively few, we set a lower service rate. If there are no customers in the system, we close it in order to save energy and reduce waste. Using quasi birth and death process and matrix-geometric solution method, we obtain the distribution of the member of customers in the system, the average member of the customers and the average sojourn time of a customer in the stationary state.

The rest of this paper is organized as follows. In Section 2 we describe the quasi birth and death process model of the system; In Section 3 we obtain the steady- state queue distribution; In Section 4 we obtain the average member of the customers and the average sojourn time of a customer in the stationary state.

2. Quasi Birth and Death Process

2.1 Model Description

Firstly, the system is in a closed state, when a customer arrives, leading to a start-up period. The start-up period U follows an exponential distribution with parameter β and the server serves at a slower rate of μ_β. After the start-up period, the system becomes a normal service state and the server serves at a normal rate of μ_b. Until there are no customers in the queue, it changes into the working vacation state. The working vacation time V follows an exponential distribution with parameter θ and the server serves at a slower rate of μ_v. When the working vacation ends, if there are customers in the queue, the server changes service rate from μ_v to μ_b and the system becomes a normal service state; if there are no customers in the queue, the system is closed. Until a customer arrives, a new cycle begins.

We assume that interarrival times, start-up period, service times, and working vacation time are mutually independent. In addition, the service order is first in first out ($FIFO$).

2.2 State Transition Rate Matrix

Let $Q(t)$ be the number of customers in the system at time t and let State variables

$$J(t) = \begin{cases} 0 & the\ system\ is\ in\ a\ working\ vacation\ period\ at\ time\ t \\ 1 & the\ system\ is\ in\ a\ start-up\ period\ at\ time\ t \\ 2 & the\ system\ is\ in\ a\ regular\ busy\ period\ at\ time\ t \end{cases}$$

Then $\{Q(t),\ J(t)\}$ is a process with the state space

$$\Omega = \{(k,\ j),\ k \geq 1,\ j = 0,\ 1,\ 2\} \cup (0,\ 0) \cup (0,\ 1)$$

Where state $(0,\ 1)$ denotes that the system is in a close-up state; state $(k,\ 0),\ k \geq 0$ indicates that the system is in working vacation state and there are k customers in the queue; state $(k,\ 1),\ k \geq 1$ indicates that the system is in start-up state and there are k customers in the queue; state $(k,\ 2),\ k \geq 1$ indicates that the system is in regular busy period state and there are k customers in the queue.

According to the lexicographical sequence, the state transition rate matrix can be written as

$$\tilde{Q} = \begin{bmatrix} A_{00} & A_{01} & & & & \\ B_{10} & A & C & & & \\ & B & A & C & & \\ & & B & A & C & \\ & & & \ddots & \ddots & \ddots \end{bmatrix}$$

where

$$A_{00} = \begin{bmatrix} -(\lambda+\theta) & \theta \\ 0 & -\lambda \end{bmatrix} \qquad A_{01} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} -(\mu_v+\theta+\lambda) & 0 & \theta \\ 0 & -(\mu_\beta+\beta+\lambda) & \beta \\ 0 & 0 & -(\lambda+\mu_b) \end{bmatrix}$$

$$B_{10} = \begin{bmatrix} \mu_n v & 0 \\ 0 & \mu_\beta \\ \mu_b & 0 \end{bmatrix} \qquad B = \begin{bmatrix} \mu_v & 0 & 0 \\ 0 & \mu_\beta & 0 \\ 0 & 0 & \mu_b \end{bmatrix} \qquad C = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

The structure of $\tilde{Q}$ indicates that $\{Q(t),\ J(t)\}$ is a quasi birth and death process (QBD), see Neuts. (1981).

Theorem 1. If $\rho = \frac{\lambda}{\mu_b} < 1$, the matrix equation

$$R^2B + RA + C = 0 \tag{1}$$

has the minimal non-negative solution

$$R = \begin{bmatrix} r & 0\frac{\theta r}{\lambda+(1-r-\rho)\mu_b} & \\ 0 & \varepsilon & \frac{\beta\varepsilon}{\lambda+(1-\varepsilon-\rho)\mu_b} \\ 0 & 0 & \rho \end{bmatrix} \tag{2}$$

Where

$$\rho = \frac{\lambda}{\mu_b} \qquad r = \frac{\lambda + \theta + \mu_v - \sqrt{(\lambda + \theta + \mu_v)^2 - 4\lambda\mu_v}}{2\mu_v}$$

$$\varepsilon = \frac{\lambda + \beta + \mu_\beta - \sqrt{(\lambda + \beta + \mu_\beta)^2 - 4\lambda\mu_\beta}}{2\mu_\beta}$$

Proof. Because $A,\ B,\ C$ are all upper-triangular, we can assume that R has the same structure as

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{bmatrix}$$

Substituting R^2 and R into equation (1), we get

$$r_{11} = \frac{\lambda + \theta + \mu_v - \sqrt{(\lambda + \theta + \mu_v)^2 - 4\lambda\mu_v}}{2\mu_v} \qquad r_{22} = \frac{\lambda + \beta + \mu_\beta - \sqrt{(\lambda + \beta + \mu_\beta)^2 - 4\lambda\mu_\beta}}{2\mu_\beta}$$

$$r_{12} = 0 \qquad r_{23} = \frac{r_{22}\beta}{\lambda + (1 - r_{22} - r_{33})}$$

$$r_{13} = \frac{r_{11}\theta}{\lambda + (1 - r_{11} - r_{33})} \qquad r_{33} = \frac{\lambda}{\mu_b}$$

To obtain the minimal non-negative solution of (1), taking $r_{11} = r$ (the other root is greater than 1), taking $r_{22} = \varepsilon$ (the other root is greater than 1), taking $r_{33} = \rho$ (the other root is $r_{33} = 1$). Using elementary method, we can prove that $0 < r < 1,\ 0 < \varepsilon < 1$. Substituting $r,\ \varepsilon,\ \rho$ into equation, we get $r_{12},\ r_{13},\ r_{23}$.

Because r satisfies the following equation

$$\mu_v r^2 - (\lambda + \theta + \mu_v) + \lambda = 0$$

equivalently, we have

$$\frac{\theta}{1-r} + \mu_v = \frac{\lambda}{r}$$

Similarly, we can have

$$\frac{\beta}{1-\varepsilon} + \mu_\beta = \frac{\lambda}{\varepsilon}$$

Theorem 2. The QBD process $\{Q(t),\ J(t)\}$ is positive recurrent if and only if $\rho < 1$. Proof. Based on the theorem of Neuts, the QBD process $\{Q(t),\ J(t)\}$ is positive recurrent if and only if the spectral radius $SP(R)$ of the rate matrix R is less than 1, and set of equations $(x_0,\ x_1,\ x_2,\ x_3,\ x_4)B[R] = 0$ has positive solution, where

$$B[R] = \begin{bmatrix} A_{00} & A_{01} \\ B_{10} & RB + A \end{bmatrix} = \begin{bmatrix} -(\lambda+\theta) & \theta & \lambda & 0 & 0 \\ 0 & \lambda & 0 & \lambda & 0 \\ \mu_v & 0 & -\frac{\lambda}{r} & 0 & \frac{\lambda}{r} - \mu_v \\ 0 & \mu_\beta & 0 & -\frac{\lambda}{\varepsilon} & \frac{\lambda}{\varepsilon} - \mu_\beta \\ \mu_b & 0 & 0 & 0 & -\mu_b \end{bmatrix} \tag{3}$$

$B[R]$ is an irreducible and a periodic generator with finite state. Therefore, $(x_0,\ x_1,\ x_2,\ x_3,\ x_4)B[R] = 0$ has positive solution. Thus, process $\{Q(t),\ J(t)\}$ is positive recurrent if and only if

$$SP(R) = \max(r,\ \varepsilon,\ \rho) < 1$$

where $0 < r < 1,\ 0 < \varepsilon < 1$, the above relation means that $\rho < 1$.

3. Steady- state Queue Length Distribution

If $\rho < 1,\ \mu_v < \mu_b,\ \mu_\beta < \mu_b$, let $(Q,\ J)$ be the stationary limit of the QBD process. Let

$$\pi_{kj} = P\{Q = k,\ J = j\},\ (k,\ j) \in \Omega$$

$$(\pi_{k0},\ \pi_{k1},\ \pi_{k2}) = \pi_k,\ k \geq 1$$

Theorem 3. If $\rho < 1,\ \mu_v < \mu_b,\ \mu_\beta < \mu_b$, the joint probability distribution of $(Q,\ J)$ is

$$\pi_{k0} = r^k \pi_{00} \qquad \pi_{k1} = \frac{\theta(1-\varepsilon)}{\beta}\varepsilon^{k-1}\pi_{00} \tag{4}$$

$$\pi_{k2} = [(\delta+\varphi)\sum_{j=0}^{k-1} r^j \rho^{k-1-j} + (\gamma-\delta-\varphi)\rho^{k-1}]\pi_{00}$$

where

$$\delta = \frac{\theta r}{\lambda + (1-r-\rho)\mu_b}, \quad \gamma = \frac{\lambda+\theta-r\mu_v}{\mu_b}, \quad \varphi = \frac{\theta(1-\varepsilon)}{\lambda+(1-\varepsilon-\rho)\mu_b}$$

$$\pi_{00} = \frac{(1-r)(1-\rho)(1-\varepsilon)}{(1+\frac{\theta}{\beta})(1-r)(1-\rho)(1-\varepsilon) + \frac{\theta}{\varepsilon}(1-r)(1-\rho) + r(1-\varepsilon)(1-\rho) + (\delta+\phi)(1-\varepsilon)} + (\gamma-\delta-\varphi)(1-\varepsilon)(1-r)$$

Proof. With the matrix-geometric solution method, we have

$$\pi_k = (\pi_{k0},\ \pi_{k1},\ \pi_{k2}) = (\pi_{10},\ \pi_{11},\ \pi_{12})R^{k-1} = 0 \tag{5}$$

and $\pi_0,\ \pi_1$ satisfy the set of equations

$$[\pi_{00}\ \ \pi_{01}\ \ \pi_{10}\ \ \pi_{11}\ \ \pi_{12}]B[R] = 0$$

Substituting $B[R]$ in (3) into the equation, we get

$$\pi_{01} = \frac{\theta(1-\varepsilon)}{\varepsilon\beta}\pi_{00} \qquad \pi_{10} = r\pi_{00}$$

$$\pi_{11} = frac\theta(1-\varepsilon)\beta\pi_{00} \qquad \pi_{12} = \frac{\lambda+\theta-r\mu_v}{\mu_b}\pi_{00}$$

note that

$$R^k = \begin{bmatrix} r^k & 0 & \frac{\theta r}{\lambda+(1-r-\rho)\mu_b}\sum_{j=0}^{k-1} r^j\rho^{k-1-j} \\ 0 & \varepsilon^k & \frac{\beta\varepsilon}{\lambda+(1-\varepsilon-\rho)\mu_b}\sum_{j=0}^{k-1} r^j\rho^{k-1-j} \\ 0 & 0 & \rho^k \end{bmatrix}, \quad k \geq 1$$

Substituting $(\pi_{10},\ \pi_{11},\ \pi_{12})$ and R^{k-1} into (5), we obtain (4). Finally, π_{00} can be determined by the normalization condition.

With (4), the probabilities of the server in various state are as follows, respectively

$$P\{the\ server\ is\ in\ close-up\ period\} = \pi_{01} = \frac{\theta(1-\varepsilon)}{\varepsilon\beta}\pi_{00}$$

$$P\{the\ server\ is\ in\ start-up\ period\} = P\{J=1\} = \sum_{k=1}^{\infty}\pi_{k1} = \frac{\theta}{\beta}\pi_{00}$$

$$P\{the\ server\ is\ in\ working\ vacation\ period\} = P\{J=0\} = \sum_{k=1}^{\infty}\pi_{k0} = \frac{1}{1-r}\pi_{00}$$

$$P\{the\ server\ is\ in\ regular\ busy\ period\} = P\{J=2\} = \sum_{k=1}^{\infty}\pi_{k2} = [(\delta+\varphi)\frac{1}{1-r}\frac{1}{1-\rho} + \frac{\gamma-\delta-\varphi}{1-\rho}]\pi_{00}$$

4. The Average of the Queue Length and the Sojourn Time in Steady State

Theorem 4. If $\rho < 1,\ \mu_v < \mu_b,\ \mu_\beta < \mu_b$, the average of the queue length in steady state

$$E(L) = [\frac{1}{(1-r)^2} + \frac{\theta}{1-\varepsilon} + \frac{(\delta+\varphi)(1-r\rho)}{(1-\rho)^2(1-r)^2} + \frac{\gamma-\delta-\varphi}{(1-\rho)^2}]\pi_{00} \tag{6}$$

the average of the sojourn time in steady state

$$E(W) = \frac{1}{\lambda}[\frac{1}{(1-r)^2} + \frac{\theta}{1-\varepsilon} + \frac{(\delta+\varphi)(1-r\rho)}{(1-\rho)^2(1-r)^2} + \frac{\gamma-\delta-\varphi}{(1-\rho)^2}]\pi_{00} \tag{7}$$

Proof. With (4), the probability generating function of Q cab be written as

$$Q(z) = \sum_{k=0}^{\infty}(\pi_{k0} + \pi_{k1} + \pi_{k2})z^k = [1 + \frac{\theta(1-\varepsilon)}{\varepsilon\beta} + \frac{rz}{1-rz} + \theta(1-\varepsilon)\frac{z}{1-\varepsilon z} + (\delta+\varphi)\frac{z}{1-\rho z}\frac{1}{1-rz} + (\gamma-\delta-\varepsilon)\frac{z}{1-\rho z}]\pi_{00}$$

Therefore

$$E(L) = Q'(z)|z = 1 = [\frac{1}{(1-r)^2} + \frac{\theta}{1-\varepsilon} + \frac{(\delta+\varphi)(1-r\rho)}{(1-\rho)^2(1-r)^2} + \frac{\gamma-\delta-\varphi}{(1-\rho)^2}]\pi_{00}$$

If the PGF of W is $W(s)$, the relationship between the PGF of Q and W is

$$Q(z) = W(\bar{\lambda} + \lambda z)$$

therefore

$$E(W) = W'(s)|s = 1 = \frac{1}{\lambda}Q'\left(\frac{s-\bar{\lambda}}{\lambda}\right)|s = 1$$

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Modeling and Analysis of an Epidemic Model with Non-monotonic Incidence Rate under Treatment

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Abstract

An epidemic model with non-monotonic incidence rate under a limited resource for treatment is proposed to understand the effect of the capacity for treatment. We have assume that treatment rate is proportional to the number of infective when it is below the capacity and is a constant when the number of infective is larger than the capacity. Existence and stability of the disease free and endemic equilibrium are investigated for both the cases. Some numerical simulations are given to illustrate the analytical results.

Keywords: Epidemic, Endemic equilibrium, Treatment, Basic reproductive number, Limit cycle

1. Introduction

The incidence in an epidemiological model is the rate at which susceptible become infectious. The form of the incidence rate that is used in the classical Kermack Mckendrick model (1927) is the simple mass action λSI where S and I denote the number of susceptible and infectious, respectively, λ is called the infection coefficient. The standard incidence is $\lambda SI|N$, where N is the total population size and λ is called the daily contact rate. Another kind of incidence is the saturation incidence $\lambda SI/(c + S)$ where c is a constant. When the number of susceptible S is large compared to c that incidence is approximately λI. This kind of incidence was proposed by Anderson and May (1979), Lourdes and Matias (1991). Many researchers (see Hethcote and Levin, 1988; Esteva and Matias, 2001; Liu *et al*., 1986; Liu *et al*., 1987) have proposed transmission laws in which the nonlinearities are more than quadratic. Ruan and Wang (2003) studied an epidemic model with a specific nonlinear incident rate $\lambda I^2S/(1 + \alpha I^2)$ and presented a detailed qualitative and bifurcation analysis of the model. They derived sufficient conditions to ensure that the system has none, one, or two limit cycles and showed that the system undergoes a Bogdanov-Takens bifurcation at the degenerate equilibrium which include a saddle-node bifurcation, a Hopf bifurcation, and homoclinic bifurcation.

A more general incidence rate $\lambda I^pS/(1 + \alpha I^q)$ were proposed by many researchers and authors (see, Liu *et al*., 1986; Derrick and Van den Driessche,1993; Hethcote andVen den Driessche, 1991, Alexander and Moghadas, 2004). Xiao and Ruan, 2007 proposed an epidemic model with non-monotonic incidence rate $\lambda IS/(1+\alpha I^2)$. Treatment plays an important role to control or decrease the spread of diseases such as flue, tuberculosis, and measles (see Feng and Thieme, 1995; Wu and Feng, 2000; Hyman and Li, 1998). In classical epidemic models, the treatment rate is assumed to be proportional to the number of the infectious, which is almost impossible in real perspective because in that case the resources for treatment should be quite large. In fact, every country or society should have a suitable capacity for treatment. If it is too large, the country or society pays for unnecessary cost. If it is too small, the country or society has the risk of the outbreak of a disease. Wang (2006) proposed a treatment function:

$$\begin{aligned} T(I) &= rI, \text{ if } 0 \le I \le I_0, \\ &= K_1, \text{ if } I > I_0, \end{aligned}$$

where $K_1 = rI_0$. This type of treatment function is more realistic because in every hospital, the number of beds is limited and also they have a certain capacity of medicines. In our proposed model we have considered an epidemic model with non monotonic incidence rate under the treatment.
Thus our model becomes

$$\frac{dS}{dt} = a - dS - \frac{\lambda IS}{I + \alpha I^2} + \beta R, \tag{1}$$

$$\frac{dI}{dt} = \frac{\lambda IS}{I + \alpha I^2} - (d + m)I - T(I), \tag{2}$$

$$\frac{dR}{dt} = mI - (d + \beta)R + T(I), \tag{3}$$

where S(t), I(t), R(t) denote the number of susceptible, infective, recovered individuals, respectively; a is the recruitment rate of the population, d is the natural death rate of the population, λ is the proportionality constant, m is the natural recovery rate of the infective individuals, β is the rate at which recovered individuals lose immunity and return to susceptible class, α is the parameter measures of the psychological or inhibitory effect. In our work we take the treatment function T (I), defined by

$$T(I) = rI, \text{if } 0 \le I \le I_0 \tag{4}$$

$$= K_1, \text{if } I > I_0, \tag{5}$$

This means that the treatment rate is proportional to the infective when the number of infective is less or equal to some fixed value I_0 and the treatment is constant when the number of infective crosses the fixed value I_0. In practical view, the above form of treatment function is justified where patients have to be hospitalized and the number of beds is limited or the medicines are not sufficient.

Part I: SIR model with $0 \le I \le I_0$.

2. Equilibrium states and their stability

In this case the system (1)-(3) reduces to

$$\frac{dS}{dt} = a - dS - \frac{\lambda IS}{I + \alpha I^2} + \beta R, \tag{6}$$

$$\frac{dI}{dt} = \frac{\lambda IS}{I + \alpha I^2} - (d + m + r)I, \tag{7}$$

$$\frac{dR}{dt} = (m + r)I - (d + \beta)R. \tag{8}$$

The system of equations (6)-(8) always has the disease free equilibrium $E_0(a/d, 0, 0)$ for any set of parameter values. The endemic equilibrium is the solution of

$a - dS - \frac{\lambda IS}{I+\alpha I^2} + \beta R = 0,$
$\frac{\lambda IS}{I+\alpha I^2} - (d + m + r)I = 0,$
$(m + r)I - (d + \beta)R = 0.$

From the third equations we get $R = \{(m + r)/(d + \beta)\}I$ and from the second equation $S = (d + m + r)(1 + \alpha I^2)/\lambda$. Now substituting R and S in the first equation, we get

$$\alpha d(d + m + r)I^2 + \lambda\{d + m + r - \beta(m + r)/(d + \beta)\}I + d(d + m + r) - \lambda a = 0. \tag{9}$$

We define the basic reproductive number as follows

$$R_0 = \frac{\lambda a}{d(d + m + r)}. \tag{10}$$

From the equation (9) we see that if $R_0 \le 1$, there is no positive solution as in that case coefficient of I^2, I and constant term are all positive, but if $R_0 > 1$, then by Descartes rule there exists a unique positive solution of (9) and consequently there exists unique positive equilibrium $E^*(S^*, I^*, R^*)$, called endemic equilibrium.

Here, $R^* = \{(m+r)/(d+\beta)\}I^*, S^* = (d+m+r)(1+\alpha I^{*2})/\lambda$ and

$$I^* = [-K\{d+m+r-\beta(m+r)/(d+\beta)\} + \sqrt{\Delta_1}]/\{2\alpha d(d+m+r)\}, \tag{11}$$

where $\Delta_1 = \lambda^2\{d+m+r-\beta(m+r)/(d+\beta)\}^2 - 4\alpha d^2(d+m+r)^2[1-R_0]$.

Obviously $\Delta_1 > 0$, when $R_0 > 1$.

To investigate the stability of the system, we first prove that $S(t)+I(t)+R(t) = a/d$ is invariant manifold of the system (6) - (8), which is attracting the first octant.
Let $N(t) = S(t)+I(t)+R(t)$, then
$\dfrac{dN}{dt} = a - dN(t)$, this imply $N(t) = A_1\, e^{-dt} + a/d$,
where $N(t_0) = A_1\, e^{-dt_0} + a/d$, therefore $N(t) = (N(t_0) - a/d)\; e^{-d(t-t_0)} + a/d$.
Thus $N(t) \to a/d$, as $t \to \infty$. So the limit set of system (6) - (8) is on the plane $S+I+R = a/d$.
Thus the reduced system is

$$\frac{dI}{dt} = \frac{\lambda I(a/d - I - R)}{1+\alpha I^2} - (d+m+r)\,I = F_1\,(I,R), \tag{12}$$

$$\frac{dR}{dt} = (m+r)\,I - (d+\beta)\,R = F_2\,(I,R). \tag{13}$$

Now to test the local stability of the above system we rescale the system by
$x = \dfrac{\lambda I}{d+\beta},\ y = \dfrac{\lambda R}{d+\beta},\ T = (d+\beta)t$
and obtain

$$\frac{dx}{dT} = \frac{x(K-x-y)}{1+vx^2} - ux, \tag{14}$$

$$\frac{dy}{dT} = wx - y, \tag{15}$$

where $K = \dfrac{a\lambda}{d(d+\beta)},\ u = \dfrac{d+m+r}{d+\beta},\ v = \dfrac{\alpha(d+\beta)^2}{\lambda^2},\ w = \dfrac{m+r}{d+\beta}$.

Here $E_0(0,0)$ is the disease free equilibrium and the unique positive equilibrium (x^*, y^*) of the system (14)-(15) is the endemic equilibrium E^* of the model (6)-(8). (x^*, y^*) exists if $u-K < 0$ and is given by $uvx^{*2} + (1+w)x^* + (u-K) = 0$; $y^* = wx^*$.

Therefore,

$$x^* = \frac{-(1+w) + \sqrt{(1+w)^2 - 4uv(u-K)}}{2uv},\ y^* = wx^*. \tag{16}$$

The jacobian matrix corresponding to $E_0(0,0)$ is

$$M_0 = \begin{bmatrix} K-u & 0 \\ w & -1 \end{bmatrix}.$$

Obviously (i) if $(K-u) > 0, (0,0)$ is an unstable saddle point;

(ii) if $K = u$, $(0,0)$ is saddle node;

(iii) if $(K-u) < 0$, $(0,0)$ is a stable node.

Here $(K-u) > 0 \Leftrightarrow R_0 > 1$ and $(K-u) < 0 \Leftrightarrow R_0 < 1$.

So, whenever E^* exists, E_0 turns to an unstable saddle point.

Now when $(K-u) > 0$ i.e. $R_0 > 1$, we discuss the stability of endemic equilibrium (x^*, y^*).

Jacobian matrix corresponding to (x^*, y^*) is

$$M_1 = \begin{bmatrix} x^*\,(vx^{*2} + 2vwx^{*2} - 2Kvx^* - 1)/(1+vx^{*2})^2 & -x^*/(1+vx^{*2}) \\ w & -1 \end{bmatrix}.$$

The sign of det $(M_1) = x^*\{1+w+2Kvx^* - v(1+w)x^{*2}\}/(1+vx^{*2})^2$ is determined by the sign of

$$P_1 = -v(1+w)x^{*2} + 2Kvx^* + (1+w). \tag{17}$$

$$\text{We have } uvx^{*2} + (1+w)x^* + (u-K) = 0. \tag{18}$$

Now $u \times (18) + (1+w) \times (17) \Rightarrow uP_1 = \{2Kuv + (1+w)^2\}\left\{x^* + \dfrac{(1+w)(2u-K)}{2Kuv + (1+w)^2}\right\}$.

Now substituting $x^* = \{-(1+w) + \Delta_1\}/2uv$, where $\Delta_1 = \sqrt{(1+w)^2 - 4uv(u-K)}$, we get
$uP_1 = \dfrac{1}{2uv}[-(1+w)\Delta_1^2 + \Delta_1\{2Kuv + (1+w)^2\}] = \dfrac{-\Delta_1}{2uv}[(1+w)\Delta_1 - \{2Kuv + (1+w)^2\}]$.

$\therefore P_1 = \dfrac{-\Delta_1}{2u^2v}[(1+w)\Delta_1 - \{2Kuv + (1+w)^2\}] = \dfrac{(1+w)\Delta_1}{2u^2v}\left[\left(1 + w + \dfrac{2Kuv}{1+w}\right) - \Delta_1\right]$.

Since
$\left(1 + w + \dfrac{2Kuv}{1+w}\right)^2 - \Delta_1^2 = \dfrac{4K^2u^2v^2}{(1+w)^2} + 4u^2v > 0,$

therefore $P_1 > 0$, and hence det (M_1) is positive for any set of parameters.

Therefore, the positive equilibrium (x^*, y^*) is either a node, a focus or a center. The eigen values of M_1 are $\lambda = \dfrac{1}{2}\left(Trace(M_1) \pm \sqrt{(Trace(M_1))^2 - 4\det(M_1)}\right)$.
The fact that det $(M_1) > 0$ implies that
$|Trace(M_1)| > \sqrt{(Trace(M_1))^2 - 4\det(M_1)}$.

The stability of the (x^*, y^*) depends on the sign of the Trace and determinant of the jacobian matrix:

(x^*, y^*) is stable if Trace $(M_1) < 0$, unstable if Trace $(M_1) > 0$. (x^*, y^*) is a node if $(Trace(M_1))^2 > 4\det(M_1)$ and a focus if $(Trace(M_1))^2 < 4\det(M_1)$.

Now Trace $(M_1) = \{-v^2x^{*4} + (1+2w)vx^{*3} - 2(1+K)vx^{*2} - x^* - 1\}/(1+vx^{*2})^2$.

So the sign of Trace (M_1) is determined by

$$P_2 = -v^2x^{*4} + (1+2w)vx^{*3} - 2(1+K)vx^{*2} - x^* - 1. \tag{19}$$

After some algebraic calculation using (18) and (19) we get, $u^3vP_2 = P_3x^* - P_4$, where

$P_3 = (1+w)[(1+w)^2 + u(1+w)(1+2w) + 2uvK] + u^2vK(1+2w) + 2u^2v(1+w)(K-u)$,

$P_4 = (K-u)[(1+w)^2 + u(1+w)(1+2w) + uv[(K-u)^2 + 2u(K-u)(1+K) + u^2]$.

Therefore P_3 and P_4 are positive for any set of parameters with $K > u$. So when (x^*, y^*) exists, the condition for the local stability of (x^*, y^*) becomes $x^* < P_4/P_3$.

The above discussion can be stated through a theorem.

Theorem 2.1. (i) When the basic reproductive number $R_0 \leq 1$, there exist no positive equilibrium of the system (14) - (15), and in that case the only disease free equilibrium $(0, 0)$ is a stable node.

(ii) When $R_0 > 1$, there exists a unique positive equilibrium of the system (14) - (15), and in that case $(0, 0)$ is an unstable saddle point. Also the condition for which the unique positive equilibrium will be locally stable is $x^* < P_4/P_3$.

Global Stability. To investigate the global stability of the disease free equilibrium it is sufficient to show that $(I(t), R(t)) \rightarrow (0, 0)$. From here, it is clear that $S(t) \rightarrow a/d$. Now from positivity of the solutions, $I(t)$ and $R(t)$ satisfy the differential inequality given by

$$\frac{dI}{dt} \leq \{\frac{\lambda a}{d} - (d+m+r)\}I = \frac{di}{dt}, \tag{20}$$

$$\frac{dR}{dt} \leq (m+r)I - (d-\beta)R = \frac{dr}{dt}. \tag{21}$$

Here $i(t), r(t)$ are linear, and $(i(t), r(t)) \rightarrow (0, 0)$ as $t \rightarrow \infty$ if $\lambda\dfrac{a}{d} - (d+m+r) < 0$ *i.e.* $R_0 < 1$.

Since $I(t) \leq i(t)$ and $R(t) \leq r(t)$, $(I(t), R(t)) \rightarrow (0, 0)$ as $t \rightarrow \infty$ by simple comparison argument. Hence disease free equilibrium is globally stable.

Now to investigate whether system (12) – (13) admits limit cycle or not, we take Dulac function $D(I, R) = (1 + \alpha I^2)/\lambda I$, then

$\partial(DF_1)/\partial I + \partial(DF_2)/\partial R = -1 - \{2\alpha(d+m+r)/\lambda\}I - \{(d+\beta)(1+\alpha I^2)\}/\lambda I < 0,$

hence the system (12)-(13) has no limit cycle in the positive quadrant, so we reach the theorem 2.2.

Theorem 2.2. If $R_0 < 1$, then the disease free equilibrium $E_0(a/d, 0, 0)$ of the system (12) - (13) is globally stable. But when $R_0 > 1$, system (12)-(13) have unique positive equilibrium and further when $x^* < P_4/P_3$ that unique positive equilibrium must be locally stable. Again since the system have no limit cycle in the positive quadrant, $E^*(x^*, y^*)$ must be globally stable under the condition $R_0 > 1$ and $x^* < P_4/P_3$.

Part II. SIR model with $I > I_0$.

3. Equilibrium states and their stability

In this case the model reduces to

$$\frac{dS}{dt} = a - dS - \frac{\lambda IS}{1+\alpha I^2} + \beta R, \tag{22}$$

$$\frac{dI}{dt} = \frac{\lambda IS}{1+\alpha I^2} - (d+m)I - K_1, \tag{23}$$

$$\frac{dR}{dt} = mI - (d+\beta)R + K_1. \tag{24}$$

Since $S + I + R = a/d$ is invariant manifold of the system (22)-(24), the model reduces to

$$\frac{dI}{dt} = \frac{\lambda I(a/d - I - R)}{1+\alpha I^2} - (d+m)I - K_1, \tag{25}$$

$$\frac{dR}{dt} = mI - (d+\beta)R + K_1. \tag{26}$$

Substituting $x = \dfrac{\lambda I}{d+\beta},\ y = \dfrac{\lambda R}{d+\beta},\ T = (d+\beta)t$
we get

$$\frac{dx}{dT} = \frac{x(L-x-y)}{(1+v_1x^2)}(L-x-y) - u_1x - c, \tag{27}$$

$$\frac{dy}{dT} = w_1x - y + c, \tag{28}$$

where $v_1 = v = \alpha(d+\beta)^2/\lambda^2$, $L = K = a\lambda/\{d(d+\beta)\}$, $u_1 = (d+m)/(d+\beta)$,
$c = \lambda K_1/(d+\beta)^2$, $w_1 = m/(d+\beta)$.
For equilibrium $x(L-x-y) - u_1x(1+v_1x^2) - c(1+v_1x^2) = 0$,

or,

$$u_1vx^3 + (1+w_1+cv)x^2 + (c+u_1-K)x + c = 0. \tag{29}$$

If $u_1 + c > K$, (29) has no positive solution, but if $u_1 + c < K$, it has either two positive roots or no positive root. By theory of equation

$$a_0x^3 + 3a_1x^2 + 3a_2x + a_3 = 0 \tag{30}$$

has all of its roots real if $G^2 + 4H^3 < 0$ and $H < 0$, where $H = a_0a_2 - a_1^2$,
$G = a_0^2a_3 - 3a_0a_1a_2 + 2a_1^3$. Comparing equation (29) to (30) we have $a_0 = u_1v$,
$a_1 = (1+w_1+cv)/3$, $a_2 = (u_1+c-K)/3$, and $a_3 = c$.
Here $H = a_0a_2 - a_1^2 = u_1v\{(u_1+c-K)/3\} - \{(1+w_1+cv)/3\}^2 < 0$, for $u_1 + c < K$.
$G^2 + 4H^3 = (a_0^2a_3 - 3a_0a_1a_2 + 2a_1^3)^2 + 4(a_0a_2 - a_1^2)^3$
$= a_0^2(a_0^2a_3^2 - 6a_0a_1a_2a_3 + 4a_3a_1^3 + 4a_0a_2^3 - 3a_1^2a_2^2)$.

Therefore

$$G^2 + 4H^3 < 0, if\ (a_0^2a_3^2 + 4a_3a_1^3 + 4a_0a_2^3) < (3a_1^2a_2^2 + 6a_0a_1a_2a_3) \tag{31}$$

To investigate the local stability of the positive equilibrium $(\bar{x}, \bar{y})$ of the system (27)-(28), we consider the jacobian matrix

$$M_2(\bar{x}, \bar{y}) = \begin{bmatrix} \dfrac{\{(1+v\bar{x}^2)(K-\bar{x}-\bar{y}-\bar{x}) - 2v\bar{x}(k\bar{x}-\bar{x}^2-\bar{x}\bar{y})}{1+v\bar{x}^2} - u_1 & \dfrac{-\bar{x}}{1+v\bar{x}^2} \\ w_1 & -1 \end{bmatrix}.$$

Now
$\det(M_2)$

$= \dfrac{-(v\bar{x}^2+1)(K-2\bar{x}-w_1\bar{x}-c)+2v\bar{x}(K\bar{x}-\bar{x}^2-w_1\bar{x}^2-c\bar{x})+u_1(v\bar{x}^2+1)^2+w_1\bar{x}(v\bar{x}^2+1)}{(1+v\bar{x}^2)^2}$

Sign of det(M_2) is determined by

$$P_5 = u_1v^2\bar{x}^4 + (Kv - vc + 2u_1v)\bar{x}^2 + (2+2w_1)\bar{x} + (c+u_1-K). \tag{32}$$

Now (32) $- v\bar{x}\times$ (29) $\Rightarrow$

$$P_5 = -v(1+w_1+cv)\bar{x}^3 + v(2K+u_1-2c)\bar{x}^2 + (2+2w_1-vc)\bar{x} + (c+u_1-K). \tag{33}$$

Again $(1+w_1+cv)\times$ (29) $+ u_1\times$ (33) $\Rightarrow$
$u_1P_5 = \bar{x}^2\{2Ku_1v + u_1^2v - 2u_1vc + (1+w_1+cv)^2\} + \bar{x}\{(2u_1+2u_1w_1-u_1vc)+$
$(1+w_1+cv)(u_1+c-K)\} + \{(cu_1+u_1^2-Ku_1)+c(1+w_1+cv)\}$.
$\therefore\ u_1P_5 = \xi_1\bar{x}^2 + \xi_2\bar{x} + \xi_3$, where $\xi_1 = \{u_1^2v + (1+w_1+cv)^2 + 2u_1v(K-c)\} > 0$ for $K > c$.
So, the sufficient condition for which $P_5 > 0$ is

$$\xi_2^2 - 4\xi_1\xi_2 \le 0. \tag{34}$$

Now Trace $(M_2) = \dfrac{\{(1+v\bar{x}^2)(K-2\bar{x}-\bar{y})-2v\bar{x}^2(K-\bar{x}-\bar{y})\}}{(1+v\bar{x}^2)^2} - (u_1+1)$
$= \dfrac{\{(1+v\bar{x}^2)(K-2\bar{x}-w_1\bar{x}-c)-2v\bar{x}^2(K-\bar{x}-w_1\bar{x}-c)-(u_1+1)(1+v\bar{x}^2)^2\}}{(1+v\bar{x}^2)^2}.$
So the sign of Trace (M_2) is determined by

$$P_6 = -(u_1+1)v^2\bar{x}^4 + vw_1\bar{x}^3 + (vc - Kv - 2vu_1 - 2v)\bar{x}^2 - (2+w_1)\bar{x} + (K-c-u_1-1). \tag{35}$$

After some algebraic calculation using (29) and (35) we get
$u_1^2P_6 = \eta_1\bar{x}^2 + \eta_2\bar{x} + \eta_3$,
where
$\eta_1 = -[(1+w_1+cv)(2u_1w_1+u_1cv+u_1+w_1+cv+1) + u_1v\{(K-c)(1+2u_1)+u_1^2+u_1\}] < 0$,
for $K > u_1 + c$,
$\eta_2 = u_1(cvu_1 + cv - 2u_1 - w_1u_1) - (u_1+c-K)(2u_1w_1+u_1cv+u_1+w_1+cv+1)$,
$\eta_3 = u_1(K-c-u_1-1) - c(2u_1w_1+u_1cv+u_1+w_1+cv+1)$.
Therefore the sufficient condition for which $P_6 < 0$ is

$$\eta_2^2 - 4\eta_1\eta_3 \le 0. \tag{36}$$

So we reach the theorem 3.1

Theorem 3.1. When $K > u_1 + c$, the system (27)-(28) has two positive equilibrium $(\bar{x}_1, \bar{y}_1)$ and $(\bar{x}_2, \bar{y}_2)$, where $\bar{x}_1$, $\bar{x}_2$ are two positive solutions of the equation (29) under the parametric restriction given by (31), moreover when the conditions (34) and (36) are satisfied at some equilibrium point, that equilibrium point must be asymptotically stable.

4. Simulation and Discussion

Case I. $0 \le I \le I_0$: If we choose the parameters as follows:

$a = 3$, $d = 0.1$, $\lambda = 0.3$, $\alpha = 0.5$, $\beta = 0.1$, $m = 0.01$, $r = 0.2$, then we get the unique positive equilibrium point (18.18827, 5.76243, 6.050551). Here the basic reproductive number $R_0 = 29.03226 > 1$. For the above choice of parameters we see that all the three components $S(t), I(t), R(t)$ approach to their steady state values as time goes to infinity, the disease becomes endemic (see figure 2).

< Figure 1 >

Again if we take $a = 15$, $d = 2.5$, $\lambda = 0.5$, $\alpha = 1$, $\beta = 0.5$, $m = 10$ and $r = 0.1$, the value of the basic reproductive number becomes $0.2380952 < 1$ and in that case we see that, the disease dies out (see figure 3).

< Figure 2-3>

By rescaling, the system (14) & (15) reduces to

$$\frac{dx}{dT} = \frac{x(45-x-y)}{1+0.2222222x^2} - 1.55x,$$
$$\frac{dy}{dT} = 1.05 - y.$$

Here $(u-K) < 0$, and hence there exists unique positive equilibrium point (x^*, y^*) where $x^* = 9.075827$ and $y^* = 8.643645$. For the above choice of parameters $P_3 = 9.162602 > 0$ $P_4 = 196.5196$, $P_4/P_3 = 21.44801$ and therefore the sufficient condition for local stability i.e. $x^* < P_4/P_3$ is satisfied here. We have drawn figures for both the system $(S(t), I(t), R(t)$ and $(x(T), y(T))$ to verify our result (see figures 4 and 5).

< Figure 4-5 >

Figures 4 and 5 also shows that there exists no limit cycle and the unique positive equilibrium (18.18827, 5.76243, 6.050551) of the system (6)-(8) or equivalently (9.075827, 8.643645) of the system (14)-(15) is globally stable.

In our model parameter α describes the psychological or inhibitory effect. From (11), we see that the steady state value I^* of the infective decreases as α increases. To verify the result we have plotted figure6 for different values of α, keeping all other values of the parameters same as for figure2.

< Figure 6-7 >

We have also plotted figure 7 for different values of m, keeping all others parameter values same as for figure 2 and see that the steady state value I^* decreases as m increases. Further we have plotted figure 8 to see the dependence of I^* on the parameter r and see that I^* decreases as r increases.

< Figure 8-9 >

To see the dependence of the steady state value S^* of the susceptible on the parameter r, we have plotted figure10 and have seen that S^* decreases as the parameter r increases, keeping all other parameters same. < Figure 10 >

Case (II): $I > I_0$ To study the system (1)-(3) numerically, where $I > I_0$, we choose our parameters as, $a = 2.8, d = 0.0453, \lambda = 0.4, \alpha = 2.0, \beta = 0.13, m = 0.01, K_1 = 0.7$.

Here $S+I+R = (a/d) = 61.810154$ is invariant manifold. So the system reduces to $dI/dt = \{(0.4)I/(1+2I^2)\}(61.810154-I-R) - (0.0553)I - 0.7$,
$dR/dt = (0.01)I - (0.1753)R + 0.7$.

To rescale the system we consider

$x = \{\lambda/(d+\beta)\}I$ $i.e.$ $x = (2.2818026)I, y = \{\lambda/(d+\beta)\}R$ $i.e.$ $y = (2.2818026)R$, $T = (0.1753)t$, which reduces the above system to $dx/dT = \{x(141.2386 - x - y)/(1 + 0.3841261x^2)\} - (0.3154592)x - 9.111591$, $dy/dT = (0.5704507)x - y + 9.111591$.

Now to find the equilibrium point of the above system we see that $u_1 + c = 9.4270502 < 141.0386, K$ and hence equation (29) has either two positive roots or no positive root. But here $a_0 = 0.1211761, a_1 = 1.519015$, $a_2 = -43.87051$, and $a_3 = c = 9.111591$. So , $H = -7.623465 < 0$, and

$(a_0^2a_3^2 + 4a_3a_1^3 + 4a_0a_2^3) - (3a_1^2a_2^2 + 6a_0a_1a_2a_3) = -53721.47 < 0$, i.e. (31)

is satisfied which imply that $G^2 + 4H^3 < 0$. Therefore for our choice of parameters the system (27)-(28) has two positive equilibrium (19.0878, 10.2004) and (0.0694, 9.1155).

< Figure 11 >

Now at (19.0878, 10.2004) the value of the $\det(M_2) = 24659.46 > 0$ and Trace $(M_2) = -42992.9 < 0$, therefore (19.0878, 10.2004) will be asymptotically stable.

Figure 13 shows that (19.0878, 10.2004) is a stable node, also figure 12 shows that the corresponding equilibrium point (48.96, 8.36, 4.47) of the system (22)-(24) is a stable node.

< Figure 12-13 >

Again at the other equilibrium point (0.694, 9.1155), value of Det $(M_2) = -131.1483 < 0$ and Trace $(M_2) = 130.1843 > 0$ and hence it becomes unstable.

Figure 14 shows the dependence of the steady state value I^* of I(t) on the parameter K_1 and we see that I^* decreases as K_1 increases.

< Figure 14-16 >

We see that basic reproductive number plays an important role to control the disease. When $R_0 \leq 1$, there exists no positive equilibrium, and in that case the disease free equilibrium is globally stable, that is the disease dies out. But when $R_0 > 1$, the unique endemic equilibrium is globally stable under some parametric condition. Also we see that the treatment rate plays a major role to control the disease. From figure 14(b), we can see that when the value of the parameter K_1 crosses a definite value 1.27, the disease dies out. Figure15 and 16 show that number of susceptible and recovered increases as the value of the parameter K_1 increases.

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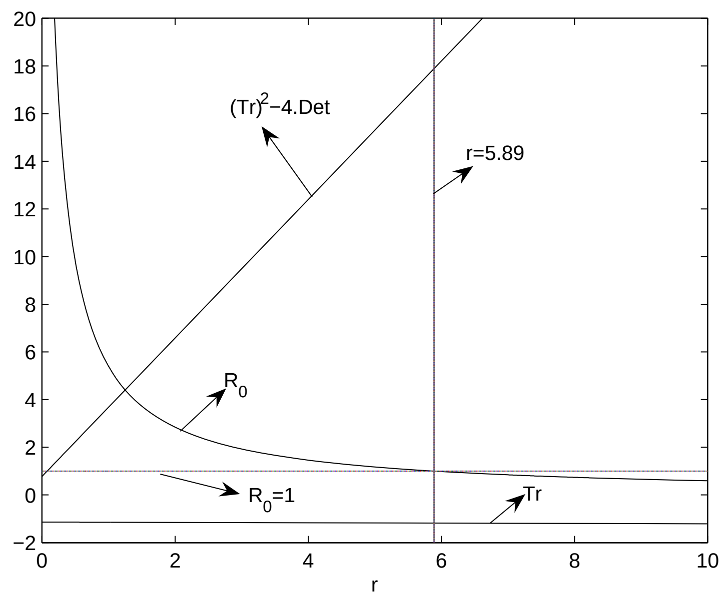


Figure 1. Bifurcation diagram for endemic and disease free equilibrium

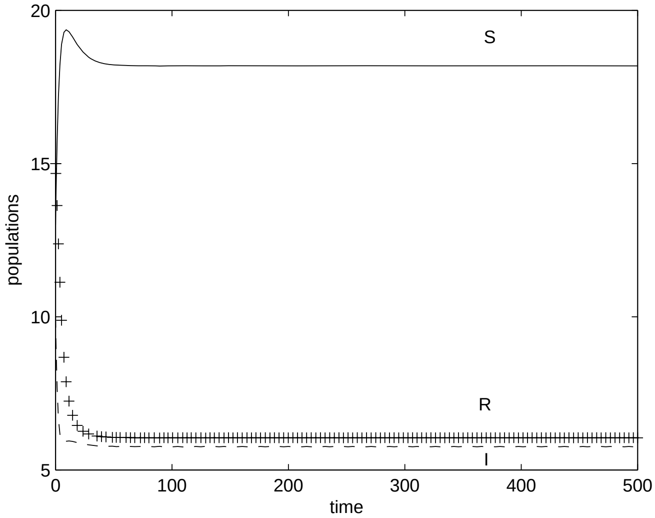


Figure 2. The plot represents that the SIR epidemic model is asymptotically stable. This plot is the numerical result of the stability analysis

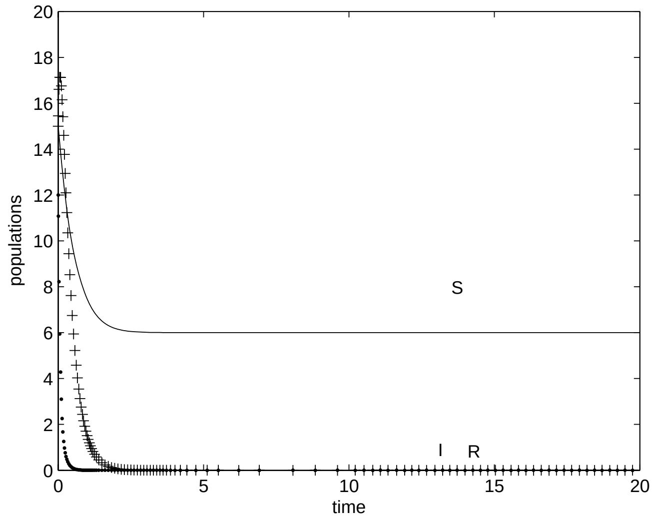


Figure 3. The plot represents that the disease dies

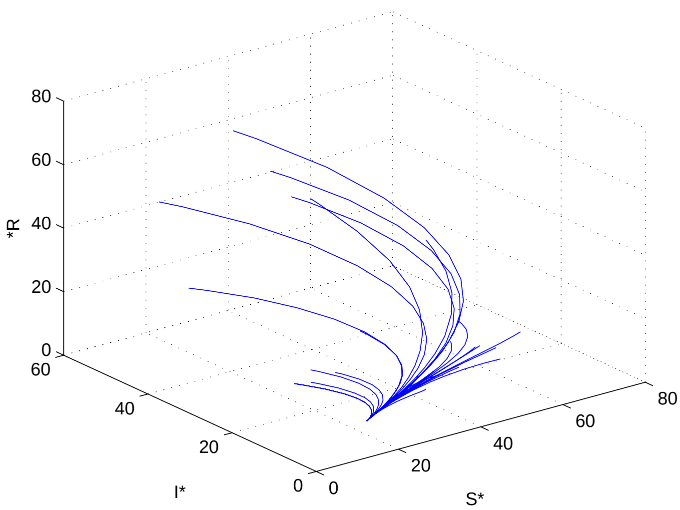


Figure 4. Equilibrium point (18.18827, 5.76243, 6.0550551) of the system (6)-(8) globally asymptotically stable

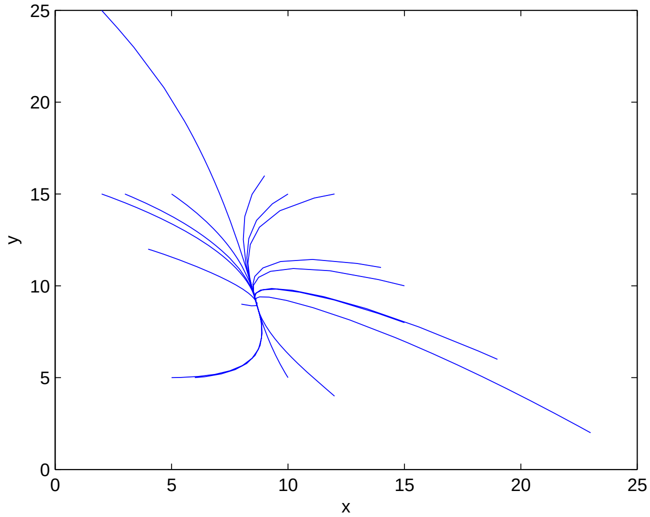


Figure 5. Equilibrium point (9.075827, 8.643645) of the system (14)-(15) is globally stable

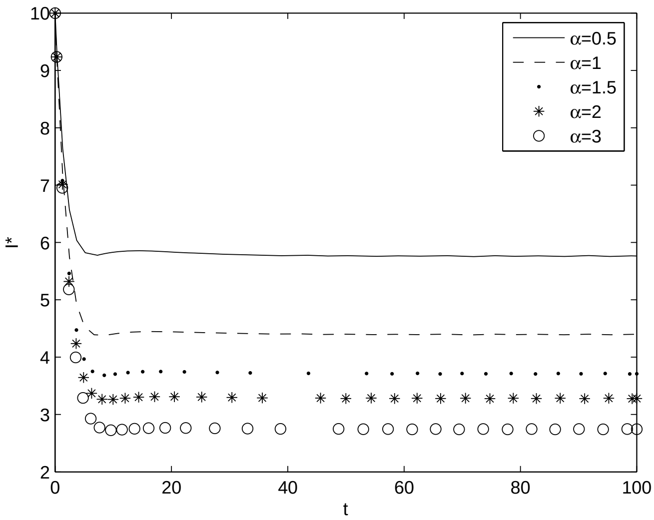


Figure 6. This figure shows the dependence of I^* on the parameter α

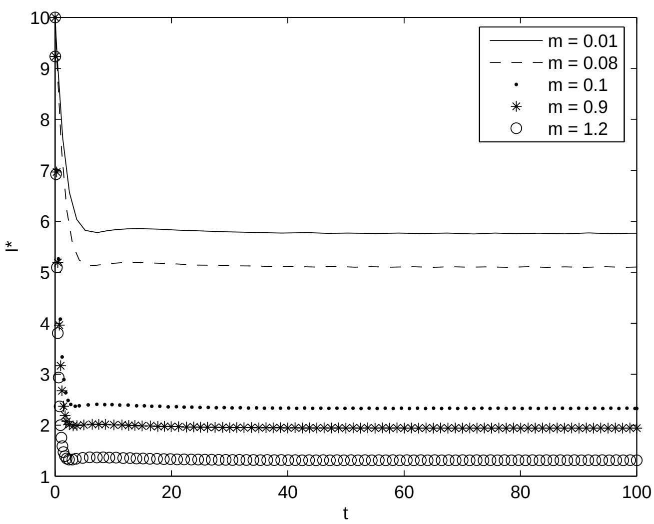


Figure 7. This figure shows the dependence of I^* on the parameter m

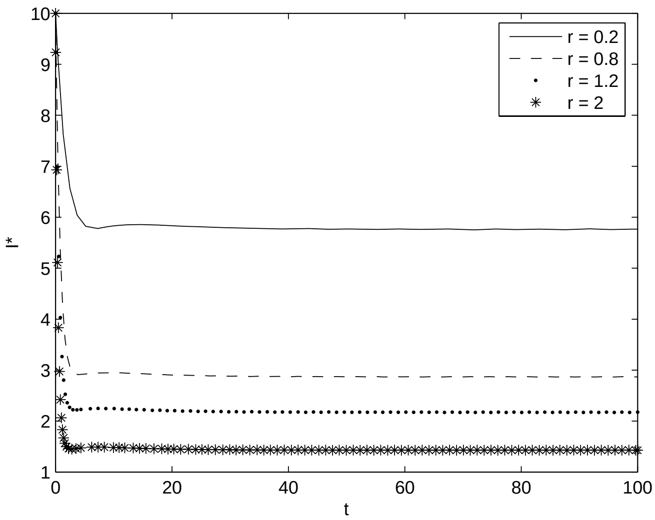


Figure 8. This figure shows the dependence of I^* on the parameter r

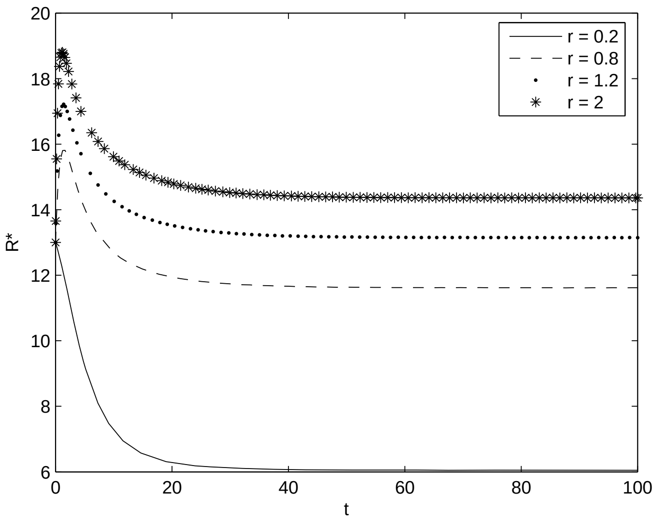


Figure 9. This figure shows the dependence of R^* on the parameter r

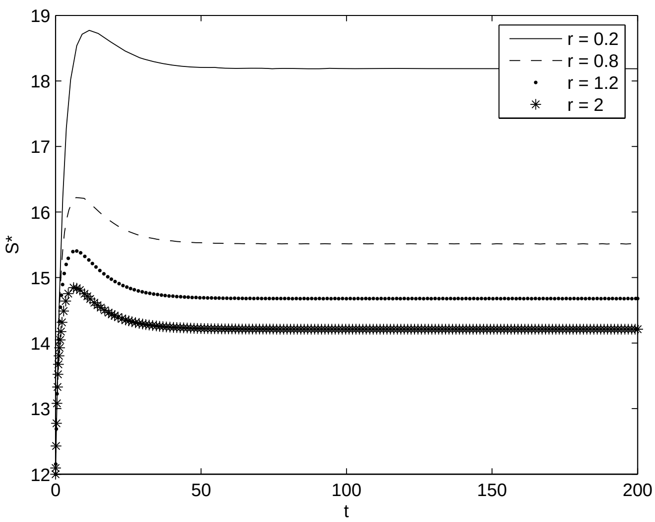


Figure 10. This figure shows the dependence of S^* on the parameter r

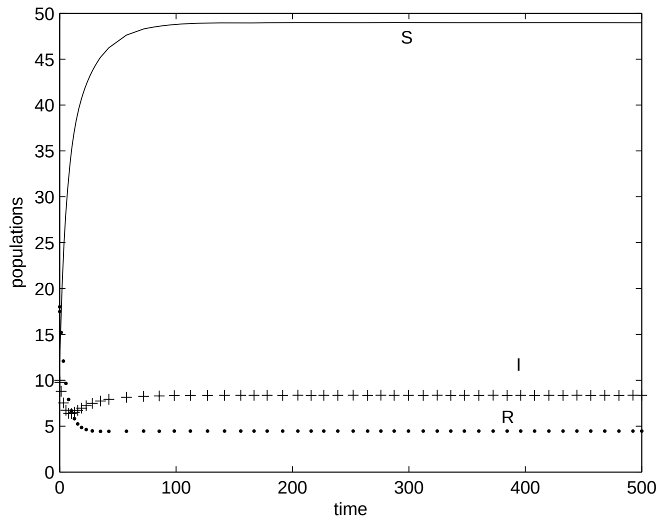


Figure 11. This figure shows that the populations approach their steady state as time goes to infinity and the disease become endemic

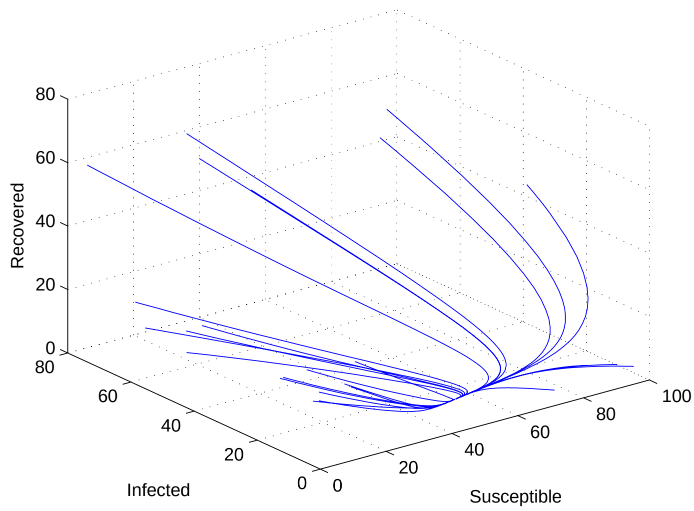


Figure 12. Equilibrium point (48.96, 8.36, 4.47) of the system (22)-(24) is globally asymptotically stable

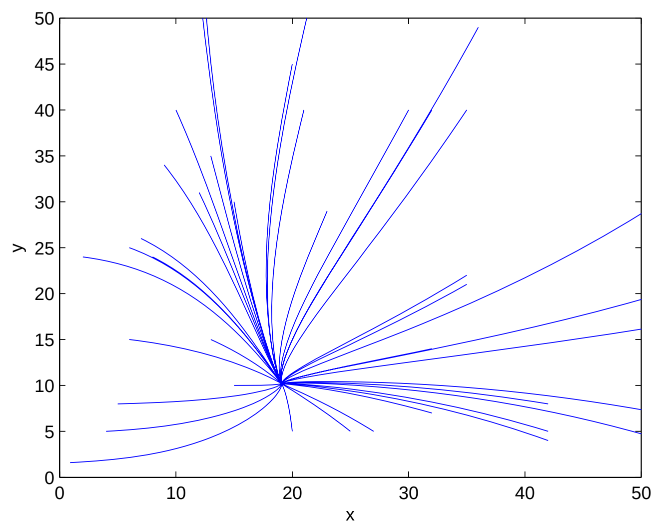
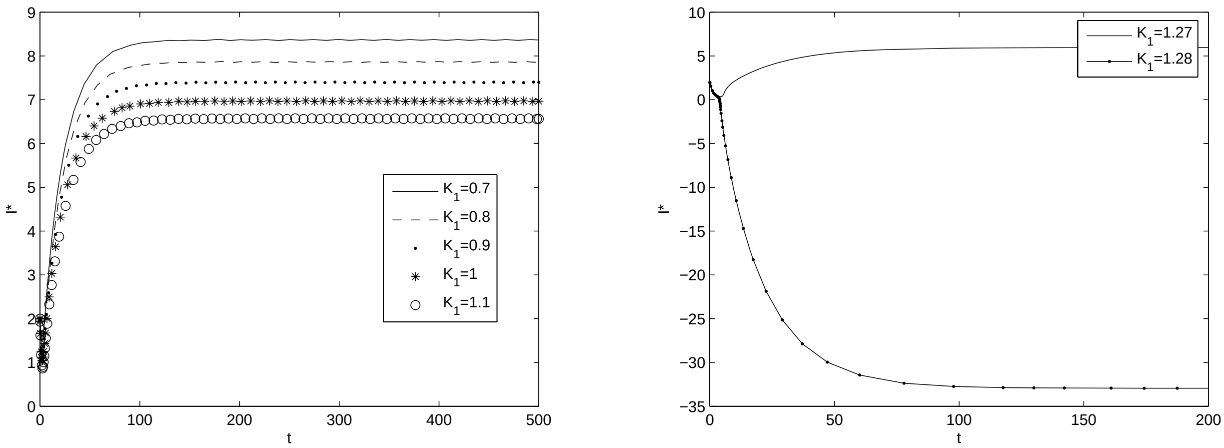


Figure 13. Equilibrium point (19.0878, 10.2004) of the system (27)-(28) is globally stable



(14a) (14b)

Figure 14. 14a and 14b show the dependence of I^* on the parameter K_1.

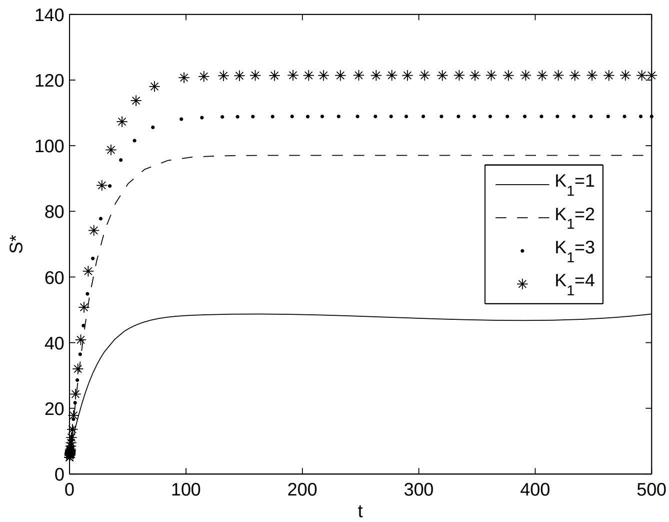


Figure 15. This figure shows the dependence of R^* on the parameter K_1.

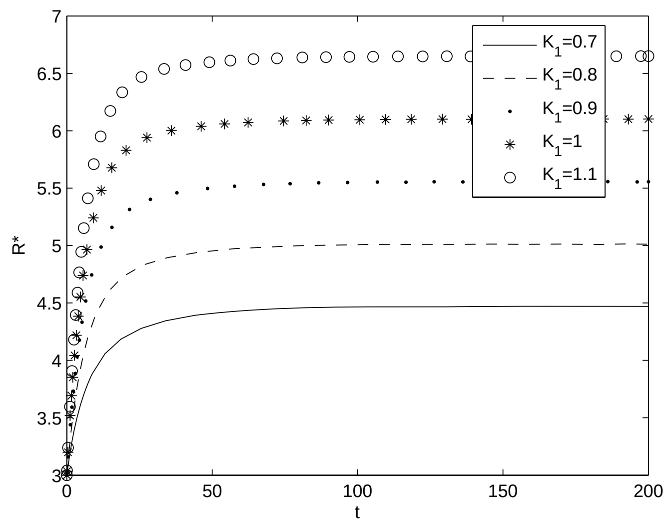


Figure 16. This figure shows the dependence of S^* on the parameter K_1.

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Augmented Lyapunov Method for BIBO Stabilization of Discrete System

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Abstract

The problem of bounded-input bounded-output (BIBO) stabilization for discrete-time uncertain system with time delay is investigated. By constructing an augmented Lyapunov function, some sufficient conditions guaranteeing BIBO stabilization and robust BIBO stabilization are established. These conditions are expressed in the forms of linear matrix inequalities (LMIs), whose feasibility can be easily checked by using Matlab LMI Toolbox. Two numerical examples are provided to demonstrate the effectiveness of the derived results.

Keywords: BIBO stabilization, Robust BIBO stabilization, Delay-dependent condition, Discrete-time system, Time delay

1. Introduction

Recently, in order to track out the reference input signal in real world, many researchers have focused their interest on the analysis of BIBO stabilization (see Guan *et al.* 1994, Xu and Zhong 1995, Wolfgang and Mecklenbräuker 1998, Michaletzky and Gerencser 2002, Partington and Bonnet 2004, Huang, Zeng and Zhong 2005). On the other hand, because of the finite switching speed, memory effects and so on, time delay is unavoidable in technology and nature. It can make the concerned control system become instable and oscillating, which cause the design and hardware implementation of the control system become difficult. Thus, BIBO stabilization analysis for delayed system is of great significance. In Li and Zhong (2008a), based on Riccati-equations, by constructing appropriate Lyapunov functions, some delay-independent BIBO stabilization criteria for a class of delayed control system with nonlinear perturbation were established. Based on Gronwall inequality, the problem of BIBO stabilization for system with multiple mixed delays and nonlinear perturbations were investigated in Li and Zhong (2008b). Similar to the method used in Li and Zhong (2008a), a class of linear delayed system with parameter uncertainty was considered in Li and Zhong (2007), and some robust BIBO stabilization criteria were derived in terms of linear matrix technique. In Li and Zhong (2009), the BIBO stabilization problem of a class of piecewise switched linear systems were further investigated. On the other hand, the problem of BIBO stabilization in mean square was also considered in Fu and Liao (2003). However, these previous results have been assumed to be in continuous time, but seldom in discrete time (see Bose and Chen 1995, Kotsios and Feely 1998). In practice, discrete-time control system is more applicable to problems that are inherently temporal in nature or related to biological realities. And it can ideally keep the dynamic characteristics, functional similarity, and even the physical or biological reality of the continuous-time systems under mild restriction. Thus, the BIBO stabilization analysis problems for discrete-time case are necessary.

Motivated by the above discussions, the main aim of this paper is to study the BIBO stabilization and robust BIBO stabilization problems for a class of discrete-time control system with time delay and parameter uncertainties. Based on linear matrix inequalities (LMIs) technique, an augmented Lyapunov function is constructed, and some sufficient conditions guaranteeing BIBO stabilization and robust BIBO stabilization are established. Finally, two numerical examples are provided to demonstrate the effectiveness of the derived results.

Notation: The notations are used in our paper except where otherwise specified. $\|\cdot\|$ denotes a vector or a matrix norm; $\mathbb{R}, \mathbb{R}^n$ are real and n-dimension real number sets, respectively; $\mathbb{N}^+$ is positive integer set. I is identity matrix; $*$ represents the elements below the main diagonal of a symmetric block matrix; Real matrix $P > 0(< 0)$ denotes P is a positive-definite (negative-definite) matrix; $\mathbb{N}[a, b] = \{a, a + 1, \cdots, b\}$; $\lambda_{min}(\lambda_{max})$ denotes the minimum (maximum) eigenvalue of a real matrix.

2. Preliminaries

Consider the following discrete-time uncertain system with time delay described by

$$\Sigma : \begin{cases} x(k+1) = A(k)x(k) + B(k)x(k-\tau) + C(k)u(k), \quad k \in \mathbb{N}^+ \\ y(k) = D(k)x(k) \\ x(k) = \varphi(k), \ -\tau \le k \le 0. \end{cases} \tag{1}$$

where $x(k) = [x_1(k), x_2(k), \cdots, x_n(k)]^T \in \mathbb{R}^n$ denotes the state vector; $u(k) = [u_1(k), u_2(k), \cdots, u_n(k)]^T \in \mathbb{R}^n$ is the control input vector; $y(k) = [y_1(k), y_2(k), \cdots, y_n(k)]^T \in \mathbb{R}^n$ is the control output vector; Positive integer τ represents the transmission delay; $\varphi(\cdot)$ is vector-valued initial function and $\|\varphi\|_\tau$ is defined by $\|\varphi\|_\tau = \sup_{i\in\mathbb{N}[-\tau,0]} \|x(i)\|$; $A(k) = A + \triangle A(k)$, $B(k) = B + \triangle B(k)$, $C(k) = C + \triangle C(k)$, $D(k) = D + \triangle D(k)$; $A, B, C, D \in \mathbb{R}^{n\times n}$ represent the weighting matrices; $\Delta A(k), \Delta B(k), \Delta C(k), \Delta D(k)$ denote the time-varying structured uncertainties which are of the following form:

$$[\Delta A(k) \quad \Delta B(k) \quad \Delta C(k) \quad \Delta D(k)] = GF(k)[E_a \quad E_b \quad E_c \quad E_d],$$

where G, E_a, E_b, E_c, E_d are known real constant matrices with appropriate dimensions; $F(k)$ is unknown time-varying matrix function satisfying $F^T(k)F(k) \le I, \forall k \in \mathbb{N}^+$.

Let $u(k)$ be linear gain local state feedback with the reference input $r(k)$ for system (1) as follows:

$$u(k) = Kx(k) + r(k) \tag{2}$$

so as to ensure stabilization of the closed-loop delayed system.

To obtain our main results, we need introduce the following definitions and lemmas.

Definition 1 *A real discrete-time vector* $r(k) \in L^n_\infty$ *if* $\|r(k)\|_\infty \triangleq \sup_{k\in\mathbb{N}[0,\infty]} \|r(k)\| < +\infty$.

Definition 2 *The control system (1) is said to be BIBO stabilized by the local control law (2) if for every solution of system (1),* $y(k)$ *satisfies*

$$\|y(k)\| \le \theta_1 \|r(k)\|_\infty + \theta_2, k \in \mathbb{N}^+.$$

where θ_1, θ_2 *are known positive constants for every reference input* $r(k) \in L^n_\infty$.

Lemma 1 *(Lee and Radovic 1987) For any given vectors* $v_i \in \mathbb{R}^n, i = 1, 2, \cdots, n$*, the following inequality holds:*

$$[\sum_{i=1}^{n} v_i]^T [\sum_{i=1}^{n} v_i] \le n \sum_{i=1}^{n} v_i^T v_i.$$

Lemma 2 *(Boyd* et al. *1994) Given constant symmetric matrices* $\Sigma_1, \Sigma_2, \Sigma_3$*, where* $\Sigma_1^T = \Sigma_1$ *and* $0 < \Sigma_2 = \Sigma_2^T$*, then* $\Sigma_1 + \Sigma_3^T \Sigma_2^{-1} \Sigma_3 < 0$ *if and only if*

$$\begin{pmatrix} \Sigma_1 & \Sigma_3^T \\ \Sigma_3 & -\Sigma_2 \end{pmatrix} < 0 \ or \ \begin{pmatrix} -\Sigma_2 & \Sigma_3 \\ \Sigma_3^T & \Sigma_1 \end{pmatrix} < 0.$$

Lemma 3 *(Liu, Wang and Liu 2008) Let N and E be real constant matrices with appropriate dimensions, matrix function $F(k)$ satisfies $F^T(k)F(k) \leq I$, then, for any $\epsilon > 0$, $EF(k)N + N^T F^T(k)E^T \leq \epsilon^{-1}EE^T + \epsilon NN^T$.*

Lemma 4 *For any real vector X, Y and positive-definite matrix $\Sigma > 0$ with appropriate dimensions, it follows that*

$$2X^T Y \leq X^T \Sigma X + Y^T \Sigma^{-1} Y.$$

For designing the linear feedback control $u(k) = Kx(k) + r(k)$ such that the closed-loop system (1) is BIBO stabilized by local control law (2), we first consider the nominal Σ_0 of Σ defined by

$$\Sigma_0 : \begin{cases} x(k+1) = Ax(k) + Bx(k-\tau) + Cu(k), \quad k \in \mathbb{N}^+ \\ y(k) = Dx(k) \\ x(k) = \varphi(k), \ -\tau \leq k \leq 0. \end{cases} \tag{3}$$

Substituting (2) into system (3) yields a closed-loop systems as follows:

$$\Sigma_0 : \begin{cases} x(k+1) = (A + CK)x(k) + Bx(k-\tau) + Cr(k), \quad k \in \mathbb{N}^+ \\ y(k) = Dx(k) \\ x(k) = \varphi(k), \ -\tau \leq t \leq 0. \end{cases} \tag{4}$$

Then, we can obtain the following BIBO stabilization results.

3. Main results

Theorem 1 *For given positive integer $\tau > 0$, local control law (2) with feedback gain matrix K stabilizes the delayed system (4), if there exist positive-definite matrices Q, R, H, P_1, P_2, positive-definite diagonal matrix Z with appropriate dimensions, such that the following LMI holds:*

$$\Xi_1 = \begin{bmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} & \Xi_{14} & \Xi_{15} \\ * & \Xi_{22} & \Xi_{23} & \Xi_{24} & \Xi_{25} \\ * & * & \Xi_{33} & \Xi_{34} & \Xi_{35} \\ * & * & * & \Xi_{44} & \Xi_{45} \\ * & * & * & * & \Xi_{55} \end{bmatrix} < 0, \tag{5}$$

where $Q = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ * & Q_{22} & Q_{23} \\ * & * & Q_{33} \end{bmatrix} > 0,$

$\Xi_{11} = A^T Q_{11} A - Q_{11} + H + P_1 + CK + K^T C^T$, $\Xi_{12} = A^T Q_{12} A - Q_{12} + B$, $\Xi_{13} = A^T(Q_{11} + Q_{13}) - I + K^T C^T$, $\Xi_{14} = A^T(Q_{12} - Q_{13}) + B^T$, $\Xi_{15} = A^T Q_{13} - Q_{13}$, $\Xi_{22} = A^T Q_{22} A - Q_{22} - H$, $\Xi_{23} = A^T(Q_{12}^T + Q_{23})$, $\Xi_{24} = A^T Q_{22} - A^T Q_{23}$, $\Xi_{25} = A^T Q_{23} - Q_{23}$, $\Xi_{33} = Q_{11} + Q_{33} + Q_{13} + Q_{13}^T - 2I + P_2 + \tau Z + R$, $\Xi_{34} = Q_{12} + Q_{23}^T - Q_{13} - Q_{33}$, $\Xi_{35} = Q_{13} + Q_{33}$, $\Xi_{44} = Q_{22} + Q_{33} - Q_{23} - Q_{23}^T - R$, $\Xi_{45} = Q_{23} - Q_{33}$, $\Xi_{55} = -\tau^{-1} Z$.

Proof. Constructing an augmented Lyapunov-Krasovskii function candidate as follows:

$$V(k) = V_1(k) + V_2(k) + V_3(k),$$

where $V_1(k) = \widetilde{X}^T(k) Q \widetilde{X}(k)$, $\widetilde{X}^T(k) = [x^T(k), x^T(k-\tau), \sum_{i=k-\tau}^{k-1} \eta^T(i)]$, $\eta(k) = x(k+1) - Ax(k)$,

$$V_2(k) = \sum_{i=k-\tau}^{k-1} x^T(i) H x(i) + \sum_{i=k-\tau}^{k-1} \eta^T(i) R \eta(i), \quad V_3(k) = \sum_{j=k-\tau}^{k-1} \sum_{i=j}^{k-1} \eta^T(i) Z \eta(i).$$

Set $X^T(k) = [x^T(k), x^T(k-\tau), \eta^T(k), \eta^T(k-\tau), \sum_{i=k-\tau}^{k-1} \eta^T(i)]$. Define $\Delta V(k) = V(k+1) - V(k)$, then along the solution of system (4) we can obtain that

$$\begin{aligned} \Delta V_1(k) &= \widetilde{X}^T(k+1) Q \widetilde{X}(k+1) - \widetilde{X}^T(k) Q \widetilde{X}(k) \\ &= x^T(k)[A^T Q_{11} A - Q_{11}]x(k) + 2x^T(k)[A^T Q_{12} A - Q_{12}]x(k-\tau) + 2x^T(k) A^T [Q_{11} + Q_{13}]\eta(k) \\ &\quad + 2x^T(k) A^T [Q_{12} - Q_{13}]\eta(k-\tau) + 2x^T(k)[A^T Q_{13} - Q_{13}](\sum_{i=k-\tau}^{k-1} \eta(i)) + x^T(k-\tau)[A^T Q_{22} A - Q_{22}]x(k-\tau) \\ &\quad + 2x^T(k-\tau) A^T [Q_{12}^T + Q_{23}]\eta(k) + 2x^T(k-\tau) A^T [Q_{22} - Q_{23}]\eta(k-\tau) + 2x^T(k-\tau)[A^T Q_{23} \\ &\quad - Q_{23}](\sum_{i=k-\tau}^{k-1} \eta(i)) + \eta^T(k)[Q_{11} + Q_{33} + Q_{13} + Q_{13}^T]\eta(k) + 2\eta^T(k)[Q_{12} + Q_{23}^T - Q_{33} - Q_{13}]\eta(k-\tau) + 2\eta^T(k) \times \\ &\quad [Q_{13} + Q_{33}](\sum_{i=k-\tau}^{k-1} \eta(i)) + \eta^T(k-\tau)[Q_{22} + Q_{33} - Q_{23}^T - Q_{23}]\eta(k-\tau) + 2\eta(k-\tau)[Q_{23} - Q_{33}](\sum_{i=k-\tau}^{k-1} \eta(i)). \end{aligned} \tag{6}$$

$$\Delta V_2(k) = x^T(k)Hx(k) - x^T(k-\tau)Hx(k-\tau) + \eta^T(k)R\eta(k) - \eta^T(k-\tau)R\eta(k-\tau). \tag{7}$$

From lemma 1, we have

$$\begin{aligned}
\Delta V_3(k) &= \sum_{j=k+1-\tau}^{k}\sum_{i=j}^{k}\eta^T(i)Z\eta(i) - \sum_{j=k-\tau}^{k-1}\sum_{i=j}^{k-1}\eta^T(i)Z\eta(i)\\
&= \sum_{j=k-\tau}^{k-1}\sum_{i=j+1}^{k}\eta^T(i)Z\eta(i) - \sum_{j=k-\tau}^{k-1}\sum_{i=j}^{k-1}\eta^T(i)Z\eta(i)\\
&= \sum_{j=k-\tau}^{k-1}[\eta^T(k)Z\eta(k) - \eta^T(j)Z\eta(j)]\\
&= \tau\eta^T(k)Z\eta(k) - \sum_{i=k-\tau}^{k-1}\eta^T(i)Z\eta(i)\\
&= \tau\eta^T(k)Z\eta(k) - \sum_{i=k-\tau}^{k-1}(\sqrt{Z}\eta(i))^T\sqrt{Z}\eta(i)\\
&\leq \tau\eta^T(k)Z\eta(k) - \frac{1}{\tau}[\sum_{i=k-\tau}^{k-1}\eta(i)]^T Z[\sum_{i=k-\tau}^{k-1}\eta(i)].
\end{aligned} \tag{8}$$

On the other hand, by lemma 4, for any positive-definite matrices P_1, P_2 with appropriate dimensions, we have

$$\begin{aligned}
0 &= 2x^T(k)[CKx(k) + Bx(k-\tau) + Cr(k) - \eta(k)]\\
&= 2x^T(k)CKx(k) + 2x^T(k)Bx(k-\tau) + 2x^T(k)Cr(k) - 2x^T(k)\eta(k)\\
&\leq x^T(k)(CK + K^TC^T + P_1)x(k) + 2x^T(k)Bx(k-\tau) - 2x^T(k)\eta(k) + r^T(k)C^TP_1^{-1}Cr(k)\\
&\leq x^T(k)(CK + K^TC^T + P_1)x(k) + 2x^T(k)Bx(k-\tau) - 2x^T(k)\eta(k) + \|r(k)\|^2\|C^TP_1^{-1}C\|,
\end{aligned} \tag{9}$$

$$\begin{aligned}
0 &= 2\eta^T(k)[CKx(k) + Bx(k-\tau) + Cr(k) - \eta(k)]\\
&= 2\eta^T(k)CKx(k) + 2\eta^T(k)Bx(k-\tau) + 2\eta^T(k)Cr(k) - 2\eta^T(k)\eta(k)\\
&\leq 2\eta^T(k)CKx(k) + 2\eta^T(k)Bx(k-\tau) + \eta^T(k)(P_2 - 2I)\eta(k) + r^T(k)C^TP_2^{-1}Cr(k)\\
&\leq 2\eta^T(k)CKx(k) + 2\eta^T(k)Bx(k-\tau) + \eta^T(k)(P_2 - 2I)\eta(k) + \|r(k)\|^2\|C^TP_2^{-1}C\|.
\end{aligned} \tag{10}$$

Combining (6)-(10), we get

$$\Delta V(k) \leq X^T(k)\Xi X(k) + \sigma\|r(k)\|^2, \tag{11}$$

where $\sigma = \|C^TP_1^{-1}C\| + \|C^TP_2^{-1}C\|$. If the LMI (5) holds, it follows that there exists a sufficient small positive scalar $\varepsilon > 0$ such that

$$\Delta V(k) \leq -\varepsilon\|x(k)\|^2 + \sigma\|r(k)\|^2. \tag{12}$$

On the other hand, it can easily to get that

$$V(k) \leq \alpha_1\|x(k)\|^2 + \alpha_2\sum_{i=k-\tau}^{k-1}\|x(i)\|^2, \tag{13}$$

where $\alpha_1 = \lambda_{max}(Q)[1 + 2\tau] + \lambda_{max}(R) + \tau\lambda_{max}(Z)$, $\alpha_2 = 2\tau[\lambda_{max}(Z) + 1] + \lambda_{max}(H) + 2\lambda_{max}(R)$.

For any $\theta > 1$, it follows from (13) that

$$\begin{aligned}
\theta^{j+1}V(j+1) - \theta^jV(j) &= \theta^{j+1}\Delta V(j) + \theta^j(\theta-1)V(j)\\
&\leq \theta^j[\sigma\theta\|r(j)\|^2 - \varepsilon\theta\|x(j)\|^2 + (\theta-1)\alpha_1\|x(j)\|^2 + (\theta-1)\alpha_2\sum_{i=j-\tau}^{j-1}\|x(i)\|^2].
\end{aligned} \tag{14}$$

Summing up both sides of (14) from 0 to $k-1$ we can obtain

$$\begin{aligned}
\theta^kV(k) - V(0) &\leq [\alpha_1(\theta-1) - \varepsilon\theta]\sum_{j=0}^{k-1}\theta^j\|x(j)\|^2 + \alpha_2(\theta-1)\sum_{j=0}^{k-1}\sum_{i=j-\tau}^{j-1}\theta^j\|x(i)\|^2 + \sum_{j=0}^{k-1}\sigma\theta^{j+1}\|r(j)\|^2\\
&\leq \mu_1(\theta)\sup_{j\in\mathbb{N}[-\tau,0]}\|x(j)\|^2 + \mu_2(\theta)\sum_{j=0}^{k}\theta^j\|x(j)\|^2 + \sum_{j=0}^{k-1}\sigma\theta^{j+1}\|r(j)\|^2,
\end{aligned} \tag{15}$$

where $\mu_1(\theta) = \alpha_2(\theta-1)\tau^2\theta^\tau$, $\mu_2(\theta) = \alpha_2(\theta-1)\tau\theta^\tau + \alpha_1(\theta-1) - \varepsilon\theta$. Since $\mu_2(1) = -\varepsilon\theta < 0$, there must exist a positive $\theta_0 > 1$ such that $\mu_2(\theta_0) < 0$. Then we have

$$\begin{aligned}
V(k) &\le \mu_1(\theta_0)(\frac{1}{\theta_0})^k \sup_{j\in\mathbb{N}[-\tau,0]} \|x(j)\|^2 + (\frac{1}{\theta_0})^k V(0) + \sigma\sum_{j=0}^{k-1}\frac{1}{\theta_0^{k-j-1}}\|r(j)\|^2 \\
&\le \mu_1(\theta_0)(\frac{1}{\theta_0})^k \sup_{j\in\mathbb{N}[-\tau,0]} \|x(j)\|^2 + (\frac{1}{\theta_0})^k V(0) + \sigma\|r(k)\|_\infty^2\sum_{j=0}^{k-1}\frac{1}{\theta_0^{k-j-1}} \\
&\le \mu_1(\theta_0)\|\varphi\|_\tau^2 + V(0) + \frac{\sigma}{\theta_0-1}\|r(k)\|_\infty^2, \forall k \ge 1.
\end{aligned} \tag{16}$$

On the other hand, set $\varpi = \alpha_1 + \tau\alpha_2$, we can obtain

$$V(0) \le \varpi \sup_{j\in\mathbb{N}[-\tau,0]} \|x(j)\|^2 \; and \; V(k) \ge \lambda_{min}(Q)\|x(k)\|^2. \tag{17}$$

It follows that $\|y(k)\| \le \theta_1\|r(k)\|_\infty + \theta_2, k \in \mathbb{N}^+$, where

$$\theta_1 = \|D\|\sqrt{\sigma(\theta_0-1)^{-1}\lambda_{min}^{-1}(Q)}, \theta_2 = \|D\|\cdot\|\varphi\|_\tau\sqrt{[\mu_1(\theta_0)+\varpi]\lambda_{min}^{-1}(Q)}.$$

By Definition 2, system (4) is BIBO stabilized by local control law (2), which complete the proof of Theorem 1.

Theorem 2 *For given positive integer $\tau > 0$, local control law (2) with feedback gain matrix K stabilizes the delayed system (4), if there exist positive-definite matrices Q, R, H, P_1, P_2, positive-definite diagonal matrix Z with appropriate dimensions, such that the following LMI holds:*

$$\Xi_2 = \begin{bmatrix} \widetilde{\Xi}_{11} & \widetilde{\Xi}_{12} & \widetilde{\Xi}_{13} & \widetilde{\Xi}_{14} & 0 \\ * & \widetilde{\Xi}_{22} & \widetilde{\Xi}_{23} & \widetilde{\Xi}_{24} & 0 \\ * & * & \Xi_{33} & \Xi_{34} & \Xi_{35} \\ * & * & * & \Xi_{44} & \Xi_{45} \\ * & * & * & * & \Xi_{55} \end{bmatrix} < 0, \tag{18}$$

where $Q = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ * & Q_{22} & Q_{23} \\ * & * & Q_{33} \end{bmatrix} > 0,$

$\widetilde{\Xi}_{11} = H + P_1 + CK + K^TC^T + A + A^T - 2I$, $\widetilde{\Xi}_{12} = B$, $\widetilde{\Xi}_{13} = Q_{11} + Q_{13} - 2I + K^TC^T + A^T$, $\widetilde{\Xi}_{14} = Q_{12} - Q_{13} + B^T$, $\widetilde{\Xi}_{22} = -H$, $\widetilde{\Xi}_{23} = Q_{12}^T + Q_{23}$, $\widetilde{\Xi}_{24} = Q_{22} - Q_{23}$.

Proof. Constructing augmented Lyapunov-Krasovskii function candidate as the same in Theorem 1. Set $\eta(k) = x(k+1) - x(k)$, one can easily obtain this result, which omitted here.

Theorem 3 *For given positive integer $\tau > 0$, local control law (2) with feedback gain matrix K robustly stabilizes the delayed system (4), if there exist positive-definite matrices Q, R, H, P_1, P_2, positive-definite diagonal matrix Z with appropriate dimensions, and positive scalar $\epsilon > 0$, such that the following LMI holds:*

$$\Xi_3 = \begin{bmatrix} \Xi_2 & \xi_1 & \epsilon\xi_2^T \\ * & -\epsilon I & 0 \\ * & * & -\epsilon I \end{bmatrix} < 0, \tag{19}$$

where $Q = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ * & Q_{22} & Q_{23} \\ * & * & Q_{33} \end{bmatrix}$, $\xi_1^T = [G^T, 0, G^T, 0, 0]$, $\xi_2 = [E_a + E_cK, E_b, 0, 0, 0]$.

Proof. Replacing A, B, C in inequality (18) with $A + GF(t)E_a$, $B + GF(t)E_b$ and $C + GF(t)E_c$, respectively. Inequality (18) for system (1) is equivalent to $\Xi_2 + \xi_1F(t)\xi_2 + \xi_2^TF^T(t)\xi_1^T < 0$. From lemma 2 and lemma 3, one can easily obtain this result, which complete the proof.

Decomposing the weighting matrix A as $A = A_1 + A_2$. Set $\eta(k) = x(k+1) - A_1x(k)$, similar to the proof of Theorem 1 and Theorem 3, we can obtain the following less conservative criteria.

Theorem 4 *For given positive integer $\tau > 0$, local control law (2) with feedback gain matrix K stabilizes the delayed system (4), if there exist positive-definite matrices Q, R, H, P_1, P_2, positive-definite diagonal matrix Z with appropriate dimensions, such that the following LMI holds:*

$$\Xi_4 = \begin{bmatrix} \Xi'_{11} & \Xi'_{12} & \Xi'_{13} & \Xi'_{14} & \Xi'_{15} \\ * & \Xi'_{22} & \Xi'_{23} & \Xi'_{24} & \Xi'_{25} \\ * & * & \Xi_{33} & \Xi_{34} & \Xi_{35} \\ * & * & * & \Xi_{44} & \Xi_{45} \\ * & * & * & * & \Xi_{55} \end{bmatrix} < 0, \tag{20}$$

where $Q = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ * & Q_{22} & Q_{23} \\ * & * & Q_{33} \end{bmatrix} > 0,$

$\Xi'_{11} = A_1^T Q_{11} A_1 - Q_{11} + H + P_1 + CK + K^T C^T + A_2^T + A_2 - 2I$, $\Xi'_{12} = A_1^T Q_{12} A_1 - Q_{12} + B$, $\Xi'_{13} = A_1^T (Q_{11} + Q_{13}) - 2I + K^T C^T + A_2^T$, $\Xi'_{14} = A_1^T (Q_{12} - Q_{13}) + B^T$, $\Xi'_{15} = A_1^T Q_{13} - Q_{13}$, $\Xi'_{22} = A_1^T Q_{22} A_1 - Q_{22} - H$, $\Xi'_{23} = A_1^T (Q_{12}^T + Q_{23})$, $\Xi'_{24} = A_1^T Q_{22} - A_1^T Q_{23}$, $\Xi'_{25} = A_1^T Q_{23} - Q_{23}$.

Theorem 5 *For given positive integer $\tau > 0$, local control law (2) with feedback gain matrix K robustly stabilizes the delayed system (4), if there exist positive-definite matrices Q, R, H, P_1, P_2, positive-definite diagonal matrix Z with appropriate dimensions, and positive scalar $\epsilon > 0$, such that the following LMI holds:*

$$\Xi_5 = \begin{bmatrix} \Xi_4 & \xi_1 & \epsilon \xi_2^T \\ * & -\epsilon I & 0 \\ * & * & -\epsilon I \end{bmatrix} < 0, \tag{21}$$

where $Q = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ * & Q_{22} & Q_{23} \\ * & * & Q_{33} \end{bmatrix}$, $\xi_1^T = [G^T, 0, G^T, 0, 0]$, $\xi_2 = [E_a + E_c K, E_b, 0, 0, 0]$.

4. Numerical examples

In this section, two numerical examples will be presented to show the validity of the main results derived above.

Example 1. Consider the delayed discrete-time system in (3) with parameters given by

$$C = \begin{bmatrix} 1.7 & 1.3 \\ 0.3 & 1.6 \end{bmatrix}, A = \begin{bmatrix} 0.5 & 0.0 \\ 0.0 & 0.4 \end{bmatrix}, B = \begin{bmatrix} 1 & 0.1 \\ 0.2 & 0.1 \end{bmatrix}, \tau = 3, A_1 = A_2 = 0.5A.$$

One can check that LMI (5) in Theorem 1, LMI (18) in Theorem 2 and LMI (20) in Theorem 4 are feasible. By the Matlab LMI Toolbox, a feasible solution to the LMI (5) is obtained as follows:

$Q_{11} = \begin{bmatrix} 2.1282 & -0.2834 \\ -0.2834 & 7.2316 \end{bmatrix}$, $Q_{12} = \begin{bmatrix} 1.0005 & 0.9379 \\ -0.2223 & 1.4758 \end{bmatrix}$, $Q_{13} = \begin{bmatrix} -0.2171 & -0.0547 \\ 0.0679 & -0.2807 \end{bmatrix}$, $Q_{22} = \begin{bmatrix} 0.8061 & 0.0016 \\ 0.0016 & 2.2791 \end{bmatrix}$, $Q_{23} = \begin{bmatrix} -0.0544 & -0.0605 \\ -0.1479 & -0.0054 \end{bmatrix}$, $Q_{33} = \begin{bmatrix} 0.1333 & -0.0821 \\ -0.0821 & 0.3975 \end{bmatrix}$, $H = \begin{bmatrix} 1.7544 & -0.5778 \\ -0.5778 & 0.7710 \end{bmatrix}$, $Z = \begin{bmatrix} 0.2787 & 0 \\ 0 & 1.2623 \end{bmatrix}$, $Q = \begin{bmatrix} 3.1153 & 0 \\ 0 & 3.1153 \end{bmatrix}$, $K = \begin{bmatrix} -2.9865 & 2.3009 \\ 0.1301 & -1.4189 \end{bmatrix}$, $R = \begin{bmatrix} 3.0110 & -0.1474 \\ -0.1474 & 7.1342 \end{bmatrix}$, $P_1 = \begin{bmatrix} 0.8821 & -0.3663 \\ -0.3663 & 0.7038 \end{bmatrix}$, $P_2 = \begin{bmatrix} 0.5416 & -0.4524 \\ -0.4524 & 1.6637 \end{bmatrix}$.

Example 2. Consider a delayed discrete-time system in (1) with parameters given by

$$E_a = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.12 \end{bmatrix}, E_b = \begin{bmatrix} 0.03 & 0.1 \\ 0.0 & 0.1 \end{bmatrix}, E_c = \begin{bmatrix} 0.02 & 0.00 \\ 0.00 & 0.02 \end{bmatrix}, G = \begin{bmatrix} 0.02 & 0.0 \\ 0.0 & 0.03 \end{bmatrix},$$

A, B, C, A_1, A_2, τ are the same as given in Example 1. One can check that LMI (19) in Theorem 3 and LMI (21) in Theorem 5 are feasible. By the Matlab LMI Toolbox, a feasible solution to the LMI (19) is obtained as follows:

$Q_{11} = \begin{bmatrix} 1.7822 & -0.2396 \\ -0.2396 & 5.5892 \end{bmatrix}$, $Q_{12} = \begin{bmatrix} 0.8793 & 0.7526 \\ -0.1401 & 1.2216 \end{bmatrix}$, $Q_{13} = \begin{bmatrix} -0.1800 & -0.0382 \\ 0.0634 & -0.2237 \end{bmatrix}$, $Q_{22} = \begin{bmatrix} 0.7020 & 0.0102 \\ 0.0102 & 1.8488 \end{bmatrix}$, $Q_{23} = \begin{bmatrix} -0.0537 & -0.0520 \\ -0.1197 & -0.0194 \end{bmatrix}$, $Q_{33} = \begin{bmatrix} 0.1061 & -0.0668 \\ -0.0668 & 0.3084 \end{bmatrix}$, $H = \begin{bmatrix} 1.4907 & -0.4467 \\ -0.4467 & 0.6227 \end{bmatrix}$, $Z = \begin{bmatrix} 0.2147 & 0 \\ 0 & 0.9402 \end{bmatrix}$,

$Q = \begin{bmatrix} 2.6640 & 0 \\ 0 & 2.6640 \end{bmatrix}$, $K = \begin{bmatrix} -2.5494 & 1.8497 \\ 0.1464 & -1.2458 \end{bmatrix}$, $R = \begin{bmatrix} 2.4877 & -0.1559 \\ -0.1559 & 5.5087 \end{bmatrix}$, $P_1 = \begin{bmatrix} 0.7114 & -0.2956 \\ -0.2956 & 0.5605 \end{bmatrix}$,
$P_2 = \begin{bmatrix} 0.4348 & -0.3614 \\ -0.3614 & 1.2885 \end{bmatrix}$.

5. Conclusion

Combined with linear matrix inequality (LMI) technique, the problem of BIBO stabilization for a class of discrete-time delayed control system is investigated. By constructing an augmented Lyapunov-Krasovskii function, some new delay-dependent conditions ensuring BIBO stabilization and robust BIBO stabilization are obtained. Numerical examples show that the new results are valid.

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The Source of Shocks and the Role of Exchange Rate as a Shock Absorber: A Comparative Study in the Crisis-hit East-Asian Countries

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Abstract

Focusing on several crisis-hit East-Asian countries, this paper seeks to uncover the main source of shocks and its link to the performance of policy regime in these countries between the two sub-periods of pre- and post-crisis. A comparative structural VAR analysis is conducted to study the dynamic of shocks. The results show that the economies of East-Asian countries are mainly driven by domestic shocks and shocks are asymmetric. External shocks have low effects on domestic variables but they are increasing over time. Given that real exchange rate reacts stronger to real economy but lower to its own shock, and that the economies tend to experience real depreciation and lower volatility in inflation in the post-crisis period, the results imply more effective policy and greater role of exchange rate to act as a shock absorber under floating exchange rate regimes aftermath the crisis.

Keywords: Exchange rate, Inflation targeting, Asymmetric shocks

1. Introduction

The understanding in the business cycle fluctuations and the economic structures are of emphasized as they provides us the information on the source and the transmission mechanism of shocks which are important in the design of effective monetary policy and also for the evaluations of policy regimes. The nature of shocks or the source of business cycle fluctuations is closely linked to the policy regimes. Economic theory tells us that floating regimes are more feasible in the presence of large external or real shocks as these regimes provide less costly adjustments through relative prices in dealing with such shocks. On the other hand, fixed regimes are preferred under more prominent domestic or nominal shocks (Cavoli & Rajan, 2003). This implies that the nature of shocks is crucial in determining the performance of policy regimes. At the other end, policy regimes could be matter in determining the transmissions and influences of shocks (Desroches (2004) and Hoffmaister *et al.* (1997)).

Focusing on several crisis-hit East-Asian countries, a structural VAR model is applied to analyze the relative importance of various shocks on domestic economy and it has two main objectives. First, it seeks to reveal the main source of shocks in the business cycle fluctuations in these emerging countries. In particular, comparisons of the results are made across countries and over time (pre- and post-crisis periods). Second, this paper also seeks to investigate if exchange rate plays a more effective role as a shock absorber aftermath the crisis or after the change to the floating regimes in East-Asia.

The results indicate that the domestic economies of East-Asian countries are driven mainly by domestic shocks, in particular the supply or real shocks. External shocks only explain a relatively low fraction in the economic fluctuations of this region. However, there is a tendency of higher effects of external shocks on domestic economies in the post-crisis period. Besides, East-Asian countries are subject to asymmetric country specific shocks where the relative importance of shocks differs across countries and the economic variables in these countries react differently to external shocks. Exchange rate is a source of shocks to itself but it is declining in the post-crisis period. The disturbance of exchange rate shock does not transmit largely to the real economy and hence it is not harmful. However, floating exchange rate regimes permit higher volatility in exchange rate and hence greater effects of external shocks on domestic economy of East-Asia. These countries also experience lower volatility in inflation and depreciation in real exchange rate, indicating higher stability in inflation and greater role for exchange rate to act as a shock absorber under the floating regimes aftermath the crisis.

The remainder of the paper proceeds as follows. Section two reviews the changes in the policy regimes in several East-

Asian countries followed by the empirical literature on the relationship between monetary policy and the nature of shocks. Section three discusses the methodology and Section four is about the data. Section five reports the results. Section six discusses the role of exchange rate as a shock absorber. Section seven concludes.

2. Monetary policy framework in East-Asia

The financial crisis of 1997-98 and the collapse of the pegged exchange rate system urged the monetary authorities in these countries to reconstruct the monetary policy frameworks and looked for better alternative implementable policy regimes. Drastic changes in the policy regimes are observed in which most of the crisis-hit East-Asian countries have moved from the rigid exchange rate regime to the flexible one and inflation targeting after the crisis of 1997-98 (see Table 1). According to the IMF classification, Korea has moved from managed floating to independently floating since November 1997 while Philippines remains the same independently floating regime. Thailand has moved from fixed regime to independently floating since July 1997. In contrast, Malaysia moved from managed floating to pegged system for the period from September 1998 to July 2005 and later shifted back to the managed floating regime. Besides, several countries have adopted the inflation targeting regime. The countries that have shifted to the inflation targeting regime include Indonesia, Korea, Philippines and Thailand. Korea was the first country in Asia that has adopted the inflation targeting regime, i.e. in April 1998, followed by Indonesia, in January, 2000, Thailand in May 2000 and later Philippines in January 2002.

The drastic change in the policy regimes in these crisis-hit emerging countries has induced many researches and debates regarding the merits and effectiveness of different policy rules and exchange rate regimes, in particular the role of exchange rate in the monetary policy framework for emerging countries aftermath the crisis. This paper takes a different approach to study the change in the structure of economy and the nature of shocks under two different policy regimes in the pre- and post-crisis in several East-Asian countries.

2.1 The source of shocks and the performances of policy in East-Asia

Empirical studies investigating the business cycle fluctuations and the structure of shocks in emerging countries are closely linked to the study in the topic of optimum currency area (OCA). These studies intend to investigate if the economic structure and the financial and politic aspects in several East-Asian countries fulfill the criteria to form the optimum currency area (OCA) in this region. The results support the formation of OCA if the business cycles or macroeconomics in these countries exhibit some similarities and shocks are symmetric. This paper has no intention to study the criteria of OCA for the region of Asia but it seeks to find the link between the structure and/ or the nature of shocks and the policy performances under different policy regimes in East-Asia. The following summarizes some empirical findings on the structure and the nature of shocks in East-Asia followed by the comparisons in the merits of policy regimes across countries.

Investigating the structural of shocks, most studies do not favor the formation of OCA for East-Asia as majority of these countries exhibit idiosyncratic and country specific shocks ((Chow & Kim (2003), Zhang *et al.* (2004), Sun & An (2008) and Hoffmaister & Rolds (1997)). Besides, East-Asian countries are mainly driven by domestic supply shocks. External shocks only explain a small part of the economic fluctuation in these countries (for example Sun & An (2008) and Hoffmaister & Rolds (1997)). On the other hand, a number of studies find significant correlations of (domestic) shocks in this region and suggest the formation of OCA in a subset of East-Asian countries ((Kwan (1994), Eichengreen & Bayoumi (1996), Bayoumi *et al.* (2000) and Zhang *et al.* (2004)). These studies find that although East-Asian countries differ in economic and financial conditions, the region is not far away from Europe in satisfying certain criteria of OCA. However, more pronounced differences are found in the degree of financial development and the lack of political commitment. Since the preconditions for the sustainable and durable regional arrangement are political criteria, the results do not favor the formation of OCA in Asia. On the other hand, Saucier (2002) finds two groups of Asian countries satisfy the factor mobility and trade criteria for OCA, i.e. Japan and Asian NIEs (Newly Industrialized Economies, i.e. Hong Kong, Korea, Singapore and Taiwan) and China and ASEAN4 (Indonesia, Malaysia, Philippines and Thailand). Focusing the analysis in five ASEAN countries from 1960 to 1996, Ramayandi (2005) finds that aggregate supply shock is not correlated in these countries. However, the results reveal significant correlation when the period of analysis is extended to 2002. On the other hand, aggregate demand shock is correlated with relatively higher magnitude under both samples. He concludes that although ASEAN countries satisfy some preconditions of OCA, it may need a lot of process and longer time to realize the idea. Investigating the business cycle features in Asian and G7 countries, Kim, Kose & Plummer (2000) find that Asian economies are more open in trade than that of G7 economies. At the same time, these economies show higher diversification in their export over time. The investment, export and import ratio to output increase significantly. However, the agricultural sector share is diminishing over time. The trade in total export within the Asian region has increased over time as well. The results also show that the main factors that drive the macroeconomic fluctuations in Asian countries are investment, government spending and consumption.

Apart from these results, some studies are interested to evaluate the merits of policy regimes following drastic shifts in exchange rate regimes in East-Asia after the Asia financial crisis of 1997-98 (for example Desroches (2004) and Edwards (2006)). Investigating the source of macroeconomic fluctuations in 22 emerging countries with different exchange rate regimes, Desroches (2004) demonstrates that exchange rate regime is a critical factor in determining the differences in the transmission mechanism of shocks across emerging countries. Using a sample of 38 developing countries, Hoffmann (2005) seeks to compare to what extent the exchange rate regimes matter in utilizing the role of exchange rate as a shock absorber. His results indicate that economies with floating exchange rate regimes tend to experience real exchange rate depreciation, hence more prominent role for the exchange rate to act as a shock absorber under floating regimes. Edwards & Yeyati (2005) analyze the effect of terms of trade or real shocks on the economic performance under different policy regimes in 183 countries over 1974 to 2000. Their results reveal that flexible exchange rate regimes help to absorb real impact of shocks in emerging and industrial economies. Output also reacts stronger to negative shocks under a pegged exchange rate. Their results also support the view where regime flexibility is positively correlated with output growth. The empirical studies based on East-Asian countries are very limited. Most of the studies are based on the developed economies and European countries. These studies does not directly link the nature of shocks to the policy performances but to evaluate and compare the role of exchange rate as a shock absorber across countries (for example, Drine & Rault (2004), Artis & Ehrmann (2006) and Alexius & Post (2008)).

3. Methodology

Most of the empirical studies on the small open economy framework apply structural vector autoregressive (SVAR) technique as this technique enables analysis on the transmission of shocks through impulse responses and variance decompositions. For simplicity, the domestic economy can be represented by the structural model:

$$A_0X_t = A_1X_{t-1} + \ldots + A_qX_{t-q} + B\varepsilon_t \tag{1}$$

$$X_t = [\Delta y_t^* \quad \Delta i_t^* \quad \Delta p_t^* \quad \Delta y_t^* \quad \Delta r_t \quad \Delta p_t]$$

where A_0 and B are the $(K \times K)$ matrices which indicate instantaneous relationship relations of variables in X_t and ε_t respectively; A_t's are $(K \times K)$ coefficient matrices given (i=1,., q) and ε_t is the vector of structural shocks. There are six endogenous variables, i.e. foreign output/ supply (y_t^*), foreign monetary policy (i_t^*), foreign price/ demand (p_t^*), domestic output/ supply (y_t), real exchange rate (r_t) and domestic demand (p_t). All the variables are in first differenced log term. ε_t consists of six shocks, i.e. foreign supply shocks, foreign policy shocks, foreign demand shocks, domestic supply shocks, real exchange rate shocks and domestic demand shocks. This structural form of equation can be transformed into reduced form by pre-multiplying both sides of variables with A_0^{-1} (see Breitung *et al.* (2004)):

$$X_t = \overline{A}_1X_{t-1} + \ldots + \overline{A}_qX_{t-q} + e_t \tag{2}$$

where $\overline{A}_j = A_0^{-1}A_j$; $(j = 1, \ldots, q)$, $e_t = A_0^{-1}B\varepsilon_t$ and $E[\varepsilon\varepsilon'] = I$.

As in Favero, (2001), equation (1) can be written in a generic form as:

$$[I_K - A(L)]X_t = B\varepsilon_t \tag{3}$$

where $A(L) = \sum_{i=1}^{q} A_iL^i$ and $A_0 = I_K$ in order to be invertible.

By inverting the term $[I_K - A(L)]$, we get the Wold moving average representation of structural form VAR process:

$$X_t = C(L)\varepsilon_t$$

$$X_t = C(L)\varepsilon_t = \Psi_0\varepsilon_t + \Psi_1\varepsilon_{t-1} + \ldots + \Psi_s\varepsilon_{t-s} \tag{4}$$

where $C(L) = [I_K - A(L)]^{-1}B$ and $\Psi_0 = B$

From equation (2), we know that $e_t = A_0^{-1}B\varepsilon_t = B\varepsilon$ given that A_0 is an identity matrix. Then, $\varepsilon_t = B^{-1}e_t$. Substitute this relationship into equation (4) gives:

$$X_t = C(L)\varepsilon_t = C(L)B^{-1}e_t \tag{5}$$

Equation (5) can be written in a Wold representation of the reduced form VAR process:

$$X_t = \Phi_0e_t + \Phi_1e_{t-1} + \ldots + \Phi_se_{t-s} \tag{6}$$

where $\Phi_i = \Psi_iB^{-1}$, $\Phi_0 = I_K$ and $i = 0,\ 1, \ldots$

3.1 Forecast error variance decomposition and impulse response function

The system equation of VAR applies two important tools in analyzing the dynamic effects of structural shocks, namely the forecast error variance decomposition (FEVD) and the (accumulated) impulse response function (IRF). The forecast error variance decomposition gives us the relative importance of shocks on determining the variation in domestic variables. This tool is constructed as the h-step forecast error from the structural innovations (Breitung et al. (2004)):

$$X_{T+h} - X_{T+h|T} = C_0\varepsilon_{T+h} + C_1\varepsilon_{T+h-1} + \ldots + C_{h-1}\varepsilon_{T+1} \quad (7)$$

Denoting the ij-th element of Ψ_n as $\Psi_{ij,n}$, the k-th element of the forecast error vector can be written as:

$$X_{k,T+h} - X_{k,T+h|T} = \sum_{n=0}^{h-1} \psi_{kK,n}\varepsilon_{K,T+h-n} \quad (8)$$

Then, the forecast error variance is constructed as the following with the pre-condition that the structural disturbances are not serially correlated:

$$\sigma_k^2(h) = \sum_{n=0}^{h-1}(\psi_{k1,n}^2 + \ldots + \psi_{kK,n}^2) = \sum_{j=1}^{K}(\psi_{kj,0}^2 + \ldots + \psi_{kj,h-1}^2) \quad (9)$$

The term in bracket of (10) indicates the contribution of variable j to the forecast error variance of variable k for h-step horizon. The contribution in percentage can be obtained in the following way:

$$\varpi_{kj}(h) = (\psi_{kj,0}^2 + \ldots + \psi_{kj,h-1}^2/\sigma_k^2(h)) \quad (10)$$

Another tool that used to interpret the VAR model is the (accumulated) impulse response function. The (accumulated) impulse response function shows the (accumulated) responses of economic variables to a one percent increase in orthogonalized shocks at given time horizons. In order to get the orthogonalized shocks, i.e. when shocks are instantaneously uncorrelated, B is written in a lower triangular matrix such that the variance covariance matrix is $\sum_e = BB'$. The orthogonalized shocks are captured by $\varepsilon_t = B^{-1}e_t$. The impulse responses to orthogonalized shocks may be obtained from equation (4) where $\Psi_i = \Phi_i B$ for $i = 0,\ 1,\ 2, \ldots$. For the accumulated long-run effects of orthogonalized shocks, they are replaced by $C(1) = \Phi B$.

3.2 Identification

In order to identify the structural parameters, we need to impose restrictions on the parameter matrices either through contemporaneous restrictions on the parameter matrices of A_0 and B or long-run restrictions on the total effects of structural shocks. This paper applies the long-run restrictions method proposed by Blanchard & Quah (1989). The long-run restrictions model sets A_0 as an identity matrix, i.e. $A_0 = I_K$. The restrictions are based on the long-run restrictions that imposed on the cumulative impulse response function. Totally $K(K-1)/2$ restrictions are imposed on the lower triangular matrix where some of the structural shocks do not have contemporaneous impacts on the other variables. The long-run impact matrix can be written in the following form:

$$e_t = C(1)\varepsilon_t$$

$$\begin{pmatrix} e_t^{y*} \\ e_t^{i*} \\ e_t^{p*} \\ e_t^{y*} \\ e_t^{r*} \\ e_t^{p*} \end{pmatrix} = \begin{pmatrix} C(1)_{11} & 0 & 0 & 0 & 0 & 0 \\ C(1)_{21} & C(1)_{22} & 0 & 0 & 0 & 0 \\ C(1)_{31} & C(1)_{32} & C(1)_{33} & 0 & 0 & 0 \\ C(1)_{41} & C(1)_{42} & C(1)_{43} & C(1)_{44} & 0 & 0 \\ C(1)_{51} & C(1)_{52} & C(1)_{53} & C(1)_{54} & C(1)_{55} & 0 \\ C(1)_{61} & C(1)_{62} & C(1)_{63} & C(1)_{64} & C(1)_{65} & C(1)_{66} \end{pmatrix} \begin{pmatrix} \varepsilon_t^{s*} \\ \varepsilon_t^{i*} \\ \varepsilon_t^{d*} \\ \varepsilon_t^{s} \\ \varepsilon_t^{r} \\ \varepsilon_t^{d} \end{pmatrix}$$

where C(1) is the long-run matrix of C(L). As the long-run impact matrix is in lower triangular choleski decomposition, the ordering of the variables matters in determining the structure of the shocks. The first variable has impacts on all variables below it but it does not receive any impacts from these variables. The second variable only receives the impacts from the first variable. It does not have any impact on the first variable but it can influence all the variables below it. This rule applies to the all subsequent variables. The foreign variables are ordered before the domestic variables by assuming domestic economy is relatively small and has no impact on the foreign economy but receives the foreign shocks exogenously. The orderings among foreign and domestic variables are based on the standard macroeconomic theory as in Sun & An (2008).

4. Data

The analysis is focused on the crisis-hit East-Asian countries. As most of these countries have shifted to more flexible exchange rate regimes following the financial crisis of 1997-98, it is of interest to investigate how the shift in the policy regimes is linked to the change in the structure and relative effect of shocks in these countries between the two sub-periods. For the purpose of this study, the data is divided into two sub-samples, i.e. before 1997M7 (as pre-crisis period or period I) and from 1999M1 and afterwards (as post-crisis period or aftermath the shift of policy regimes or period II). The foreign country is represented by the US and the domestic country refers to individual East-Asian country. The industrial production index is used as the proxy for output variable (which is seasonally adjusted) as most of the East-Asian countries do not have long enough series for GDP. The monetary policy variable is proxied by the interest rate data. In the case of US interest rate, the Federal Fund Rate (FFR) is used to represent the monetary policy. The real exchange rate is defined as the relative price of non-traded goods (CPI) in terms of traded goods (PPI) as in Desroches (2004) and Hoffmaister & Rolds (1997). This series is constructed as the ratio of domestic CPI over foreign PPI (in term of domestic currency, i.e. foreign PPI series multiplied by nominal exchange rate per USD series):

$$r_t = \frac{CPI_t}{er_t PPI_t^*}$$

The increase in real exchange rate implies appreciation while the decline in it means depreciation. All the data are in monthly and are obtained from the International Financial Statistics (IFS), IMF. Due to the data availability problem, only four East-Asian countries are included in this study. These countries are Korea, Malaysia, Philippines and Thailand. The data span from 1981M1-2008M4 in the case of Korea and Malaysia. Philippines takes the range from 1985M1-2008M4 and Thailand 1987M1-2008M4.

5. Results

The structural VAR system equation is estimated using the data for the two sub-periods (the pre- and post-crisis periods) for each country. All the series are in logarithms (except the interest rate series) in order to capture the percentage change in the variables. Applying the unit-root test of Augmented Dicky-Fuller (ADF) to the two sub-periods sample shows that in most cases, these variables are not stationary in their levels but they are stationary in differenced terms (see Table 2). In order to generate efficient estimators, the system equation is estimated in differenced form (Note 1). The long-run relationship of variables is identified using the Blanchard & Quah technique. Akaike Info Criterion, Final Prediction Error and Schwarz Criterion suggest different length of lags to be included in the analysis of each country. As in Bayoumi & Eichengreen (1994), this paper includes the same length of lags (six lags) in the model for all countries in order to preserve symmetric of specifications (Note 2). The constant and seasonal dummies are assumed in each case. Impulse dummy is considered in case significant impulse or break of series is detected/suggested by the unit-root with structural break test (Note 3).

Before discussing the results, the data for each country are studied. As observed from the statistic for domestic variables (see Table 3), all the four countries in the analysis experience a decline in the change in price or inflation and growth rate in the post-crisis period (Note 4). The data reveal the trade-off between the output growth and the inflation. Although these countries have improved the inflation rate, they face the trade-off in the form of lower output growth. On the other hand, these countries are moving from appreciation in real exchange rate to depreciation (with the exception of Korea). As discussed later, the move from appreciation to depreciation indicates a positive outcome, i.e. exchange rate plays a more effectively role as shocks absorber (see section 5.2).

5.1 Forecast error variance decompositions (FEVD)

The forecast error variance decomposition shows the percentage relative explanatory power of each orthogonalized shock on the variation of each domestic variable. In line with the results reported in previous studies, the business cycle fluctuation (output) in East-Asia is driven by domestic shocks. External shocks explain a relatively low economic fluctuation in these countries. The results hold in the two sub-periods (see Table 4).

Comparing the results of FEVD across countries and over time, it is observed that the relative impact of external shocks is increasing while that of domestic shocks is declining (with the exception of Philippines's real exchange rate). The explanations for this phenomenon include higher trade openness of East-Asian countries, integration in international trades and the moved to more flexible exchange rate regimes which permit greater foreign effects on domestic economy. Previous studies show that the degree of economic integration and the shares of investment, exports and imports in Asia have increased significantly over the last three decades. These countries also exhibit higher degree of trade openness which explains the reason why they are more prone to external shocks (Kim, Kose & Plummer (2003)).

The results also indicate that the main source to the output fluctuations is domestic supply or real shock while the main factors that contribute to changes in real exchange rate are domestic demand and real exchange rate shocks (nominal shocks). Domestic supply and demand shocks are the main determinants to the variation in domestic inflation.

5.2 Impulse response function (IRF)

The (accumulated) impulse response function shows the responses of each variable in the system equation to a positive one standard deviation of each shock. It provides us the information on the size of shocks and also the dynamic of shocks, i.e. how the shocks induce different reactions of economic variables over time.

Table 5 summarizes the results of accumulated IRF (in numerical values) for each domestic variable in response to each shock under the 1^{st}, 6^{th}, 12^{th} and 18^{th} month horizons of several East-Asian countries. Table 7 shows the figures of impulse response function of shocks to the change in real exchange rate. The negative sign for the change in output and price indicate the decline in both variables under a 1% increase of each shock while the negative sign in the response of the change in real change rate implies depreciation. The results show that the output growth and the change in real exchange rate are more sensitive to shocks in compare to the domestic inflation. Both variables are more volatile. The domestic output growth is mainly determined by the domestic supply shock. However, the size of domestic supply shock is declining but the size of external shocks is increasing in the period aftermath the crisis in East-Asia. The change in real exchange rate seems to be more volatile in period II in response to shocks. The increase in the volatility of real exchange rate reflects the abandon of fixed exchange rate regimes and the adoption of flexible exchange rate and inflation targeting regimes in most of the East-Asian countries aftermath the crisis. The shift in the policy regimes and the implementation of inflation targeting in several East-Asian countries aftermath the crisis also help to maintain the low and stable inflation rate. These changes are demonstrated in the results here where the response of inflation to shocks has declined in period II.

Besides experiencing different sizes and impacts of shocks, the economic variables in East-Asia also react differently to shocks between the two sub-periods. An increase in the foreign supply shock leads to the decline in the output growth in the pre-crisis period but it tends to increase the domestic output growth in the post-crisis period (see Table 5). The theoretical prediction is when there is an increase in the foreign supply such as the increase in the foreign productivity; the foreign price tends to be lower. Since domestic price is relatively higher than foreign price, this induces a rise in domestic interest rate and domestic currency appreciates. The opposite outcome is possible if the authority controls the movements in exchange rate. The public will revise the expectation on future interest rate to be lower as they expect the authority to keep the exchange rate target. This leads to the decline in interest rate and domestic currency depreciates.

The increase in the foreign interest rate means lower foreign price. This generates two effects on domestic output and price level (Kim & Roubini (2000)). The domestic economy tends to follow the step of foreign economy by increasing the domestic interest rate not only foreign country is large and has large effect on the small country but it is also to avoid inflationary effect. This leads to lower money supply and hence lower price. Since price is sticky and adjusts slowly, exchange rate shows a jump appreciation and depreciates back to the new equilibrium level. The depreciation leads to higher price and stimulates higher output or production. On the other hand, higher interest rate tends to dampen the demand and leads to the decline in output (Kim & Roubini (2000)). In general, the results show that East-Asian countries tend to experience higher output in the pre-crisis period but lower output in the post-crisis period. Nominal depreciation is translated into real exchange rate depreciation in both periods.

A one percent increase in the foreign demand shock, i.e. the increase in the foreign price means the domestic price is relatively lower than the foreign price. Lower domestic price is associated with lower domestic interest rate and domestic currency depreciates. This later leads to higher domestic price and output. On the other hand, under the exchange rate targeting regime, the public anticipate a rise in the future interest rate as they expect the authority will increase the interest rate in the future to maintain the exchange rate target. The expectation on higher interest rate leads to higher price and domestic currency appreciates. The results demonstrate that the intervention of the authorities to control the exchange rate movements which leads to higher price and lower output (in some cases) in both periods. However, nominal appreciation is translated into real depreciation.

Domestic supply shock, for example the increase in productivity leads to higher output or production and lower domestic price. This causes to lower interest rate and domestic currency is expected to depreciate (Goo (2008)). On the other hand, a positive supply shock may also lead to higher interest rate if the increase in the aggregate demand is greater than the increase in aggregate supply. The price tends to be higher and the exchange rate appreciates. In the pre-crisis period with exchange rate targeting regime, domestic economy tends to experience appreciation as the public anticipates the increase in the expected interest. This leads to higher price and exchange rate appreciates. The results are mixed but changes in nominal exchange rate are translated into real depreciation in period II.

Under a negative exchange rate shock, domestic currency is expected to depreciate. This improves the trade balance and induces higher output. On the other hand, depreciation in domestic currency can lead to a fall in output when prices of import and export adjust faster than the increase in the quantity of trade (Goo (2008)). The results show that a positive exchange rate leads to appreciation in domestic currency with higher price. This causes to lower output.

When there is an increase in the demand on domestic goods, domestic price tends to be higher. This induces higher price and production (higher output). Higher demand also leads to higher interest rate and domestic currency appreciates. However, a positive demand shock can also lead to depreciation in domestic currency, especially in the pre-crisis period. This is due to the anticipation of the market participants that revise expectation on the future interest rate. The decline in the expected interest rate is associated with lower price (Goo (2008)). The results here show that domestic demand shock leads to higher price but its effect on output and real exchange rate is not significant. Overall, the responses of domestic variables to shocks are different across countries and between the two sub-periods. However, the economies of Malaysia, Philippines and Thailand tend to experience real depreciation hitting by external shocks in the post-crisis period. This indicates that exchange rate plays a greater role as a shock absorber in these countries aftermath the crisis or after the shift to floating regime.

5.3 Correlation of shocks

The analysis on the correlation of shocks is conducted by constructing the disturbances of shocks in the system equation. The shocks in East-Asian countries are symmetric if the correlation of shocks is positive among countries but they are asymmetric if the correlation is negative or insignificant (Zhang, Sato & McAleer (2004)). This analysis provides the information on the degree of linkage and similarity in the structural of shocks that faced by the domestic economies of East-Asia.

Table 6 displays the correlations of domestic shocks among East-Asian countries. As observed, the correlations of domestic shocks (domestic supply, real exchange rate and domestic demand shocks) are very low among East-Asian countries. However, the correlation between the real exchange rate and supply shocks show the tendency to increase in the post-crisis period. This may explained by higher regional trade and cooperation among these countries and the move to the same direction in the policy regime, i.e. flexible exchange rate and inflation targeting regimes. As mentioned in Kim, Kose & Plummer (2000 and 2003), Asian countries exhibit higher intra-Asian trade and closer economic cooperation over time.

6. The role of exchange rate: a shock absorber or generator?

The floating exchange rate regime is effective when exchange rate adjusts to external shocks and acts as a shock absorber. The precondition for the exchange rate to act as a shock absorber is changes in nominal exchange rate should be transmitted in real exchange rate changes (Edwards (2006)) (Note 5). However, exchange rate does not always act as a shock absorber. In some circumstances, it even generates larger shocks.

Previous studies show that real exchange rate changes and the effectiveness role of exchange rate as a shock absorber are crucially determined by the source of shocks. Exchange rate has a room for stabilizing and can act as a shock absorber only when an economy experiences asymmetric shocks compare to its trading partner. Therefore, under the existence of asymmetric shocks, the cost of relinquish the exchange rate will be high (Artis & Ehrmann (2006)).

The analysis on the source and the structure of shocks is conducted by imposing identifications on the structural of shocks in the SVAR model. Shocks are categorized as asymmetric (symmetric) when the domestic and foreign interest rates react differently (similarly) to real shocks (Artis & Ehrmann (2006)). Exchange rate plays a significant role as a shock absorber in case it reacts strongly to asymmetric shocks. Exchange rate is a shock generator if it reacts mainly to its own shock. However, if the disturbance does not transmit largely to the real economy, exchange rate is not destabilizing (Alexius & Post (2008) and Artis & Ehrmann (2006)).

The results of impulse response function (see Table 5 and 7) demonstrate that hitting by external shocks, three economies (except Korea) tend to experience real exchange rate depreciation in the post-crisis period. This implies that exchange rate plays a greater role as shocks absorber in these economies under more flexible regime.

The same evidence also found in the results of forecast error variance decomposition for output and real exchange rate. The FEVD for real exchange rate (see Table 4) shows that real exchange rate reacts strongly to its own shock, indicating that exchange rate is a source of shocks to itself. However, the effect of exchange rate shock has declined in the post-crisis period. At the same time, the effect of foreign demand shock and domestic supply shock on this variable have increased, implying some significant of external and domestic real shocks on real exchange rate. Next, I examine the FEVD for output to see how the disturbance of exchange rate is transmitted to the real economy and if output is determined by the same shocks as in real exchange rate. The FEVD for output shows that output in East-Asian countries are driven mainly by the domestic supply or the real shock. Since both output and real exchange rate variables are affected by different main source of shocks, the results reveal asymmetric shocks i.e. exchange rate can act as a shock absorber. On the other hand, the exchange rate shock has low impact on the movements of output in period I but it is increasing in period II after the move to the flexible regimes (with the exception of Thailand). The results indicate that the disturbance of exchange rate transmits lowly to the real economy in period I, hence exchange rate is not destabilizing. However, the effect is increasing in period II especially in Philippines and exchange rate could be destabilizing in this country. The effect of exchange rate shock on domestic inflation is low in all countries. The results of FEVD are confirmed by the results of impulse response function (see Table 5).

Overall, the results indicate that there is a room for exchange rate to act as a shock absorber given that shocks are asymmetric in this region. Although exchange rate is a source of shocks to itself, it is not harmful as the disturbance of exchange rate shock does not transmit largely to the real economy. However, the effect of exchange rate shock on real economy has increased in the post-crisis period. This change is consistent to the shift in exchange rate regime from the more rigid one to the flexible one and inflation targeting in the post-crisis period. As the real exchange rate reacts stronger to output shock and the domestic economies tend to experience real depreciation caused by the external shocks in period II, the results imply more effective monetary policy and greater role of exchange rate to be a shock absorber under the flexible exchange rate regimes aftermath the crisis.

7. Conclusion

This paper conducts a structural VAR analysis to examine the source of business cycle fluctuations in several East-Asian countries. The main focus of this analysis is to compare the relative importance of various shocks, the real exchange rate fluctuation and the structure and transmission of shocks across East-Asian countries in the pre- and post-crisis periods/ after the shift of policy regimes. The results provide information for the selection of more appropriate policy regimes and also for the evaluation of monetary policy.

The results indicate that the main sources to the economic fluctuations in East-Asia are domestic shocks, in particular the supply or real shock. External shocks although explain a relatively low fraction in the economic fluctuations of this region, they show a tendency to increase over time. Therefore, the increasingly relative effect of external shocks in East-Asia is one of the aspects that should not be ignored in the monetary policy framework for (the East-Asia) emerging countries.

The results also show that East-Asian countries are subject to asymmetric country specific shocks. The relative importance of shocks differs across countries and the economic variables in these countries react differently to shocks. However, these countries show more similar results in the post-crisis period, indicating more symmetric in the structure of economics in these countries over time. The results are confirmed by the analysis in the correlation of shocks i.e. the correlation in the structure of domestic shocks across countries is higher in the post-crisis period.

Comparing the results of the two sub-periods with the shift of policy regimes from more rigid to flexible one, it is observed that the move to the flexible and inflation targeting regimes results in higher exchange rate volatility. Exchange rate is a source of shocks to itself but it is not destabilizing since its effect on the real economy is very low. At the same time, inflation targeting regime maintains stability in price with lower volatility in inflation. Besides, the domestic economy tends to experience depreciation in real exchange rate with greater rate in response to external shocks in the period aftermath the shift of the policy regimes. Real exchange rate also reacts stronger to output shock in the post-crisis period. These changes imply more effective monetary policy and greater role of exchange rate as a shock absorber in these countries after the shift to the flexible exchange rate and inflation targeting regimes.

Although the analysis on the structure and the source of shocks indicate greater role of exchange rate to act as a shock absorber under flexible regimes in East-Asia, there are some circumstances that the flexible regimes may not work well. For instance, countries that hold large foreign currency denominated liabilities may experience higher debt when the currency depreciates. This may leads to bankruptcies and reduction in the economic growth (Eichengreen & Hausmann (1999)). Besides, Devereux (2004) also suggests the choice for the peg regime for emerging countries given that international financial market is imperfect. Under the imperfect international financial market, floating regimes do not generate welfare maximization as they do not entail output to react efficiently to external demand shock. Hence, besides the source of shocks, the policy maker should consider other aspects (for example the structure of financial market and exchange rate pass-through) in making the choice of policy regimes.

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Notes

Note 1. The estimation applies top-down subset restrictions to improve the outcome, i.e. using an elimination algorithm to conduct a checking procedure from the last regressor to see if exclude this term in the equation improves the criterion value. If yes, the regressor will be eliminate and the process is continue to check the second last regressor and so on.

Note 2. Applying different number of lags in the model does not change the main results. This paper follows previous studies on monthly data to include six lags in the model, for example Kim & Roubini (2000) and Artis & Ehrmann (2006).

Note 3. Impulse dummies are assumed for the following cases: Philippines, period II (2000M1 and 2000M10) and Malaysia, period I (1984M3 and 1988M4).

Note 4. However, Thailand experiences a small increase in growth rate.

Note 5. Real depreciation generates relative price effects where domestic price is cheaper than foreign price. This leads to higher demand on domestic goods and improve the balance sheet of domestic country.

Table 1. Monetary Policy Framework

No	Countries	Monetary Policy Framework
1	Korea	Three main periods: 1. Monetary targeting Since 1957, M1 was pre-announced quarterly or yearly as a macroeconomics policy. In 1979, monetary target changed to a M2 growth rate till mid 1990s After crisis 1997-98, accepted IMF rescue financing plan, used M3 as reference value of monetary base, at the same time, adopted inflation targeting (two pillar system) In 2001, M3 growth rate only monitored, and the monitoring ended in 2003 with a pure inflation targeting 2. Interest rate as an operational target After 1997-98, the interest rate was accepted as an operational target. Since 1999, Monetary Policy Committee (MPC) announced the target call rate for interest rate. 3. Inflation targeting Since 2000, core CPI inflation rate has been chosen as the benchmark inflation indicator. The target rate is determined annually with the range of +/-1%. Official exchange rate regimes: 1. March 1980-October 1997----Managed floating 2. November 1997-present----Independently floating
2	Malaysia	Official exchange rate regimes: 1. January 1986-February 1990----Limited flexibility 2. March 1990-November 1992----Fixed 3. December 1992-September 1998----Managed floating 4. September 1998-July 2005----Pegged arrangement 5. July 2005-present ----Managed floating
3	Philippines	Two periods: 1. Monetary targeting In the past, monetary policy framework based on base or reserve money programming. 2. Inflation targeting (2002 onwards) Inflation targeting policy adopted formally in January 2000 and the implementation started in January 2002. CPI or headline inflation is used as its monetary policy target and overnight repurchase rate and reverse repurchase rate are used as the main instrument of monetary policy. Official exchange rate regimes: 1. January 1988-present----Independently floating
4	Thailand	Three main periods: 1. Pegged exchange rate regime (2nd World War-June 1997) The value of Baht was pegged to a major currency/ gold or to a basket of currencies 2. Monetary targeting regime (July 1997-May 2000) Beginning the periods of floating exchange rate. Received assistance from IMF, targeted at domestic money supply. Set daily and quarterly monetary base targets. 3. Inflation targeting regime (May 2000-present) Inflation targeting is more effective as the relationship between money supply and output growth was becoming less stable after financial crisis. Official exchange rate regimes: 1. January 1970-June 1997-----fixed 2. July 1997-present----Independently floating

Table 2. Augmented Dicky-Fuller (ADF) Unit-root test

Variable	Korea		Malaysia		Philippines		Thailand		US	
	Period I	Period II	Period I	Period II	Period I	Period II	Period I	Period II	Period I	Period II
y_t^*									-2.7714	-1.4200
Δy_t^*									-5.0909***	-10.4592***
i_t^*									-2.0925	-2.1286
Δi_t^*									-3.6192***	-2.9074**
p_t^*									-1.8934	-1.1419
Δp_t^*									-6.7386***	-8.6924***
y_t	-2.1331	-3.7930**	-2.3829	-3.1423*	-2.0635	-2.1313	-1.2068	-2.3300		
Δy_t	-21.8048***	-11.2026***	-21.8043***	-6.3390***	-9.5618***	-8.3034***	-12.9193***	-6.8498***		
r_t	-0.5716	-1.7960	-0.8499	-1.9419	0.1314	-1.9146	0.0096	-1.9675		
Δr_t	-9.7418***	-9.2745***	-12.7499***	-9.4366***	-13.7909***	-9.8324***	-8.8962***	-9.7887***		
p_t	-1.1804	-2.6683	-1.3682	-1.1688	-2.2560	-2.0930	-1.9543	-0.9396		
Δp_t	-9.3978	-9.3256***	-11.4576***	-11.5337***	-6.9011***	-4.9143***	-9.3439***	-6.3199***		

Notes: All the series are in log form except the series for interest rate. The series for the level of y^*, y, p_t and p_t are assumed to have constant term, trend and seasonal dummies while the differenced of these variables are assumed to exhibit constant and seasonal dummies. The other variables are assumed to have constant term. The selection on the number of lags for each series is based on the number suggested by Akaike Info Criterion, Final Prediction Error and Schwarz Criterion.

Δ denotes the first differenced operator. $***$ denotes the significant statistic at 1% level; $**$ the significant statistic at 5% level and * the significant statistic at 10% level.

Table 3. Descriptive statistics

	Korea		Malaysia		Philippines		Thailand	
	Period I	Period II	Period I	Period II	Period I	Period II	Period I	Period II
Δp_t								
mean	0.0044	0.0025	0.0027	0.0017	0.0093	0.0039	0.0036	0.0022
std. dev.	0.0053	0.0041	0.0040	0.0027	0.0123	0.0058	0.0051	0.0042
Δy_t								
mean	0.0082	0.0069	0.0079	0.0054	0.0105	0.0060	0.0071	0.0079
std. dev.	0.0348	0.0397	0.0630	0.0367	0.0501	0.0510	0.0381	0.0321
Δr_t								
mean	0.0014	0.0002	0.0005	-0.0006	0.0014	-0.0068	0.0010	-0.0004
std. dev.	0.0103	0.0240	0.0127	0.0120	0.0331	0.0232	0.0166	0.0206

The variables are in first differenced log form. The data are in monthly and spanning from 1980's-1997M6 (period I) and 1999M1-2008M4 (period II). A one percentage is equivalent to 0.01.

Table 4. Forecast Error Variance Decomposition (FEVD)

	Period I								Period II							
	FEVD for the change in output								FEVD for the change in output							
	ε_t^{s*}	ε_t^{i*}	ε_t^{d*}	ε_t^{s}	ε_t^{r}	ε_t^{d}	Total (f)	Total (d)	ε_t^{s*}	ε_t^{i*}	ε_t^{d*}	ε_t^{s}	ε_t^{r}	ε_t^{d}	Total (f)	Total (d)
Korea	0.01	0.00	0.00	0.98	0.00	0.01	0.01	0.99	0.05	0.00	0.00	0.54	0.03	0.38	0.05	0.95
	0.03	0.02	0.06	0.86	0.01	0.01	0.11	0.88	0.05	0.05	0.09	0.29	0.10	0.42	0.19	0.81
Malaysia	0.02	0.05	0.01	0.92	0.00	0.00	0.08	0.92	0.24	0.03	0.01	0.60	0.10	0.02	0.28	0.72
	0.06	0.05	0.04	0.83	0.02	0.00	0.15	0.85	0.20	0.05	0.05	0.51	0.10	0.09	0.30	0.70
Philippines	0.00	0.00	0.10	0.87	0.02	0.00	0.10	0.89	0.03	0.00	0.24	0.42	0.25	0.06	0.27	0.73
	0.04	0.02	0.12	0.71	0.07	0.04	0.18	0.82	0.08	0.10	0.17	0.22	0.31	0.11	0.35	0.64
Thailand	0.00	0.03	0.00	0.93	0.04	0.01	0.03	0.97	0.02	0.01	0.00	0.96	0.00	0.00	0.03	0.96
	0.03	0.07	0.03	0.54	0.23	0.10	0.13	0.87	0.09	0.13	0.03	0.64	0.03	0.09	0.25	0.76
	FEVD for the change in real exchange rate								FEVD for the change in real exchange rate							
	ε_t^{s*}	ε_t^{i*}	ε_t^{d*}	ε_t^{s}	ε_t^{r}	ε_t^{d}	Total (f)	Total (d)	ε_t^{s*}	ε_t^{i*}	ε_t^{d*}	ε_t^{s}	ε_t^{r}	ε_t^{d}	Total (f)	Total (d)
Korea	0.01	0.01	0.01	0.01	0.97	0.00	0.03	0.98	0.15	0.04	0.25	0.12	0.45	0.00	0.44	0.57
	0.02	0.08	0.01	0.05	0.82	0.02	0.11	0.89	0.16	0.20	0.18	0.10	0.34	0.03	0.54	0.47
Malaysia	0.03	0.00	0.02	0.00	0.90	0.04	0.05	0.94	0.00	0.00	0.05	0.19	0.72	0.04	0.05	0.95
	0.03	0.01	0.02	0.01	0.85	0.08	0.06	0.94	0.06	0.04	0.13	0.18	0.51	0.08	0.23	0.77
Philippines	0.38	0.04	0.03	0.00	0.52	0.02	0.45	0.54	0.06	0.07	0.08	0.42	0.35	0.02	0.21	0.79
	0.32	0.10	0.05	0.04	0.46	0.05	0.47	0.55	0.12	0.13	0.13	0.31	0.27	0.03	0.38	0.61
Thailand	0.01	0.00	0.00	0.12	0.86	0.01	0.01	0.99	0.01	0.12	0.16	0.01	0.69	0.01	0.29	0.71
	0.03	0.02	0.02	0.17	0.74	0.02	0.07	0.93	0.20	0.12	0.16	0.02	0.45	0.04	0.48	0.51
	FEVD for the change in price								FEVD for the change in price							
	ε_t^{s*}	ε_t^{i*}	ε_t^{d*}	ε_t^{s}	ε_t^{r}	ε_t^{d}	Total (f)	Total (d)	ε_t^{s*}	ε_t^{i*}	ε_t^{d*}	ε_t^{s}	ε_t^{r}	ε_t^{d}	Total (f)	Total (d)
Korea	0.11	0.12	0.06	0.04	0.20	0.47	0.29	0.71	0.18	0.02	0.00	0.38	0.01	0.41	0.20	0.80
	0.11	0.13	0.08	0.05	0.19	0.44	0.32	0.68	0.15	0.07	0.03	0.31	0.07	0.36	0.25	0.74
Malaysia	0.02	0.03	0.00	0.00	0.09	0.86	0.05	0.95	0.01	0.00	0.02	0.36	0.06	0.54	0.03	0.96
	0.03	0.05	0.07	0.02	0.14	0.69	0.15	0.85	0.04	0.04	0.08	0.32	0.10	0.42	0.16	0.84
Philippines	0.07	0.01	0.01	0.01	0.31	0.59	0.09	0.91	0.00	0.03	0.06	0.00	0.10	0.80	0.09	0.90
	0.22	0.05	0.18	0.01	0.18	0.36	0.45	0.55	0.14	0.07	0.32	0.07	0.07	0.34	0.53	0.48
Thailand	0.04	0.17	0.00	0.00	0.26	0.53	0.21	0.79	0.15	0.00	0.55	0.00	0.29	0.00	0.70	0.29
	0.04	0.15	0.01	0.03	0.34	0.43	0.20	0.79	0.24	0.03	0.40	0.01	0.20	0.13	0.67	0.34

The numerical figures show the 1st and 20th horizons of variance decompositions; the column total (f) shows the total FEVD for foreign/ external shocks while the column total (d) shows the FEVD for domestic shocks

Table 5. The size of shocks under Impulse Response Functions (IRF)

	Period I Output growth						Period II Output growth					
Korea	ε_t^{s*}	ε_t^{i*}	ε_t^{d*}	ε_t^{s}	ε_t^{r}	ε_t^{d}	ε_t^{s*}	ε_t^{i*}	ε_t^{d*}	ε_t^{s}	ε_t^{r}	ε_t^{d}
	-16	40	-28	143	10	30	45	8	16	31	44	-9
	-9	40	-15	164	6	0	125	-79	47	98	-11	9
	1	44	-28	169	2	3	98	-65	20	85	-5	-16
	6	39	-27	166	1	-2	82	-90	24	99	11	-2
Malaysia												
	-3	55	59	243	-47	7	110	30	27	15	7	-10
	141	64	22	267	28	5	221	21	14	89	0	-30
	164	61	0	277	9	5	184	-12	-22	108	10	6
	170	63	-11	276	2	2	166	-39	-12	106	3	4
Philippines	-42	-21	79	196	-23	-3	-16	37	113	82	-43	-22
	-23	43	55	147	-13	-8	4	-2	127	139	-53	22
	-10	63	64	167	0	-8	23	39	94	112	17	-13
	-14	64	72	164	0	0	36	31	84	106	-5	3
Thailand	-10	-2	15	240	38	44	30	-7	15	50	10	13
	-40	0	27	158	-54	-4	64	-7	-18	58	0	2
	-71	7	56	188	-17	3	110	-24	-33	69	2	-9
	-87	-1	62	188	-6	-1	71	-23	-15	55	-3	-7
	The change in real exchange rate						The change in real exchange rate					
Korea	ε_t^{s*}	ε_t^{i*}	ε_t^{d*}	ε_t^{s}	ε_t^{r}	ε_t^{d}	ε_t^{s*}	ε_t^{i*}	ε_t^{d*}	ε_t^{s}	ε_t^{r}	ε_t^{d}
	-9	12	-9	5	106	9	59	46	-94	-38	135	8
	-23	58	-5	36	119	20	153	182	-44	-44	137	2
	-15	68	-2	42	129	3	127	188	-43	-34	172	0
	-14	79	-5	46	127	2	72	142	-40	-28	186	-9
Malaysia	21	-9	-19	-4	116	-10	-19	-14	-53	-46	93	-18
	9	-13	-8	3	134	1	-51	-27	-62	-54	55	-2
	9	-13	-4	-1	132	-1	-72	-18	-47	-70	71	-2
	10	-14	-4	0	133	0	-71	-8	-43	-73	74	0
Philippines	127	-14	-27	-10	123	-10	42	-46	-44	-109	101	-15
	144	-25	-7	19	173	2	-47	-124	45	-106	96	-9
	149	-27	-5	14	167	-1	-79	-140	74	-116	115	-3
	148	-28	-3	16	168	0	-95	-153	95	-115	117	-2
Thailand	-5	-6	-3	17	76	5	-20	-57	-65	13	136	-17
	5	-17	-11	-22	44	-6	-22	-76	-62	-4	140	-22
	-1	-16	-5	-9	50	2	-133	-88	-35	5	137	-5
	0	-14	-7	-11	51	1	-160	-104	-38	2	147	4
	The change in price/ inflation						The change in price/ inflation					
Korea	ε_t^{s*}	ε_t^{i*}	ε_t^{d*}	ε_t^{s}	ε_t^{r}	ε_t^{d}	ε_t^{s*}	ε_t^{i*}	ε_t^{d*}	ε_t^{s}	ε_t^{r}	ε_t^{d}
	-15	-16	11	-9	20	31	-8	-2	5	-14	-3	17
	-22	-18	21	-7	18	34	-14	-9	4	-8	-2	9
	-25	-22	24	-7	20	35	-15	-10	4	-11	-4	12
	-26	-22	26	-7	21	35	-13	-9	2	-10	-4	11
Malaysia	2	5	7	-2	10	30	0	5	3	-10	4	12
	-1	-1	18	4	20	37	4	8	10	-14	7	13
	0	-1	22	3	19	38	4	10	11	-14	8	13
	1	-1	22	3	20	39	4	10	11	-13	8	14
Philippines	-25	1	13	1	27	46	0	-5	16	7	3	22
	-69	24	60	12	21	46	-24	-14	44	11	3	25
	-90	34	82	7	20	58	-23	-19	50	9	6	26
	-106	39	99	9	19	63	-19	-19	52	7	7	27
Thailand	-7	-13	1	0	16	23	-15	4	27	0	19	11
	-9	-8	3	-2	6	25	-21	-3	40	-3	15	27
	-5	0	2	0	8	24	-4	-4	28	-3	19	24
	-3	2	1	-2	8	23	-3	-1	27	-3	17	20

The numerical figures show the 1^{st}, 6^{th}, 12^{th} and 18^{th} month responses of domestic variables to the 1% or 0.01 increase in the standard deviation of various orthogonalized shocks. The numerical responses are indicated in ($x10^{-4}$).

Table 6. Correlations of domestic shocks across East-Asian countries

	Period I Korea	Mal	Phil	Thai		Period II Korea	Mal	Phi	Thai
	Domestic supply shock					**Domestic supply shock**			
Korea	1.0000				Korea	1.0000			
Malaysia	-0.0542	1.0000			Malaysia	0.0938	1.0000		
Philippines	0.1595	-0.0805	1.0000		Philippines	-0.0854	0.1357	1.0000	
Thailand	0.2506	0.0027	-0.0218	1.0000	Thailand	0.2924	0.3250	0.2586	1.0000
	Exchange rate shock					**Exchange rate shock**			
Korea	1.0000				Korea	1.0000			
Malaysia	0.0209	1.0000			Malaysia	0.4594	1.0000		
Philippines	0.1521	0.1180	1.0000		Philippines	0.3720	0.4828	1.0000	
Thailand	0.2093	0.2849	0.0864	1.0000	Thailand	0.5142	0.3329	0.4046	1.0000
	Domestic demand shock					**Domestic demand shock**			
Korea	1.0000				Korea	1.0000			
Malaysia	0.0606	1.0000			Malaysia	-0.2653	1.0000		
Philippines	0.0968	0.0965	1.0000		Philippines	0.0807	-0.1673	1.0000	
Thailand	0.0269	0.0274	0.1691	1.0000	Thailand	0.0672	0.0018	0.1895	1.0000

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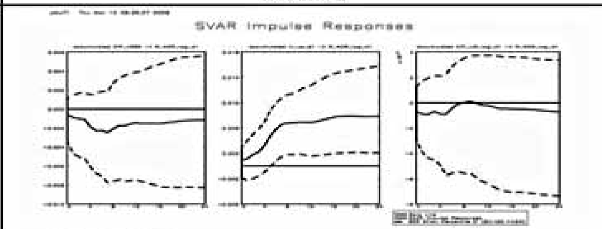
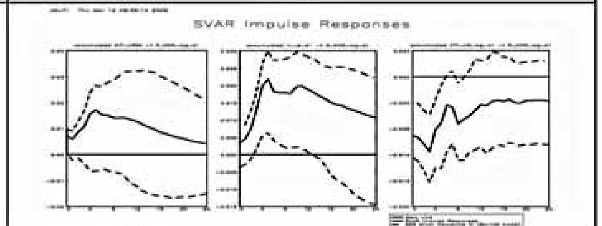
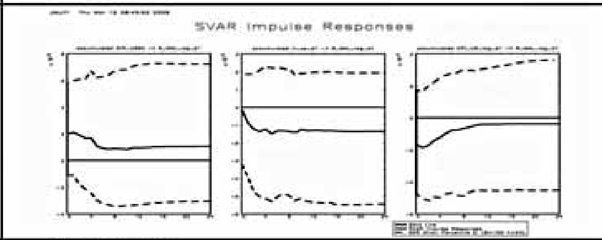
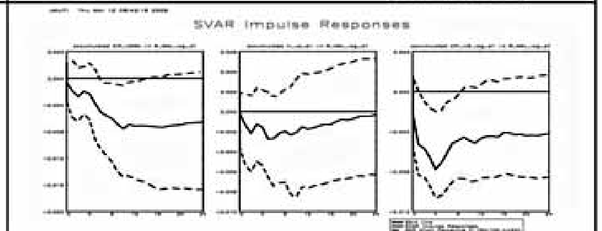
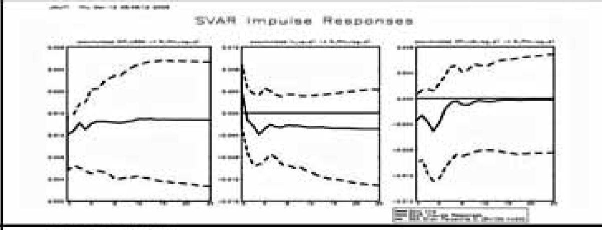
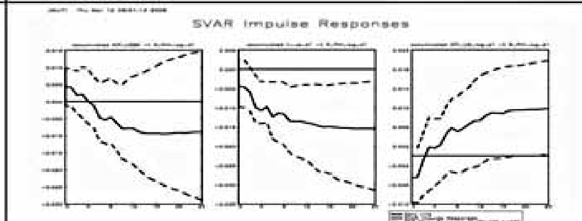
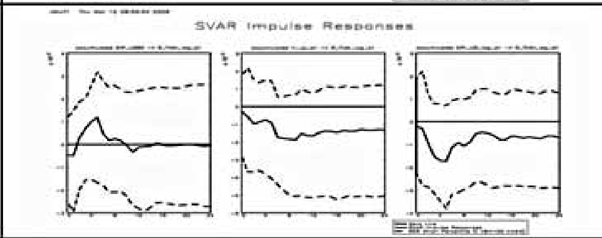
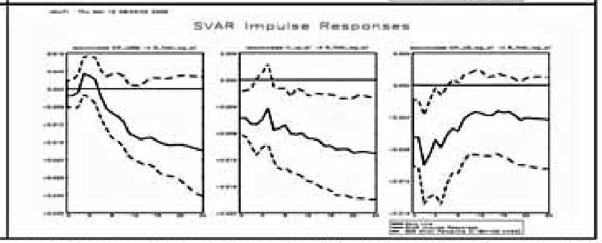
Table 7. Impulse response functions – Responses of real exchange rate to external shocks

Country	Period I	Period II
Korea		
Malaysia		
Philippines		
Thailand		

The table shows the impulse response functions of real exchange rate under a one percentage increase in orthogonalized foreign supply, foreign monetary policy and foreign demand shocks (from left to right).

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Application of Radial Point Interpolation Method to Temperature Field

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Abstract

A point interpolation which bases on the radial function is a new meshless method. It is advantageous over the original PIM with polynomial basis in avoiding singularity when shape functions are constructed. It is also easy to deal with essential boundary for its property of Kronecher Delta function. To verify it's valid, this paper introduced the basic principle of RPIM. In addition, numerical example of heat conduction showed that the new methods possessed several advantages, such as high efficiency, high accuracy, and high stability. It is a promising method in physics.

Keywords: Meshless methods, Point interpolation methods, Radial basis functions, Radial point interpolation method

1. Introduction

Meshless method is a new numerical analysis method which has rapidly developed in recent years (Zhang Xiong and Liu Yan, 2004). Because it bases of nodes information and thoroughly or partially eliminates the grids, and it also has high efficiency, easy to implement, therefore it is more flexible and effective in dealing with large deformation, high-gradient and other advantages than the traditional finite element.

Recent meshless approximations mainly have the following types of programs (LIU G.R, 2007), element-free Galerkin method (EFGM), local Peter-Galerkin method (MLPG), Point Interpolation (PIM) and so on, but mostly shape functions don't have Kronecker Delta function property, which makes the essential boundary conditions hard to deal with. Radial Point Interpolation Method (RPIM) is a new meshless method (XIONG Yuan-bo, LONG Shu-yao and LIU Kai-yuan, 2007, p135-138. XIA Mao-hui , JIA Yan and LIU Cai, 2006, p112-117), its shape function is constructed by the combination of radial and polynomial basis functions. Because it has the property of delta functions, so it is convenient to implement the boundary conditions. At the same time its interpolation process is very similar to the finite element, so a lot of finite element procedures can be applied directly. Select the appropriate shape parameters is the key to Radial point interpolation method, which is usually determined by empirical formula.

At present, meshless method appling to the temperature field mainly use EFGM(GAO Zhi-hua, ZHANG Ming-yi and LIU Zhi-qiang, 2006, p545-550. YUAN Su-ling, GE Yong-qing and WANG Zhang-qi, 2003, p82-86) and MLPG (LI Qing-hua, CHEN Shen-shen and XIONG Yong-gang, 2006, p22-24). This article introduced the basic theory of the radial point interpolation (RPIM), and used this method to construct the interpolation function, and applied it to two-dimensional steady-state temperature field, and example verified that RPIM is a less time-consuming and high precision, simple and effective computing method.

2. Point Interpolation Method

Consider a scalar function $u(x)$ defined in problem domain Ω, set several nodes in and on the domain randomly, use total of n field nodes included in the local support domain Ω of the point of interest at x_q to interpolate, and the $u(x)$ at x_q is approximated in the form of

$$u^h(x, x_q) = \sum_{i=1}^{n} R_i(x)a_i + \sum_{j=1}^{n} P_j(x)b_j = R^T(x)a + P^T(x)b \tag{1}$$

Where $R_i(x)$ is a radial basis function, $P_j(x)$ is monomial in the space coordinates $x^T = \{x, y\}$, m is the number of polynomial basis functions. Coefficients a_i, b_j are constants yet to be determined. We also choose $m < n$ to have better stability of the interpolating function. In two-dimensional problems, the general linear-based $P^T(x) = [1, x, y]$ is used.

For given x, we have

$$a = [a_1, a_2, \cdots, a_n]^T$$
$$b = [b_1, b_2, \cdots, b_m]^T$$
$$R^T(x) = [R_1(x), R_2(x), \cdots, R_n(x)] \tag{2}$$
$$P^T(x) = [P_1(x), P_2(x), \cdots, P_m(x)]$$

Typically, in two-dimensional problems

$$R_i(x) = R_i(r_i) = R_i(x, y) \tag{3}$$

$$r_i(x) = [(x - x_i)^2 + (y - y_i)^2]^{1/2} \tag{4}$$

Now enforcing equation (1) to be satisfied at nodes to determine the coefficients a_i, b_j, the matrix form is

$$U_S = R_Q a + P_m b \tag{5}$$

Where $U_S = [u_1, u_2, \cdots, u_n]^T$, Matrix R_Q is given by

$$R_Q = \begin{bmatrix} R_1(x_1) & R_2(x_1) & \cdots & R_n(x_1) \\ R_1(x_2) & R_2(x_2) & \cdots & R_n(x_2) \\ \vdots & \vdots & \vdots & \vdots \\ R_1(x_n) & R_2(x_n) & \cdots & R_n(x_n) \end{bmatrix} \tag{6}$$

The matrix P_m is a $n \times m$ matrix given by

$$P_m = \begin{bmatrix} P_1(x_1) & P_2(x_1) & \cdots & P_m(x_1) \\ P_1(x_2) & P_2(x_2) & \cdots & P_m(x_2) \\ \vdots & \vdots & \vdots & \vdots \\ P_1(x_n) & P_2(x_n) & \cdots & P_m(x_n) \end{bmatrix} \tag{7}$$

However, there are $n + m$ variables in equation (5), but only have n equations, so it is an undetermined equations, solving the above equation (5) needs to impose a constraint equation

$$P_m^T a = 0 \tag{8}$$

Combing equations (5) and (8), the matrix form becomes

$$\begin{bmatrix} R_Q & P_m \\ P_m & 0 \end{bmatrix} \begin{Bmatrix} a \\ b \end{Bmatrix} = \begin{Bmatrix} U_S \\ 0 \end{Bmatrix} \tag{9}$$

Solving equation (9), we can obtain

$$b = S_B U_S$$

$$a = S_a U_s \tag{10}$$

Where $S_b = [P_m^T R_Q^{-1} P_m]^{-1} P_m R_Q^{-1}, S_a = R_Q^{-1} - R_Q^{-1} P_m S_b$.

Substituting a, b back into equation (1), we obtain

$$u^h(x, x_q) = [R^T S_A + P^T S_b] U_S = \sum_{i=1}^{n} \Phi_i(x) u_i = \Phi(x) U_S \tag{11}$$

Where shape function $\Phi(x)$ is given by

$$\Phi(x) = [\Phi_1(x), \Phi_2(x), \cdots, \Phi_n(x)] \tag{12}$$

The derivatives of shape functions can be easily obtained as

$$\begin{cases} \frac{\partial \Phi_k}{\partial x} = \sum_{i=1}^{n} \frac{\partial R_i}{\partial x} S^a_{ik} + \sum_{j=1}^{n} \frac{\partial P_j}{\partial x} S^b_{jk} \\ \frac{\partial \Phi_k}{\partial y} = \sum_{i=1}^{n} \frac{\partial R_i}{\partial y} S^a_{ik} + \sum_{j=1}^{n} \frac{\partial P_j}{\partial y} S^b_{jk} \end{cases} \tag{13}$$

Here are three often used globally supported radial basis functions:

(1)Multi-quadrics(MQ):

$$R_i(x) = (r_i^2 + c^2)^q = [(x - x_i)^2 + (y - y_i)^2 + c^2]^q \qquad c > 0 \tag{14}$$

(2)Gaussian(EXP):

$$R_i(x) = exp(-cr^2) = exp(-c[(x - x_i)^2 + (y - y_i)^2]) \tag{15}$$

(3)Thin plate spline:

$$R_i(x) = r_i^\eta = [(x - x_i)^2 + (y - y_i)^2]^\eta \qquad \eta \in N \tag{16}$$

In above equations, c, q, η are all called shape-parameters, $r = \|x - x_i\|$. Normally, the choosing of these parameters will effect the result. J.G. Wang and G.R. Liu have discussed in their papers. They also discovered that $q = 1.03, c = 1.42$ performs best in MQ. We will use this result in the following example, the derivative of MQ function is:

$$\begin{cases} \frac{\partial R_i}{\partial x} = 2q(r_i^2 + c^2)^{q-1}(x - x_i) \\ \frac{\partial R_i}{\partial y} = 2q(r_i^2 + c^2)^{q-1}(y - y_i) \end{cases} \tag{17}$$

3. Discretized equation of Temperature field

Consider the issue of two-dimensional steady-state temperature field:

$$\begin{cases} \frac{\partial}{\partial x}(k_x \frac{\partial \Phi}{\partial x}) + \frac{\partial}{\partial y}(k_y \frac{\partial \Phi}{\partial y}) + \rho Q = 0 & \text{in the } \Omega \\ \Phi = \overline{\Phi} & \text{on the } \Gamma_\varphi \\ k_x \frac{\partial \Phi}{\partial x} n_x + k_y \frac{\partial \Phi}{\partial y} n_y = \overline{q} & \text{on the } \Gamma_q \end{cases} \tag{18}$$

Where Φ indicates temperature, Γ_φ is the Dirichlet border, Γ_q is Neumann border, ρ is material density, k_x is the thermal conductivity coefficient along the direction of x, k_y is the thermal conductivity material coefficient along the direction of y, Q is the object density of internal heat source, n_x and n_y is the boundary normal direction cosine.

The standard variational (weak) form of equation (18) is posed as follows:

$$F(A) = \int_\Omega [\frac{1}{2} k_x (\frac{\partial \Phi}{\partial x})^2 + \frac{1}{2} k_y (\frac{\partial \Phi}{\partial y})^2 - \Phi \rho Q] d\Omega - \int_{\Gamma_q} \Phi \overline{q} d\Gamma \tag{19}$$

Substituting equation (11) back into equation (19), we can obtain

$$K\Phi = F \tag{20}$$

Where

$$K_{IJ} = \int_\Omega (k_x N_{I,x} N_{J,x} + k_y N_{I,y} N_{J,y}) d\Omega$$

$$F_I = \int_\Omega N_I (\rho Q) d\Omega + \int_{\Gamma_q} N_I \overline{q} d\Gamma$$

4. Numerical example

As shown in Figure 1, 5×5 square region, given constant temperature $\Phi = 0$°C on the edge of $x = 0, x = 5$ and $y = 0$, along the edge of $y = 5$ set a constant temperature of $\Phi = 10$°C, with no heat source, thermal conductivity coefficients $k_x = k_y = 1$. RPIM used to calculate the temperature distribution, the problem domain is represented by 225(15×15) regularly distributed nodes, 14×14 rectangular Gaussian background cells are used for numerical interactions. In each background cell, 3×3 Gaussian points are employed. Compare RPIM solution to the exact solution on the cross section $x = 2.5$, Figure 2 gives the trends of cross section on the meshless solution , exact solution and FEM solution. Through figure 2 we can see that the meshless results are basically consistent with the exact solution, which verified the accuracy and effectiveness of RPIM.

5. Conclusions

(1). Meshless method only needs nodes of information and boundary conditions, which cast off the restricted units of the finite element method and decrease the work of finite element method in the complex mesh generation and re-dividing.

(2). RPIM method is more advanced than the element-free Galerkin method based on the mobile least squares, as long as R_Q^{-1} determined, shape function and its derivative will be able to determined. Shape function possess δ function characteristics, it is easier to handle essential boundary conditions.

(3). This paper attempted to promote this method to the temperature field problem, example results show that using this method to deal with the issue of temperature field receives satisfactory results, which further validated the meshless method RPIM is accuracy and effectiveness.

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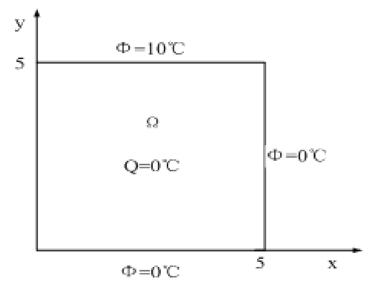

Figure 1. Heat conduction model of two-dimension

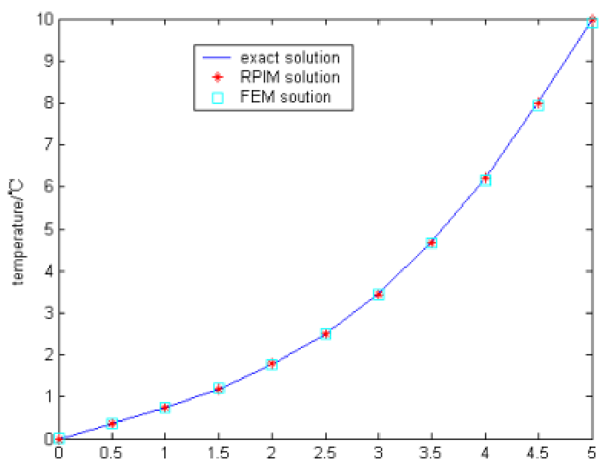

Figure 2. The change of temperature along x = 2.5

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The Algebraic Construction of Commutative Group

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Abstract

The construction of the integers introduced by Dedekind is an algebraic one. Subtraction can not be done without restriction in natural numbers N. If we consider the definition of multiplication of integral domain Z, N with respect to subtraction is needed. It is necessary to give the definition of subtraction in N. Instead of starting from natural numbers, one could begin with any commutative semi-group and construct from it as the construction of the integers to obtain a commutative group. If the cancellation law does not hold in the commutative semi-group, some modifications are required. The mapping from the commutative semi-group to the commutative group is not injective and compatible with addition. In the relation between real numbers and decimals, N also plays an important role.

Keywords: Well-defined, Equivalence relation, Commutative group, Cancellation law, Injective, Compatible, Archimedean property

1. The construction and application of subtraction of natural numbers

1.1 Subtraction of natural numbers N

Definition $\quad a = b - c \Longleftrightarrow a + c = b. \quad \forall a, b, c \in N.$

If $a = b - c$ and also $a' = b - c$, then $a + c = b$ and $a' + c = b$. And we have $a = a'$ from $a + c = a' + c$, according to the cancellation law of N. Hence, subtraction of N is well-defined.

Besides, commutative law, association law and distribution law with respect to subtraction of N are satisfied.

Commutative law: $a = b - c \Leftrightarrow a + c = b. \quad a' = c - b \Leftrightarrow a' + b = c$. We have $a' + (a + c) = c \Rightarrow a' + a = 0$.
Namely, $(c - b) + (b - c) = 0, (c - b) = -(b - c)$.

Association law: $a = b-c \Leftrightarrow a+c = b. \Rightarrow d+b = d+(a+c) \Rightarrow d+b = (d+a)+c \Rightarrow d+a = (d+b)-c \Rightarrow d+(b-c) = (d+b)-c$.

Distribution law: $a = b - c \Leftrightarrow a + c = b. \Rightarrow d(a + c) = db \Rightarrow da + dc = db \Rightarrow da = db - dc \Rightarrow d(b - c) = db - dc$.

Similarly, $a = b - c \Leftrightarrow a + c = b. \Rightarrow (a + c)d = bd \Rightarrow ad + cd = bd \Rightarrow ad = bd - cd \Rightarrow (b - c)d = bd - cd$.

According to the operations of N, we can prove multiplication in Z is well-defined and integers form an integral domain with respect to addition and multiplication.

1.2 The integral domain Z

We should like $(a - b) \cdot (c - d)$ to be equal to $(ac + bd) - (ad + bc)$ and accordingly this leads to the following definition:
$[a, b] \cdot [c, d] = [ac + bd, ad + bc]$ for $a, b, c, d \in N$

This definition is independent of the particular choice of the representative pairs.

Next we will prove $[a, b] \cdot [c, d] = [ac + bd, ad + bc]$ for $a, b, c, d \in N$ is well-defined.

If $[a, b] = [a', b'], [c, d] = [c', d']$, then $[a, b] = [a', b'] \Rightarrow a + b' = a' + b \Rightarrow a = a' + b - b'$.
$[c, d] = [c', d'] \Rightarrow c+d' = c'+d \Rightarrow c = c'+d-d'$. $[a, b]\cdot[c, d] = [ac+bd, ad+bc], [a', b']\cdot[c', d'] = [a'c'+b'd', a'd'+b'c']$.

We have $ac + bd + a'd' + b'c' = (a' + b - b')(c' + d - d') + bd + a'd' + b'c' = a'c' + a'd - a'd' + bc' + bd - bd' - b'c' - b'd + b'd' + bd + a'd' + b'c' = a'c' + a'd + bc' + 2bd - bd' - b'd + b'd'$

$a'c' + b'd' + ad + bc = a'c' + b'd' + (a' + b - b')d + b(c' + d - d') = a'c' + b'd' + a'd + bd - b'd + bc' + bd - bd' = a'c' + b'd' + a'd + 2bd - b'd + bc' - bd'$

then $(ac+bd)+(a'd'+b'c')=(a'c'+b'd')+(ad+bc)$.Namely, $[ac+bd, ad+bc]=[a'c'+b'd', a'd'+b'c']$.
That is to say, $[a,b]\cdot[c,d]=[a',b']\cdot[c',d']$.

Theorem The integers form an integral domain with respect to addition and multiplication. (that is, a commutative ring without zero divisors and with identity element).

We have proved Z is a commutative group with respect to addition. Next we will consider Z with respect to multiplication.

Commutative law: $[a,b]\cdot[c,d]=[ac+bd, ad+bc]=[ca+db, cb+da]=[c,d]\cdot[a,b]. \quad \forall[a,b],[c,d]\in Z$

Associative law:
$([a,b]\cdot[c,d])\cdot[e,f]=[ac+bd, ad+bc]\cdot[e,f]=[(ace+bde)+(adf+bcf),(acf+bdf)+(ade+bce)]=[(ace+adf)+(bcf+bde),(acf+ade)+(bce+bdf)]=[a,b]\cdot[ce+df, cf+de]=[a,b]\cdot([c,d]\cdot[e,f]) \quad \forall[a,b],[c,d],[e,f]\in Z$

Distribution law:
$[a,b]\cdot([c,d]+[e,f])=[a,b]\cdot[c+e,d+f]=[a(c+e)+b(d+f),a(d+f)+b(c+e)]=[ac+ae+bd+bf, ad+af+bc+be]=[(ac+bd)+(ae+bf),(ad+bc)+(af+be)]=[ac+bd, ad+bc]+[ae+bf, af+be]=[a,b]\cdot[c,d]+[a,b]\cdot[e,f]$

Here we know Z is a commutative ring.

Besides, $[1,0]\cdot[a,b]=[a,b]\cdot[1,0]=[a\cdot 1+b\cdot 0, a\cdot 0+b\cdot 1]=[a,b]. \quad \forall[a,b]\in Z.$

Next we assume there exist zero-divisors in Z, that is to say, $\exists[a,b]\neq[0,0]$, and $\exists[c,d]\neq[0,0]$.

$[a,b]\cdot[c,d]=[ac+bd, ad+bc]=[0,0], \quad \forall[a,b],[c,d]\in Z.$
Then $ac+bd+0=0+ad+bc, ac+bd=ad+bc, ac-ad=bc-bd, a(c-d)=b(c-d), a(c-d)-b(c-d)=0, (c-d)(a-b)=0. \Rightarrow c=d$ or $a=b$.

Which is contradictory to the assumption $[a,b]\neq[0,0],[c,d]\neq[0,0]$.

Hence, the assumption is not satisfied, there is no zero-divisors in Z.

Here we should also prove "If $m,n\in N$ and $mn=0$ then $m=0$ or $n=0$. $\Leftrightarrow$ If $m\neq 0$ and $n\neq 0$, then $mn\neq 0$. " by induction.

Firstly, we should prove "If $m\neq 0$ and $n\neq 0$, then $m+n\neq 0$. " by induction.

If $m=1, 1+n=S(n)\neq 0$.
If $m=k, k+n\neq 0$.
When $m=k+1,(k+1)+n=(k+n)+1=S(k+n)\neq 0$.

So, "If $m\neq 0$ and $n\neq 0$, then $m+n\neq 0$. " is proved.

If $m=1, 1\cdot n=n\neq 0$.
If $m=k, k\cdot n\neq 0$.
When $m=k+1,(k+1)\cdot n=k\cdot n+n\neq 0$.

Hence, " ""If $m,n\in N$ and $mn=0$ then $m=0$ or $n=0$." is proved.

2.The construction of commutative group

We begin with any commutative semi-group H and construct from it as the construction of the integers to obtain a commutative group G. If the cancellation law does not hold in H, we define$(a,b)\sim(c,d)$ if and only if there is an e such that $a+d+e=b+c+e$. However, in this case $\iota: H\longrightarrow G$ is not injective.

2.1 The relation defined on $H\times H$

We consider the relation $\sim$, defined on $H\times H$, by $(a,b)\sim(c,d)$ if and only if there is an e such that $a+d+e=b+c+e$.We then establish that this is an equivalence relation.

It may be proved as follows:

Reflexivity: There is an e such that $a+b+e=b+a+e\Rightarrow(a,b)\sim(a,b). \quad \forall(a,b)\in H$

Symmetry: If $(a,b)\sim(c,d)$, then there is an e such that $a+d+e=b+c+e$.
Hence, $c+b+e=d+a+e\Rightarrow(c,d)\sim(a,b). \quad \forall(a,b),(c,d)\in H$

Transitivity: If $(a,b)\sim(c,d)$ and $(c,d)\sim(e,f)$ then by definition, there are g and h such that $a+d+g=b+c+g$ and $c+f+h=d+e+h$. $\forall(a,b),(c,d),(e,f)\in H$

By addition we obtain $a+d+g+c+f+h=b+c+g+d+e+h$. And by letting $i=g+h+c+d$, we obtain there is an i such that $a+f+i=b+e+i$, that is $(a,b)\sim(e,f)$. (We have also made use of the commutativity and associativity of addition.)

G may now be defined as equivalence classes of the relation $\sim$. The class represented by (a,b) is denoted by $[a,b]$. G is a set of equivalence classes.

2.2 Addition on $H \times H$

We can define on $H \times H$ a component-wise addition, $(a, b) + (c, d) := (a + c, b + d)$.

The commutative and associative laws hold, and the zero element is $(0, 0)$.

Commutative law: $(a, b) + (c, d) := (a + c, b + d) = (c + a, d + b) = (c, d) + (a, b)$.

Associative law:
$((a, b) + (c, d)) + (e, f) = (a + c, b + d) + (e, f) = (a + (c + e), b + (d + f)) = (a, b) + (c + e, d + f)) = (a, b) + ((c, d) + (e, f))$.

Zero element: $(0, 0) + (a, b) = (a, b) + (0, 0) = (a, b)$.

This addition is compatible with the relation $\sim$, that is to say, if $(a', b') \sim (a, b)$ and $(c', d') \sim (c, d)$ then $(a' + c', b' + d') \sim (a + c, b + d)$.

$(a', b') \sim (a, b), (c', d') \sim (c, d) \Rightarrow a' + b = b' + a, c' + d = d' + c \Rightarrow (a' + c') + (b + d) = (b' + d') + (a + c)$
$\Rightarrow (a' + c', b' + d') \sim (a + c, b + d)$.

It is therefore meaningful to introduce in G, an addition $G \times G \longrightarrow G, [a, b] + [c, d] := [a + c, b + d]$,which is likewise commutative and associative and which has $[0, 0]$ as zero element.

Commutative law: $[a, b] + [c, d] := [a + c, b + d] = [c + a, d + b] = [c, d] + [a, b]$.

Associative law:
$([a, b] + [c, d]) + [e, f] = [a + c, b + d] + [e, f] = [a + (c + e), b + (d + f)] = [a + (c + e), b + (d + f)] = [a, b] + ([c, d] + [e, f])$.

Zero element: $[0, 0] + [a, b] = [a, b] + [0, 0] = [a, b]$.

Next we will prove the addition in G is well-defined.

If $[a, b] = [a', b']$ and $[c, d] = [c', d']$, we should check $[a, b] + [c, d] = [a', b'] + [c', d']$.

Solution: $[a, b] = [a', b'] \Rightarrow$ there is an e such that $a + b' + e = a' + b + e$.

$[c, d] = [c', d'] \Rightarrow$there is an f such that $c + d' + f = c' + d + f$.

Then there is a $g = e + f$ such that $a + c + b' + d' + g = a' + c' + b + d + g$.
And $[a, b] + [c, d] = [a + c, b + d], [a', b'] + [c', d'] = [a' + c', b' + d']$. Hence$[a, b] + [c, d] = [a', b'] + [c', d']$.

By passing to equivalence classes we have gained more. Each $[a, b]$ has an inverse, namely $[b, a]$. We have established the following.

2.3 Commutative group G

Theorem G forms a commutative group with respect to addition.

The element inverse to $\alpha \in G$ is uniquely determined, and is denoted by $-\alpha$. Subtraction in G is defined by $\alpha-\beta := \alpha+(-\beta)$.

Proof: (1)$\forall [a, b], [c, d] \in G, [a, b] + [c, d] \in G$.

(2)$\forall [a, b], [c, d] \in G, [a, b] + [c, d] = [c, d] + [a, b]$.

(3)$\forall [a, b], [c, d], [e, f] \in G, ([a, b] + [c, d]) + [e, f] = [a, b] + ([c, d] + [e, f])$.

(4)$\forall [a, b] \in G, \exists [0, 0] \in G, [0, 0] + [a, b] = [a, b] + [0, 0] = [a, b]$. And $[0, 0] = [0, 0] + [0, 0]' = [0, 0]'$, the zero element is unique.

(5)$\forall [a, b] \in G, \exists [b, a] = -[a, b] \in G, [a, b] + (-[a, b]) = (-[a, b]) + [a, b] = [0, 0]$.

In fact, $[a, b] + [b, a] = [a + b, b + a]$ and $a + b + 0 = b + a + 0$. Then there exists an $e = 0$ such that $a + b + 0 + 0 = b + a + 0 + 0$. Hence $[a, b] + [b, a] = [0, 0]$.

Besides, $[a, b] + [c, d] = [0, 0] \Rightarrow [b, a] + ([a, b] + [c, d]) = [b, a] + [0, 0] \Rightarrow [b, a] + [a, b] + [c, d] = [b, a] \Rightarrow [c, d] = [b, a]$. The inverse of $[a, b]$ is also unique.

2.4 The mapping from H to G

The mapping $\iota : H \longrightarrow G, a \longrightarrow [a, 0]$ is not injective and compatible with addition.

If the cancellation law does not hold in H, that is to say, there are $a, b, c \in H$ such that $a + c = b + c$ and $a \neq b$.
Then $[a, 0] = [b, 0]$ and $a \neq b$, namely, $\iota(a) = \iota(b)$ and $a \neq b$. Hence, ι is not injective.
Besides, ι is compatible with addition, because of $\iota(a) = [a, 0], \iota(b) = [b, 0], \iota(a + b) = [a + b, 0] = [a, 0] + [b, 0] \Rightarrow \iota(a + b) = \iota(a) + \iota(b)$.

3. The relation between real numbers and decimals

The relation between real numbers and decimals has been generally pointed out in $The\ principle\ of\ mathematical$ *analysis*. Since the importance of application of Archimedean property of R and the relationship between real numbers and decimals, the method of how to choose $n_1, ..., n_{k-1}$ of "Having chosen $n_0, n_1, ..., n_{k-1}$, let n_k be the largest integer such that $n_0 + \frac{n_1}{10} + \cdots + \frac{n_{k-1}}{10^{k-1}} + \frac{n_k}{10^k} \le x$"as been given, and the proof of "Let E be the set of these numbers $n_0 + \frac{n_1}{10} + \cdots + \frac{n_{k-1}}{10^{k-1}} + \frac{n_k}{10^k}(k = 0, 1, 2, \cdots)(5)$ .Then $x = supE$."as been indicated, which Rudin have not mentioned totally. In the proof of the two questions natural numbers also play an important role.

3.1 The existence of n_0.

Theorem 1.20 (a) If $x \in R, y \in R$ and $x > 0$ then there is a positive integer n such that $nx > y$.
Part (a) is usually referred to as the Archimedean property of R.

Let $x > 0$ be real.

According to the Archimedean property of $R, x \in R, 1 \in R, 1 > 0$, then there is a positive integer n such that $n \cdot 1 > x$.

Hence $x \in [0, 1) \cup [1, 2) \cup \cdots \cup [n-1, n)$, then there is $n_0 \in Z^+ \cup 0$ such that $x \in [n_0, n_0 + 1)$.
And n_0 is the largest integer such that $n_0 \le x < n_0 + 1$.

3.2 The method of choosing $n_1, \cdots, n_{k-1}$.

$0 \le x - n_0 < 1, 0 \le 10(x - n_0) < 10, 10(x - n_0) \in [0, 1) \cup [1, 2) \cup \cdots \cup [9, 10)$.

Then there exists $n_1 \in Z$ and $0 \le n_1 < 10$ such that $10(x - n_0) \in [n_1, n_1 + 1)$, and n_1 is the largest integer such that
$n_1 \le 10\,(x - n_0) < n_1 + 1, \frac{n_1}{10} \le x - n_0 < \frac{n_1}{10} + \frac{1}{10}, 0 \le x - n_0 - \frac{n_1}{10} < \frac{1}{10}, 0 \le 100(x - n_0 - \frac{n_1}{10}) < 10.$

In the similar way, there is a largest integer n_2 such that
$0 \le n_2 \le 100(x-n_0-\frac{n_1}{10}) < n_2+1, \frac{n_2}{100} \le x-n_0-\frac{n_1}{10} < \frac{n_2}{100}+\frac{1}{100}, 0 \le x-n_0-\frac{n_1}{10}-\frac{n_2}{100} < \frac{1}{100}, 0 \le 1000(x-n_0-\frac{n_1}{10}-\frac{n_2}{100}) < 10.$

There is a largest integer n_3 such that $0 \le n_3 \le 1000(x - n_0 - \frac{n_1}{10} - \frac{n_2}{100}) < n_3 + 1$.

Do the same actions till we obtain n_{k-1} such that $0 \le n_{k-1} \le 10^{k-1}(x - n_0 - \frac{n_1}{10} - \cdots - \frac{n_{k-2}}{10^{k-2}}) < n_{k-1} + 1$.

3.3 The proof of $x = supE$.

Let n_k be the largest integer such that $0 \le n_k \le 10^k(x - n_0 - \frac{n_1}{10} - \cdots - \frac{n_{k-1}}{10^{k-1}}) < n_k + 1$.

$x - n_0 - \frac{n_1}{10} - \cdots - \frac{n_{k-1}}{10^{k-1}} \ge \frac{n_k}{10^k}, x \ge n_0 + \frac{n_1}{10} + \cdots + \frac{n_{k-1}}{10^{k-1}} + \frac{n_k}{10^k}$.
Let E be the set of these numbers $n_0 + \frac{n_1}{10} + \cdots + \frac{n_{k-1}}{10^{k-1}} + \frac{n_k}{10^k}(k = 0, 1, 2, \cdots)(5)$

$E = \left\{n_0 + \frac{n_1}{10} + \cdots + \frac{n_{k-1}}{10^{k-1}} + \frac{n_k}{10^k} | k = 0, 1, 2, \cdots\right\}$

We have known x is an upper bound of E . Next we will prove x is the smallest upper bound of E.

$\forall y < x, x - y > 0 \Rightarrow \frac{1}{x-y} > 0.$
According to Archimedean property $\frac{1}{x-y} \in R, 1 \in R, 1 > 0$, then there is a positive integer n such that $1 \cdot n > \frac{1}{x-y}$.
We let $a_k = n_0 + \frac{n_1}{10} + \cdots + \frac{n_{k-1}}{10^{k-1}} + \frac{n_k}{10^k}, k = 0, 1, 2, \cdots$.
$n(x - y) > 1, nx - ny > 1, nx - 1 > ny \Rightarrow y < x - \frac{1}{n}(*)$

We have known $10^k(x - (n_0 + \frac{n_1}{10} + \cdots + \frac{n_{k-1}}{10^{k-1}})) < n_{k+1} \Rightarrow x - (n_0 + \frac{n_1}{10} + \cdots + \frac{n_{k-1}}{10^{k-1}} + \frac{n_k}{10^k}) < +\frac{1}{10^k} \Rightarrow x - a_k < \frac{1}{10^k}(**)$

Next we will proof $10^k \ge n$ by the principle of complete induction to complete the proof. If a certain property is possessed by the number 0(the commencement of the induction) and if, for every number n which has the property, its successor also has the property(the induction step), then the property is possessed by all the natural numbers.

Step 1 When $n = 0, 10^k \ge 0$, it is satisfied.

Step 2 Assume $n = k, 10^k \ge k$, is satisfied.

Step 3 Then $10^{k+1} - k + 1 = 10 \cdot 10^k - k - 1 = 10^k - k + 9 \cdot 10^k - 1 > 9 \cdot 1 - 1 = 8 > 0.10^{k+1} \ge k + 1.$

Hence, the proof of $10^k \ge n$ is complete.

Then $\frac{1}{10^k} \le \frac{1}{n} \Rightarrow x - a_k < \frac{1}{n}$(because of $(**)$) $\Rightarrow x - \frac{1}{n} < a_k \Rightarrow y < a_k$(because of $(*)$).
Namely, $\forall y < x$. y is not an upper bound of E .

Hence, x is the smallest upper bound of E. $x = supE$.

The decimal expansion of x is $n_0 \cdot n_1 n_2 n_3 \cdots (6)$.

Conversely, for any infinite decimal(6) the set of number (5)is bounded above, ($0 \le n_1, n_2, \cdots, n_k < 10$ and $n_1, n_2, \cdots, n_k \in Z$)
$(n_0 + 1) - a_k = n_0 + 1 - (n_0 + \frac{n_1}{10} + \cdots + \frac{n_k}{10^k}) = 1 - \frac{n_1}{10} - \frac{n_2}{100} - \cdots - \frac{n_k}{10^k} \ge 1 - \frac{9}{10} - \frac{9}{100} - \cdots - \frac{9}{10^k} = 1 - 9(\frac{1}{10} + \frac{1}{100} + \cdots + \frac{1}{10^k})$

$= 1 - 9 \cdot \frac{\frac{1}{10} \cdot (1 - \frac{1}{10^k})}{1 - \frac{1}{10}} = 1 - (1 - \frac{1}{10^k}) = \frac{1}{10^k} > 0$

We have $\forall k = 0, 1, 2, \cdots, a_k < n_0 + 1$.

And (6) is the decimal expansion of $supE$.

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Abstract

This paper is in continuation of earlier work of Denis *et. al.* associated with Ramanujan's Seventh Entry of Chapter XVII of Second Notebook.

Keywords: Pochhammer symbol, Gaussian hypergeometric function, Complete elliptic integrals, Kampé de fériet double hypergeometric function and srivastava's triple hypergeometric function

1. Introduction and preliminaries

The Pochhammer's symbol or Appell's symbol or shifted factorial or rising factorial or generalized factorial function is defined by

$$(b,\ k) = (b)_k = \frac{\Gamma(b+k)}{\Gamma(b)} = \begin{cases} b(b+1)(b+2)\cdots(b+k-1); & \text{if } k = 1, 2, 3, \cdots \\ 1 & ; \ \text{if } k = 0 \\ k! & ; \ \text{if } b = 1,\ k = 1, 2, 3, \cdots \end{cases}$$

where b is neither zero nor negative integer and the notation Γ stands for Gamma function.

1.1 Generalized gaussian hypergeometric function

Generalized ordinary hypergeometric function of one variable is defined by

$$ {}_AF_B\left[\begin{array}{ll} a_1, a_2, \cdots, a_A & ; \\ & z \\ b_1, b_2, \cdots, b_B & ; \end{array}\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_A)_k z^k}{(b_1)_k (b_2)_k \cdots (b_B)_k k!} $$

or

$$ {}_AF_B\left[\begin{array}{ll} (a_A) & ; \\ & z \\ (b_B) & ; \end{array}\right] \equiv {}_AF_B\left[\begin{array}{ll} (a_j)_{j=1}^A & ; \\ & z \\ (b_j)_{j=1}^B & ; \end{array}\right] = \sum_{k=0}^{\infty} \frac{((a_A))_k z^k}{((b_B))_k k!} \tag{1.1} $$

where denominator parameters $b_1, b_2, \cdots, b_B$ are neither zero nor negative integers and $A,\ B$ are non-negative integers.

1.2 Kampé de fériet's general double hypergeometric function

In 1921, Appell's four double hypergeometric functions F_1, F_2, F_3, F_4 and their confluent forms $\Phi_1, \Phi_2, \Phi_3, \Psi_1, \Psi_2, \Xi_1, \Xi_2$ were unified and generalized by Kampé de Fériet.
We recall the definition of general double hypergeometric function of Kampé de Fériet in slightly modified notation of H.M.Srivastava and R.Panda:

$$ \mathrm{F}^{A:B;D}_{E:G;H}\left[\begin{array}{ll} (a_A) : (b_B) ; (d_D) & ; \\ & x,\ y \\ (e_E) : (g_G) ; (h_H) & ; \end{array}\right] = \sum_{m,n=0}^{\infty} \frac{((a_A))_{m+n}\, ((b_B))_m\, ((d_D))_n\, x^m\, y^n}{((e_E))_{m+n}\, ((g_G))_m\, ((h_H))_n\, m!\, n!} \tag{1.2} $$

where for convergence

(i) $A + B < E + G + 1,\ A + D < E + H + 1\ \ ; |x| < \infty,\ \ |y| < \infty,$ or

(ii) $A + B = E + G + 1,\ A + D = E + H + 1,$ and

$$ \begin{cases} |x|^{\frac{1}{(A-E)}} + |y|^{\frac{1}{(A-E)}} < 1 & ,\text{if } E < A \\ \max\ \{|x|, |y|\} < 1 & ,\text{if } E \geq A \end{cases} $$

1.3 Srivastava's general triple hypergeometric function

In 1967, H. M. Srivastava defined a general triple hypergeometric function $F^{(3)}$ in the following form

$$ F^{(3)}\left[\begin{array}{ll} (a_A) :: (b_B); (d_D); (e_E) : (g_G); (h_H); (l_L); & \\ & x,\ y,\ z \\ (m_M) :: (n_N); (p_P); (q_Q) : (r_R); (s_S); (t_T); & \end{array}\right] $$

$$ = \sum_{i,j,k=0}^{\infty} \frac{((a_A))_{i+j+k}\ ((b_B))_{i+j}\ ((d_D))_{j+k}\ ((e_E))_{k+i}\ ((g_G))_i\ ((h_H))_j\ ((l_L))_k\ x^i\ y^j\ z^k}{((m_M))_{i+j+k}\ ((n_N))_{i+j}\ ((p_P))_{j+k}\ ((q_Q))_{k+i}\ ((r_R))_i\ ((s_S))_j\ ((t_T))_k\ i!\ j!\ k!} \tag{1.3} $$

1.4 Wright's generalized hypergeometric function

$$ {}_p\Psi_q\left[\begin{array}{ll} (\alpha_1, A_1), \cdots, (\alpha_p, A_p) & ; \\ & x \\ (\lambda_1, B_1), \cdots, (\lambda_q, B_q) & ; \end{array}\right] = \sum_{m=0}^{\infty} \frac{\Gamma(\alpha_1 + mA_1)\Gamma(\alpha_2 + mA_2) \cdots \Gamma(\alpha_p + mA_p) x^m}{\Gamma(\lambda_1 + mB_1)\Gamma(\lambda_2 + mB_2) \cdots \Gamma(\lambda_q + mA_q) m!} \tag{1.4} $$

$$ {}_p\Psi^*_q\left[\begin{array}{ll} (\alpha_1, A_1), \cdots, (\alpha_p, A_p) & ; \\ & x \\ (\lambda_1, B_1), \cdots, (\lambda_q, B_q) & ; \end{array}\right] = \sum_{m=0}^{\infty} \frac{(\alpha_1)_{mA_1} (\alpha_2)_{mA_2} \cdots (\alpha_p)_{mA_p} x^m}{(\lambda_1)_{mB_1} (\lambda_2)_{mB_2} \cdots (\lambda_q)_{mB_q} m!} \tag{1.5} $$

2. Some integrals of ramanujan and erdélyi

Entry 7 (ix). If $|x| < 1$, then

$$ \frac{\pi}{2} \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\phi}{\sqrt{(1 + x \sin\phi)}} = \int_0^{\frac{\pi}{2}} \frac{\cos^{-1}(x \sin^2\phi)\mathrm{d}\phi}{\sqrt{(1 - x^2 \sin^4\phi)}} = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\theta \mathrm{d}\phi}{(1 + x \sin\theta \sin^2\phi)} \tag{2.1} $$

Entry 7 (x). If $|x| < 1$, then

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\theta\mathrm{d}\phi}{\sqrt{(1 - x\sin^2\theta)(1 - x\sin^2\theta\sin^2\phi)}} = \left(\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\phi}{\sqrt{(1 - x\sin^4\phi)}}\right)^2 \tag{2.2}$$

Entry 7 (xi). If $|x| < 1$, then

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{x\sin\phi\mathrm{d}\theta\mathrm{d}\phi}{\sqrt{(1 - x^2\sin^2\phi)(1 - x^2\sin^2\theta\sin^2\phi)}} = \int_0^{\frac{\pi}{2}}\int_0^{\sin^{-1}x} \frac{\mathrm{d}\theta\mathrm{d}\phi}{\sqrt{(1 - x^2\sin^2\phi - \sin^2\theta\cos^2\phi)}}$$

$$= \frac{1}{2}\left(\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\phi}{\sqrt{(1 - \frac{(1+x)}{2}\sin^2\phi)}}\right)^2 - \frac{1}{2}\left(\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\phi}{\sqrt{(1 - \frac{(1-x)}{2}\sin^2\phi)}}\right)^2 \tag{2.3}$$

Erdélyi *et. al.*[p.315(7)]

$$\Pi^*(\phi, \psi, k) = \left(\int_0^{\phi} \frac{k^2\cos\psi\sin\psi\sqrt{(1 - k^2\sin^2\psi)}\ \sin^2 t}{(1 - k^2\sin^2\psi\sin^2 t)\sqrt{(1 - k^2\sin^2 t)}}\ \mathrm{d}t\right) \tag{2.4}$$

Kyrala [p.287(Q.27)]

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{[k^2\cos^2\theta + (1 - k^2)\cos^2\phi]\ \mathrm{d}\theta\mathrm{d}\phi}{\sqrt{(1 - k^2\sin^2\theta)}\sqrt{(1 - (1 - k^2)\sin^2\phi)}} \tag{2.5}$$

Above integral was considered to prove Legendre relation $E(k)K'(k) + E'(k)K(k) - K'(k)K(k) = \frac{\pi}{2}$

3. Evaluation of integrals

$$\frac{\pi}{2}\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\phi}{\sqrt{(1 + x\sin\phi)}} = \frac{\pi^2}{4}\,{}_2F_1\left[\begin{array}{ccc} \frac{1}{4}, \frac{3}{4} & ; & \\ & & x^2 \\ 1 & ; & \end{array}\right] - \frac{\pi\,x}{4}\,{}_3F_2\left[\begin{array}{ccc} \frac{3}{4}, \frac{5}{4}, 1 & ; & \\ & & x^2 \\ \frac{3}{2}, \frac{3}{2} & ; & \end{array}\right] \tag{3.1}$$

$$\int_0^{\frac{\pi}{2}} \frac{\cos^{-1}(x\sin^2\phi)\mathrm{d}\phi}{\sqrt{(1 - x^2\sin^4\phi)}} = \frac{\pi^2}{4}{}_2F_1\left[\begin{array}{ccc} \frac{1}{4}, \frac{3}{4} & ; & \\ & & x^2 \\ 1 & ; & \end{array}\right] - \frac{x\pi}{4}{}_3F_2\left[\begin{array}{ccc} 1, \frac{3}{4}, \frac{5}{4} & ; & \\ & & x^2 \\ \frac{3}{2}, \frac{3}{2} & ; & \end{array}\right] \tag{3.2}$$

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\theta\mathrm{d}\phi}{(1 + x\sin\theta\sin^2\phi)} = \frac{\pi^2}{4}{}_2F_1\left[\begin{array}{ccc} \frac{1}{4}, \frac{3}{4} & ; & \\ & & x^2 \\ 1 & ; & \end{array}\right] - \frac{x\pi}{4}{}_3F_2\left[\begin{array}{ccc} 1, \frac{3}{4}, \frac{5}{4} & ; & \\ & & x^2 \\ \frac{3}{2}, \frac{3}{2} & ; & \end{array}\right] \tag{3.3}$$

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\theta\mathrm{d}\phi}{\sqrt{(1 - x\sin^2\theta)(1 - x\sin^2\theta\sin^2\phi)}} = \frac{\pi^2}{4}\ F^{1:1;2}_{1:0;1}\left[\begin{array}{ccc} \frac{1}{2}:\frac{1}{2}\ ;\frac{1}{2}, \frac{1}{2} & ; & \\ & & x,\ x \\ 1: -; 1 & ; & \end{array}\right] \tag{3.4}$$

$$\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\phi}{\sqrt{(1 - x\sin^4\phi)}} = \frac{\pi}{2}{}_2F_1\left[\begin{array}{ccc} \frac{1}{4}, \frac{3}{4} & ; & \\ & & x \\ 1 & ; & \end{array}\right] \tag{3.5}$$

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{x\sin\phi\mathrm{d}\theta\mathrm{d}\phi}{\sqrt{(1 - x^2\sin^2\phi)(1 - x^2\sin^2\theta\sin^2\phi)}} = \frac{\pi x}{2}\ F^{1:1;2}_{1:0;1}\left[\begin{array}{ccc} 1:\frac{1}{2}\ ;\frac{1}{2}, \frac{1}{2} & ; & \\ & & x^2,\ x^2 \\ \frac{3}{2}: -; 1 & ; & \end{array}\right] \tag{3.6}$$

$$\int_0^{\frac{\pi}{2}}\int_0^{\sin^{-1}x} \frac{\mathrm{d}\theta\mathrm{d}\phi}{\sqrt{(1 - x^2\sin^2\phi - \sin^2\theta\cos^2\phi)}} = \frac{\pi\sin^{-1}x}{2}\ F^{1:1;2}_{1:0;1}\left[\begin{array}{ccc} \frac{1}{2}:\frac{1}{2}\ ;\frac{1}{2}, \frac{1}{2} & ; & \\ & & x^2,\ 1 \\ 1: -; 1 & ; & \end{array}\right] -$$

$$- \frac{\pi x\sqrt{(1 - x^2)}}{16}\ F^{(3)}\left[\begin{array}{ccc} \frac{3}{2}:: -;\frac{3}{2}, \frac{3}{2};-:\ \frac{1}{2};\ 1;1, 1 & ; & \\ & & x^2,\ 1,\ x^2 \\ 2:: -; 2,\ 2\ ; -:-;-;\frac{3}{2} & ; & \end{array}\right] \tag{3.7}$$

$$\frac{1}{2}\Bigg(\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\phi}{\sqrt{(1-\frac{(1+x)}{2}\sin^2\phi)}}\Bigg)^2-\frac{1}{2}\Bigg(\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\phi}{\sqrt{(1-\frac{(1-x)}{2}\sin^2\phi)}}\Bigg)^2$$

$$=\frac{\pi^2}{8}\left\{{}_2F_1\left[\begin{array}{lll}\frac{1}{2},\frac{1}{2} & ; & \\ & & \frac{(1+x)}{2}\\ 1 & ; & \end{array}\right]\right\}^2-\frac{\pi^2}{8}\left\{{}_2F_1\left[\begin{array}{lll}\frac{1}{2},\frac{1}{2} & ; & \\ & & \frac{(1-x)}{2}\\ 1 & ; & \end{array}\right]\right\}^2 \tag{3.8}$$

$$\int_0^{\phi}\frac{k^2\cos\psi\sin\psi\sqrt{(1-k^2\sin^2\psi)}\ \sin^2 t}{(1-k^2\sin^2\psi\sin^2 t)\sqrt{(1-k^2\sin^2 t)}}\,\mathrm{d}t=-\frac{k^2\sin(2\psi)\sin(2\phi)\sqrt{(1-k^2\sin^2\psi)}}{8}\times$$

$$\times F^{(3)}\left[\begin{array}{ll}\frac{3}{2}::-;-;-:1;\frac{1}{2};1,1; & \\ & k^2\sin^2\psi,k^2,\ k^2\sin^2\psi\sin^2\phi\\ 2::-;-;-:-;-;\frac{3}{2}; & \end{array}\right]-$$

$$-\frac{k^4\sin(2\psi)\sin(2\phi)\sqrt{(1-k^2\sin^2\psi)}}{32}\times$$

$$\times F^{(3)}\left[\begin{array}{ll}\frac{5}{2}::-;\frac{3}{2};2:1;1;1; & \\ & k^2\sin^2\psi\sin^2\phi,k^2,\ k^2\sin^2\phi\\ 3::-;2;\frac{5}{2}:-;-;-; & \end{array}\right]+$$

$$+\frac{\phi\, k^2\sin(2\psi)\sqrt{(1-k^2\sin^2\psi)}}{4}F^{1:1;1}_{1:0;0}\left[\begin{array}{ll}\frac{3}{2}:1\ ;\frac{1}{2}; & \\ & k^2\sin^2\psi,\ k^2\\ 2:-;-; & \end{array}\right] \tag{3.9}$$

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\frac{[k^2\cos^2\theta+(1-k^2)\cos^2\phi]\ \mathrm{d}\theta\mathrm{d}\phi}{\sqrt{(1-k^2\sin^2\theta)}\sqrt{(1-(1-k^2)\sin^2\phi)}}$$

$$=\frac{k^2\pi^2}{8}{}_2F_1\left[\begin{array}{lll}\frac{1}{2},\frac{1}{2} & ; & \\ & & k^2\\ 2 & ; & \end{array}\right]{}_2F_1\left[\begin{array}{lll}\frac{1}{2},\frac{1}{2} & ; & \\ & & (1-k^2)\\ 1 & ; & \end{array}\right]+$$

$$+\frac{(1-k^2)\pi^2}{8}{}_2F_1\left[\begin{array}{lll}\frac{1}{2},\frac{1}{2} & ; & \\ & & k^2\\ 1 & ; & \end{array}\right]{}_2F_1\left[\begin{array}{lll}\frac{1}{2},\frac{1}{2} & ; & \\ & & (1-k^2)\\ 2 & ; & \end{array}\right] \tag{3.10}$$

4. Derivation

4.1 Evaluation of integrals involved in entry 7(ix)

$$\text{Let}\ \ \frac{\pi}{2}\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\phi}{\sqrt{(1+x\sin\phi)}}=\frac{\pi}{2}\sum_{m=0}^{\infty}\frac{(\frac{1}{2})_m(-x)^m}{m!}\int_0^{\frac{\pi}{2}}\sin^m\phi\ \mathrm{d}\phi=\frac{\pi^2}{4}\sum_{m=0}^{\infty}\frac{(\frac{1}{2})_m(\frac{1}{2})_{m/2}(-x)^m}{m!(1)_{m/2}}$$

$$=\frac{\pi^2}{4}\,{}_2\Psi_1^*\left[\begin{array}{lll}(\frac{1}{2},1),(\frac{1}{2},\frac{1}{2}) & ; & \\ & & -x\\ (1,\frac{1}{2}) & ; & \end{array}\right]=\frac{\pi}{4}\,{}_2\Psi_1\left[\begin{array}{lll}(\frac{1}{2},1),(\frac{1}{2},\frac{1}{2}) & ; & \\ & & -x\\ (1,\frac{1}{2}) & ; & \end{array}\right]$$

$$=\frac{\pi^2}{4}\,{}_2F_1\left[\begin{array}{lll}\frac{1}{4},\frac{3}{4} & ; & \\ & & x^2\\ 1 & ; & \end{array}\right]-\frac{\pi\,x}{4}\,{}_3F_2\left[\begin{array}{lll}\frac{3}{4},\frac{5}{4},1 & ; & \\ & & x^2\\ \frac{3}{2},\frac{3}{2} & ; & \end{array}\right] \tag{4.1}$$

$$\text{Again}\qquad\int_0^{\frac{\pi}{2}}\frac{\cos^{-1}(x\sin^2\phi)\mathrm{d}\phi}{\sqrt{(1-x^2\sin^4\phi)}}=\int_0^{\frac{\pi}{2}}\frac{\left(\frac{\pi}{2}-\ \sin^{-1}(x\sin^2\phi)\right)\mathrm{d}\phi}{\sqrt{(1-x^2\sin^4\phi)}}$$

$$=\int_0^{\frac{\pi}{2}}\frac{(\frac{\pi}{2})\mathrm{d}\phi}{\sqrt{(1-x^2\sin^4\phi)}}-\int_0^{\frac{\pi}{2}}\frac{\sin^{-1}(x\sin^2\phi)\mathrm{d}\phi}{\sqrt{(1-x^2\sin^4\phi)}}$$

$$=\frac{\pi}{2}\int_0^{\frac{\pi}{2}}(1-x^2\sin^4\phi)^{-\frac{1}{2}}\mathrm{d}\phi\ -\ \int_0^{\frac{\pi}{2}}(x\sin^2\phi)\ {}_2F_1\left[\begin{array}{lll}1,\ 1 & ; & \\ & & x^2\sin^4\phi\\ \frac{3}{2} & ; & \end{array}\right]\mathrm{d}\phi$$

$$= \frac{\pi}{2}\int_0^{\frac{\pi}{2}} {}_1F_0\left[\begin{array}{ll} \frac{1}{2} & ; \\ & x^2\sin^4\phi \\ - & ; \end{array}\right]\mathrm{d}\phi - x\int_0^{\frac{\pi}{2}}\sin^2\phi\ {}_2F_1\left[\begin{array}{ll} 1,\,1 & ; \\ & x^2\sin^4\phi \\ \frac{3}{2} & ; \end{array}\right]\mathrm{d}\phi$$

$$= \frac{\pi}{2}\sum_{m=0}^{\infty}\frac{(\frac{1}{2})_m x^{2m}}{m!}\int_0^{\frac{\pi}{2}}\sin^{4m}\phi\mathrm{d}\phi \ - \ x\sum_{m=0}^{\infty}\frac{(1)_m(1)_m x^{2m}}{(\frac{3}{2})_m m!}\int_0^{\frac{\pi}{2}}\sin^{4m+2}\phi\mathrm{d}\phi$$

$$= \frac{\pi}{2}\sum_{m=0}^{\infty}\frac{(\frac{1}{2})_m x^{2m}\Gamma(\frac{4m+1}{2})\Gamma(\frac{0+1}{2})}{m!\ 2\ \Gamma(\frac{4m+0+2}{2})} \ - \ x\sum_{m=0}^{\infty}\frac{(1)_m(1)_m x^{2m}\Gamma(\frac{4m+3}{2})\Gamma(\frac{0+1}{2})}{(\frac{3}{2})_m m!\ 2\ \Gamma(\frac{4m+4}{2})}\mathrm{d}\phi$$

$$= \frac{\pi^2}{4}\sum_{m=0}^{\infty}\frac{x^{2m}(\frac{1}{4})_m(\frac{3}{4})_m}{m!m!} \ - \ \frac{x\pi}{4}\sum_{m=0}^{\infty}\frac{(1)_m x^{2m}(\frac{3}{4})_m(\frac{5}{4})_m}{(\frac{3}{2})_m m!(\frac{3}{2})_m}$$

$$= \frac{\pi^2}{4}\ {}_2F_1\left[\begin{array}{ll} \frac{1}{4},\frac{3}{4} & ; \\ & x^2 \\ 1 & ; \end{array}\right] - \frac{x\pi}{4}\ {}_3F_2\left[\begin{array}{ll} 1,\,\frac{3}{4},\,\frac{5}{4} & ; \\ & x^2 \\ \frac{3}{2},\,\frac{3}{2} & ; \end{array}\right] \tag{4.2}$$

Also $$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\theta\mathrm{d}\phi}{(1+x\sin\theta\sin^2\phi)} = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}(1+x\sin\theta\sin^2\phi)^{-1}\mathrm{d}\theta\mathrm{d}\phi$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\sum_{m=0}^{\infty}(-1)^m x^m\sin^m\theta\sin^{2m}\phi\mathrm{d}\theta\mathrm{d}\phi = \sum_{m=0}^{\infty}(-1)^m x^m\Big(\int_0^{\frac{\pi}{2}}\sin^m\theta\mathrm{d}\theta\Big)\Big(\int_0^{\frac{\pi}{2}}\sin^{2m}\phi\mathrm{d}\phi\Big)$$

$$= \sum_{m=0}^{\infty}x^{2m}\Big(\int_0^{\frac{\pi}{2}}\sin^{2m}\theta\mathrm{d}\theta\Big)\Big(\int_0^{\frac{\pi}{2}}\sin^{4m}\phi\mathrm{d}\phi\Big) - \ \sum_{m=0}^{\infty}x^{2m+1}\Big(\int_0^{\frac{\pi}{2}}\sin^{2m+1}\theta\mathrm{d}\theta\Big)\Big(\int_0^{\frac{\pi}{2}}\sin^{4m+2}\phi\mathrm{d}\phi\Big)$$

$$= \sum_{m=0}^{\infty}x^{2m}\Big(\frac{\Gamma(\frac{2m+1}{2})\Gamma(\frac{0+1}{2})}{2\Gamma(\frac{2m+2}{2})}\Big)\Big(\frac{\Gamma(\frac{4m+1}{2})\Gamma(\frac{0+1}{2})}{2\Gamma(\frac{4m+2}{2})}\Big) - \sum_{m=0}^{\infty}x^{2m+1}\Big(\frac{\Gamma(\frac{2m+2}{2})\Gamma(\frac{0+1}{2})}{2\Gamma(\frac{2m+3}{2})}\Big)\Big(\frac{\Gamma(\frac{4m+3}{2})\Gamma(\frac{0+1}{2})}{2\Gamma(\frac{4m+4}{2})}\Big)$$

$$= \sum_{m=0}^{\infty}x^{2m}\Big(\frac{\Gamma(\frac{1}{2}+m)\sqrt{\pi}}{2\ m!}\Big)\Big(\frac{\Gamma(\frac{1}{2}+2m)\sqrt{\pi}}{4\ m!}\Big) - \sum_{m=0}^{\infty}x^{2m+1}\Big(\frac{(1)_m\sqrt{\pi}}{2\ \Gamma(\frac{3}{2}+m)}\Big)\Big(\frac{\Gamma(\frac{3}{2}+2m)\sqrt{\pi}}{2\ (1)_{2m+1}}\Big)$$

$$= \frac{\pi^2}{4}\sum_{m=0}^{\infty}\frac{(\frac{1}{4})_m(\frac{3}{4})_m(x^2)^m}{m!(1)_m} - \frac{x\pi}{4}\sum_{m=0}^{\infty}\frac{(1)_m(\frac{3}{4})_m(\frac{5}{4})_m(x^2)^m}{m!(\frac{3}{2})_m(\frac{3}{2})_m}$$

$$= \frac{\pi^2}{4}\ {}_2F_1\left[\begin{array}{ll} \frac{1}{4},\frac{3}{4} & ; \\ & x^2 \\ 1 & ; \end{array}\right] - \frac{\pi\ x}{4}\ {}_3F_2\left[\begin{array}{ll} 1,\,\frac{3}{4},\,\frac{5}{4} & ; \\ & x^2 \\ \frac{3}{2},\,\frac{3}{2} & ; \end{array}\right] \tag{4.3}$$

4.2 Evaluation of integrals involved in entry 7(x)

$$\text{L.H.S.} = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\theta\mathrm{d}\phi}{\sqrt{(1-x\sin^2\theta)(1-x\sin^2\theta\sin^2\phi)}}$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}(1-x\sin^2\theta)^{-\frac{1}{2}}(1-x\sin^2\theta\sin^2\phi)^{-\frac{1}{2}}\mathrm{d}\theta\mathrm{d}\phi$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\sum_{m=0}^{\infty}\frac{(\frac{1}{2})_m x^{2m}\sin^{2m}\theta}{m!}\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n x^n\sin^2\theta\sin^{2n}\phi}{n!}\mathrm{d}\theta\mathrm{d}\phi$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_m x^m(\frac{1}{2})_n x^n}{m!n!}\Big(\int_0^{\frac{\pi}{2}}\sin^{2m+2n}\theta\mathrm{d}\theta\Big)\Big(\int_0^{\frac{\pi}{2}}\sin^{2n}\phi\mathrm{d}\phi\Big)$$

$$= \pi\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_m x^m(\frac{1}{2})_n x^n}{m!n!}\Big(\frac{\Gamma(\frac{2m+2n+1}{2})\Gamma(\frac{0+1}{2})}{2\Gamma(\frac{2m+2n+0+2}{2})}\Big)\Big(\frac{\Gamma(\frac{2n+1}{2})\Gamma(\frac{0+1}{2})}{2\Gamma(\frac{2n+0+2}{2})}\Big)$$

$$= \frac{\pi}{4}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_m x^m(\frac{1}{2})_n x^n}{m!n!}\Big(\frac{\Gamma(\frac{1}{2}+m+n)}{(1)_{m+n}}\Big)\Big(\frac{\Gamma(\frac{1}{2}+n)}{(1)_n}\Big)$$

$$= \frac{\pi^2}{4}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_m x^m(\frac{1}{2})_n x^n(\frac{1}{2})_{m+n}\frac{1}{2})_n}{m!n!(1)_{m+n}(1)_n} = \frac{\pi^2}{4}\ F^{1:1;2}_{1:0;1}\left[\begin{array}{ll} \frac{1}{2}:\frac{1}{2}\ ;\frac{1}{2},\frac{1}{2} & ; \\ & x,\ x \\ 1:-;1 & ; \end{array}\right] \tag{4.4}$$

$$\text{R.H.S.} = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{(1 - x\sin^4\phi)}} = \int_0^{\frac{\pi}{2}} \left(1 - x\sin^4\phi\right)^{-\frac{1}{2}} d\phi = \int_0^{\frac{\pi}{2}} \sum_{m=0}^{\infty} \frac{(\frac{1}{2})_m x^m \sin^{4m}\phi}{m!} d\theta$$

$$= \sum_{m=0}^{\infty} \frac{(\frac{1}{2})_m x^m}{m!} \Big(\frac{\Gamma(\frac{4m+1}{2})\Gamma(\frac{0+1}{2})}{2\Gamma(\frac{4m+2}{2})} \Big) = \frac{\pi}{2} \sum_{m=0}^{\infty} \frac{(\frac{1}{4})_m(\frac{3}{4})_m x^m}{m!m!} = \frac{\pi}{2}\, {}_2F_1 \left[\begin{array}{ccc} \frac{1}{4},\ \frac{3}{4} & ; & \\ & & x \\ 1 & ; & \end{array} \right] \tag{4.5}$$

4.3 Evaluation of integrals involved in entry 7(xi)

Let
$$\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{x \sin\phi \, d\theta d\phi}{\sqrt{(1 - x^2\sin^2\phi)(1 - x^2\sin^2\theta\sin^2\phi)}}$$

$$= x \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \sin\phi (1 - x^2\sin^2\phi)^{-\frac{1}{2}} (1 - x^2\sin^2\phi\sin^2\theta)^{-\frac{1}{2}} d\theta d\phi$$

$$= x \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_m(\frac{1}{2})_n\, x^{2m+2n}}{m!\, n!} \Big(\int_0^{\frac{\pi}{2}} \sin^{2m+2n+1}\phi d\phi \Big)\Big(\int_0^{\frac{\pi}{2}} \sin^{2n}\theta d\theta \Big)$$

$$= \frac{\pi x}{2} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(1)_{m+n}(\frac{1}{2})_m(\frac{1}{2})_n(\frac{1}{2})_n\, x^{2m+2n}}{(\frac{3}{2})_{m+n}(1)_n m!\, n!} = \frac{\pi x}{2} F^{1:1;2}_{1:0;1} \left[\begin{array}{ccc} 1 : \frac{1}{2}\ ; \frac{1}{2}, \frac{1}{2} & ; & \\ & & x^2,\ x^2 \\ \frac{3}{2} : -; 1 & ; & \end{array} \right] \tag{4.6}$$

Again
$$\int_0^{\frac{\pi}{2}} \int_0^{\sin^{-1}x} \frac{d\theta d\phi}{\sqrt{(1 - x^2\sin^2\phi - \sin^2\theta\cos^2\phi)}}$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\sin^{-1}x} (1 - x^2\sin^2\phi - \sin^2\theta\cos^2\phi)^{-\frac{1}{2}} d\theta d\phi$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\sin^{-1}x} \Big(\sum_{p=0}^{\infty} \frac{(\frac{1}{2})_p (x^2\sin^2\phi + \sin^2\theta\cos^2\phi)^p}{p!} \Big) d\theta d\phi$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\sin^{-1}x} \sum_{m,n=0}^{\infty} \frac{(\frac{1}{2})_{m+n} (x^2\sin^2\phi)^m (\sin^2\theta\cos^2\phi)^n}{m!\, n!} d\theta d\phi$$

$$= \sum_{m,n=0}^{\infty} \frac{(\frac{1}{2})_{m+n} x^{2m}}{m!\, n!} \Big(\int_0^{\frac{\pi}{2}} \sin^{2m}\phi\cos^{2n}\phi d\phi \Big)\Big(\int_0^{\sin^{-1}x} \sin^{2n}\theta d\theta \Big)$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_{m+n} x^{2m}}{m!\, n!} \frac{\Gamma(\frac{2m+1}{2})\Gamma(\frac{2n+1}{2})}{2\Gamma(\frac{2m+2n+2}{2})} \Big[-\ \frac{(\frac{1}{2})_n x\sqrt{(1 - x^2)}}{n!} \sum_{r=0}^{n-1} \frac{(1)_r\, x^{2r}}{(\frac{3}{2})_r} + \frac{(\frac{1}{2})_n \sin^{-1}x}{(1)_n} \Big]$$

$$= -\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_{m+n} x^{2m}}{m!\, n!} \frac{\Gamma(\frac{2m+1}{2})\Gamma(\frac{2n+1}{2})}{2\Gamma(\frac{2m+2n+2}{2})}\ \frac{(\frac{1}{2})_n x\sqrt{(1 - x^2)}}{n!} \sum_{r=0}^{n-1} \frac{(1)_r\, x^{2r}}{(\frac{3}{2})_r} +$$

$$+ \frac{\pi\sin^{-1}x}{2} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_{m+n}(\frac{1}{2})_m(\frac{1}{2})_n(\frac{1}{2})_n\, x^{2m}}{m!\, n!(1)_n(1)_{m+n}}$$

$$= -\frac{\pi\, x\sqrt{(1 - x^2)}}{2} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{n-1} \frac{(\frac{1}{2})_{m+n}(\frac{1}{2})_m(\frac{1}{2})_n(\frac{1}{2})_n(1)_r\, x^{2m+2r}}{m!\, n!(1)_n(1)_{m+n}(\frac{3}{2})_r} +$$

$$+ \frac{\pi\sin^{-1}x}{2} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_{m+n}(\frac{1}{2})_m(\frac{1}{2})_n(\frac{1}{2})_n\, x^{2m}}{m!\, n!(1)_n(1)_{m+n}}$$

$$= -\frac{\pi\, x\sqrt{(1 - x^2)}}{2} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty} \frac{(\frac{1}{2})_{m+n+r+1}(\frac{1}{2})_m(\frac{1}{2})_{n+r+1}(\frac{1}{2})_{n+r+1}(1)_r\, x^{2m}x^{2r}}{m!(1)_{n+r+1}(1)_{m+n+r+1}(1)_{n+r+1}(\frac{3}{2})_r} +$$

$$+ \frac{\pi\sin^{-1}x}{2} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_{m+n}(\frac{1}{2})_m(\frac{1}{2})_n(\frac{1}{2})_n\, x^{2m}}{m!\, n!(1)_n(1)_{m+n}}$$

$$= -\frac{\pi\, x\sqrt{(1-x^2)}}{16}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\frac{(\frac{3}{2})_{m+n+r}(\frac{3}{2})_{n+r}(\frac{3}{2})_{n+r}(\frac{1}{2})_m(1)_r(1)_r(1)_n\; x^{2m}x^{2r}}{(2)_{n+r}(2)_{m+n+r}(2)_{n+r}(\frac{3}{2})_r m!n!r!}+$$

$$+\frac{\pi\sin^{-1}x}{2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_{m+n}(\frac{1}{2})_m(\frac{1}{2})_n(\frac{1}{2})_n\; x^{2m}}{m!\; n!(1)_n(1)_{m+n}}$$

$$= -\frac{\pi x\sqrt{(1-x^2)}}{16}\, F^{(3)}\left[\begin{array}{ll} \frac{3}{2}::\,-;\frac{3}{2},\,\frac{3}{2};-:\,\frac{1}{2};\,1;1,1\quad ; & \\ & x^2,\;1,\;x^2 \\ 2::\,-;\,2,\;2\;;\,-:-;-;\frac{3}{2}\quad ; & \end{array}\right]+$$

$$+\frac{\pi\sin^{-1}x}{2}\, F^{1:1;2}_{1:0;1}\left[\begin{array}{ll} \frac{1}{2}:\frac{1}{2}\;\;;\frac{1}{2},\,\frac{1}{2}\quad ; & \\ & x^2,\;1 \\ 1:\,-;\,1\quad ; & \end{array}\right] \tag{4.7}$$

Similarly we can prove

$$\frac{1}{2}\Big(\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\phi}{\sqrt{(1-\frac{(1+x)}{2}\sin^2\phi)}}\Big)^2-\frac{1}{2}\Big(\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\phi}{\sqrt{(1-\frac{(1-x)}{2}\sin^2\phi)}}\Big)^2$$

$$= \frac{\pi^2}{8}\left\{{}_2F_1\left[\begin{array}{ll} \frac{1}{2},\frac{1}{2}\quad ; & \\ & \frac{(1+x)}{2} \\ 1\quad ; & \end{array}\right]\right\}^2-\frac{\pi^2}{8}\left\{{}_2F_1\left[\begin{array}{ll} \frac{1}{2},\frac{1}{2}\quad ; & \\ & \frac{(1-x)}{2} \\ 1\quad ; & \end{array}\right]\right\}^2 \tag{4.8}$$

4.4 Evaluation of integral given by Erdélyi et. al.

$$\Pi^{*}(\phi,\psi,k) = \Big(\int_0^{\phi}\frac{k^2\cos\psi\sin\psi\sqrt{(1-k^2\sin^2\psi)}\;\sin^2 t}{(1-k^2\sin^2\psi\sin^2 t)\sqrt{(1-k^2\sin^2 t)}}\,\mathrm{d}t\Big) = k^2\sin\psi\cos\psi\sqrt{(1-k^2\sin^2\psi)}\times$$

$$\times\int_0^{\phi}{}_1F_0\left[\begin{array}{ll} 1\;\; ; & \\ & k^2\sin^2\psi\sin^2 t \\ -\;\; ; & \end{array}\right]{}_1F_0\left[\begin{array}{ll} \frac{1}{2}\;\; ; & \\ & k^2\sin^2 t \\ -\;\; ; & \end{array}\right]\sin^2 t\,\mathrm{d}t$$

$$= k^2\sin\psi\cos\psi\sqrt{(1-k^2\sin^2\psi)}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(1)_m(k^2\sin^2\psi)^m(\frac{1}{2})_n(k^2)^n}{m!n!}\int_0^{\phi}\sin^{2m+2n+2}t\,\mathrm{d}t$$

$$= k^2\sin\psi\cos\psi\sqrt{(1-k^2\sin^2\psi)}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(1)_m(k^2\sin^2\psi)^m(\frac{1}{2})_n(k^2)^n}{m!n!}\times$$

$$\times\Big[-\Big(\frac{(\frac{1}{2})_{m+n+1}\sin\phi\cos\phi}{(1)_{m+n+1}}\sum_{r=0}^{m+n}\frac{(1)_r\sin^{2r}\phi}{(\frac{3}{2})_r}\Big)+\Big(\frac{\phi(\frac{1}{2})_{m+n+1}}{(1)_{m+n+1}}\Big)\Big]$$

$$= -\frac{k^2\sin(2\psi)\sin(2\phi)\sqrt{1-k^2\sin^2\psi}}{8}\times$$

$$\times\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{m+n}\frac{(1)_m(k^2\sin^2\psi)^m(\frac{1}{2})_n(k^2)^n(\frac{3}{2})_{m+n}(1)_r\sin^{2r}\phi}{(1)_m(1)_n(2)_{m+n}(\frac{3}{2})_r}+$$

$$+\frac{\phi\, k^2\sin(2\psi)\sqrt{1-k^2\sin^2\psi}}{4}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(1)_m(k^2\sin^2\psi)^m(\frac{1}{2})_n(k^2)^n(\frac{3}{2})_{m+n}}{(1)_m(1)_n(2)_{m+n}}$$

$$= -\frac{k^2\sin(2\psi)\sin(2\phi)\sqrt{1-k^2\sin^2\psi}}{8}\times$$

$$\times\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\frac{(k^2\sin^2\psi)^{m+r}(\frac{1}{2})_n(k^2)^n(\frac{3}{2})_{m+n+r}(1)_r\sin^{2r}\phi(1)_m(1)_r}{(2)_{m+n+r}(\frac{3}{2})_r\; m!n!r!}-$$

$$-\frac{k^2\sin(2\psi)\sin(2\phi)\sqrt{1-k^2\sin^2\psi}}{8}\times$$

$$\times \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty} \frac{(1)_m (k^2\sin^2\psi)^m (\frac{1}{2})_{n+r+1} (k^2)^{n+r+1} (\frac{3}{2})_{m+n+r+1} (1)_{r+m+1} \sin^{2(m+r+1)\phi}}{(1)_m (1)_{n+r+1} (2)_{m+n+r+1} (\frac{3}{2})_{m+r+1}} +$$

$$+ \frac{\phi\, k^2\sin(2\psi)\sqrt{1-k^2\sin^2\psi}}{4} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(1)_m (k^2\sin^2\psi)^m (\frac{1}{2})_n (k^2)^n (\frac{3}{2})_{m+n}}{(2)_{m+n} m! n!}$$

$$= -\frac{k^2\sin(2\psi)\sin(2\phi)\sqrt{(1-k^2\sin^2\psi)}}{8} \times$$

$$\times F^{(3)}\left[\begin{array}{ll} \frac{3}{2} :: -; -; -: 1 ; \frac{1}{2} ; 1, 1; & \\ & k^2\sin^2\psi, k^2,\ k^2\sin^2\psi\sin^2\phi \\ 2 :: -; - ; -: - ; -; \frac{3}{2}; & \end{array}\right] -$$

$$- \frac{k^4\sin(2\psi)\sin(2\phi)\sqrt{(1-k^2\sin^2\psi)}}{32} \times$$

$$\times F^{(3)}\left[\begin{array}{ll} \frac{5}{2} :: -; \frac{3}{2} ; 2 : 1 ; 1 ; 1; & \\ & k^2\sin^2\psi\sin^2\phi, k^2,\ k^2\sin^2\phi \\ 3 :: -; 2 ; \frac{5}{2} : -; -; -; & \end{array}\right] +$$

$$+ \frac{\phi\, k^2\sin(2\psi)\sqrt{(1-k^2\sin^2\psi)}}{4} F^{1:1;1}_{1:0;0}\left[\begin{array}{ll} \frac{3}{2} : 1\ ; \frac{1}{2}; & \\ & k^2\sin^2\psi,\ k^2 \\ 2 : -; -; & \end{array}\right] \qquad (4.9)$$

4.5 Evaluation of (3.10)

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{[k^2\cos^2\theta + (1-k^2)\cos^2\phi]\, d\theta d\phi}{\sqrt{(1-k^2\sin^2\theta)}\sqrt{(1-(1-k^2)\sin^2\phi)}}$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} (1-k^2\sin^2\theta)^{-\frac{1}{2}} (1-(1-k^2)\sin^2\phi)^{-\frac{1}{2}} (k^2\cos^2\theta + (1-k^2)\cos^2\phi)\, d\theta d\phi$$

$$= k^2 \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_m(\frac{1}{2})_n\, k^{2m}(1-k^2)^n}{m!n!} \int_{\theta=0}^{\frac{\pi}{2}} \sin^{2m}\theta\cos^2\theta\, d\theta \int_{\phi=0}^{\frac{\pi}{2}} \sin^{2n}\phi\, d\phi +$$

$$+ (1-k^2) \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_m(\frac{1}{2})_n\, k^{2m}(1-k^2)^n}{m!n!} \int_{\theta=0}^{\frac{\pi}{2}} \sin^{2m}\theta\, d\theta \int_{\phi=0}^{\frac{\pi}{2}} \sin^{2n}\phi\cos^2\phi\, d\phi$$

$$= \frac{k^2\pi^2}{8} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_m(\frac{1}{2})_m(\frac{1}{2})_n(\frac{1}{2})_n\, k^{2m}(1-k^2)^n}{(2)_m(1)_n\, m!n!} +$$

$$+ \frac{(1-k^2)\pi^2}{8} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_m(\frac{1}{2})_m(\frac{1}{2})_n(\frac{1}{2})_n\, k^{2m}(1-k^2)^n}{(2)_n(1)_m\, m!n!}$$

$$= \frac{k^2\pi^2}{8} \sum_{m=0}^{\infty} \frac{(\frac{1}{2})_m(\frac{1}{2})_m\, k^{2m}}{(2)_m m!} \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n(\frac{1}{2})_n(1-k^2)^n}{(1)_n n!} +$$

$$+ \frac{(1-k^2)\pi^2}{8} \sum_{m=0}^{\infty} \frac{(\frac{1}{2})_m(\frac{1}{2})_m\, k^{2m}}{(1)_m\, m!} \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n(\frac{1}{2})_n\, (1-k^2)^n}{(2)_n\, n!}$$

$$= \frac{k^2\pi^2}{8} {}_2F_1\left[\begin{array}{ll} \frac{1}{2}, \frac{1}{2} & ; \\ & \quad k^2 \\ 2 & ; \end{array}\right] {}_2F_1\left[\begin{array}{ll} \frac{1}{2}, \frac{1}{2} & ; \\ & \quad (1-k^2) \\ 1 & ; \end{array}\right] +$$

$$+ \frac{(1-k^2)\pi^2}{8} {}_2F_1\left[\begin{array}{ll} \frac{1}{2}, \frac{1}{2} & ; \\ & \quad k^2 \\ 1 & ; \end{array}\right] {}_2F_1\left[\begin{array}{ll} \frac{1}{2}, \frac{1}{2} & ; \\ & \quad (1-k^2) \\ 2 & ; \end{array}\right] \qquad (4.10)$$

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Differential Models for Integrated Drought Risk Management

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Abstract

We exam the Vitae System from a mathematic point of view. Taking communities' capacity, disaster factor and management efforts or policies into account, a differential model for Integrated Drought Risk Management is introduced in this paper. The effects of the management strength on the water consumption are studied first on a constant then on a periodic background. They are slow-fast model and chaotic one, respectively. Geometric and numerical method are applied to those models.

Keywords: Differential system, Integrated risk management, Drought management, Vitae system, Slow-fast model, Chaos

1. Introduction

The Integrated Disaster Risk Management (IDRM) plays a key role in Natural Disaster Reduction. The Vitae System Model was presented by Okada as a conceptual framework for IDRM (Okada, N., 2006; Okada, N., 2004; Jiquan, Zhang, 2007, p19-23). It claims to view cities, regions and communities as vital integrity with robustness and resiliency in its coping capacity. Based on the Vitae System Model, Okada suggests that disaster planning and management problems be integrated with urban planning and management ones in a unified framework (Fig. 1).

There are some research works on the Vitae System framework (Okada, N., 2008; Xu, Wei, 2008, p59-65). In (Xu, Wei, 2008, p59-65), some basic model for disaster shelter planning and evaluation based on the Vitae system were established. In (Okada, N., 2008), within the framework of Vitae System for IDRM, the questions of resources allocation under high uncertain in a community factor and among communities were considered, and the disaster recovery process was interpreted.

The main purpose of this paper is to set an analysis model for IDRM. This kind of attempt has not been read before. We choose drought disaster mitigation management for the convenience of quantifying. On one hand, the relevant factor involved in disaster mitigation management can be chosen as water consumption while water consumption in a city and region has been documented in quantitative fashion. On the other hand attempts have been made to control or mitigate drought by many governments.

The rest of this paper is organized as follows. In Section 2, we present the Integrated drought risk management model which is a general three-dimension differential system. Two special cases of the model are considered in the next two sections. In Section 3, under a constant risk, geometric method is applied to analyze the water consumption trend of different management efforts. Numerical method is used in Section 4 under a period risk. The last section is the conclusion.

Insert < Figure 1 > here

2. Integrated drought risk management model

We will display in detail how the general differential model for Integrated drought risk management is established in this section.

Consider a city or region faced with rapid population growth and uncertain climate future challenges such as drought. As a Vitae System, three basic ingredients can be considered as the community's coping capacity capability, disaster risk factor and the management factor. While in (Okada, N., 2006), the community's coping capacity capability is 10 day average water saving percentage, we select water consumption. The two variables are inverse proportioned.

Let $w(t)$ be the total water consumption of the whole city at time t. If there are no disaster nor management effort carried in the region, the rate of change $\frac{dw}{dt}$ is simply the natural consumption rate G which is here assumed to be a function of water consumption only. It is nature when there is no water or consumption reaches the maximal water reserve W, the consumption rate must be zero. Let us consider what the function looks like. The Logistic model is the simplest one (Fig. 2).

In the Logistic model, $\frac{dG(w)}{dw} < 0$, that means the consumption rate per unit water consumption $\frac{G(w)}{w}$ is a decreasing function of water consumption w. However, sometimes it is possible that the consumption rate per unit water consumption $\frac{G(w)}{w}$ is first increasing with w then decreasing. The shape of $G(w)$ is first convex then concave. As a special case, in the limit, $G(w)$ is first initially negative then positive. Then we get the different types of natural consumption rate functions as shown in Fig. 2. This kind of function can be seen in other fields (Gatto, M., 1987).

Insert < Figure 2 > here

Disaster risk function can be measured in different ways. We suggest it is the damage or possible damage the disaster causes to cities, measured in money. r can be quite complex.

Let m represents, in suitable unites, the amount of labor and capital invested.In real world situations M may be a rather complex function of w, r, m. When management policy and disaster factor are considered, the rate of change $\frac{dw}{dt}$ of water consumption is the difference between natural consumption rate and them.

Based on the above analysis process, we present the following differential model for IDRM

$$\begin{cases} \frac{dw}{dt} = G(w) - ar - bm, \\ \frac{dr}{dt} = R(w, r, m), \\ \frac{dm}{dt} = M(w, r, m), \end{cases} \tag{1}$$

where a and b are constants representing the impact strength the disaster risk and the management have on the coping capability.

System (1) is quite subtle. In order to know its actually evolution, every function and parameter must be given. This is a big challenge. However in many occasions what we want to know is just the trend not the exact mathematical numbers. Thus in the coming sections we will only study two special cases.

3. Slow-fast Vital model

Assume the city or region is always under the threat of drought, that is $r(t) = const.$. So the management function M is specified as a function of water consumption and the disaster risk strength, i.e. $M = M(w, m)$. Since the draught exists as a constant, it is reasonable to take a steady policy to deal with the drought. Mathematical speaking, the rates at which the management function evolute in Vitae System can be much slower than that of the water consumption. Hence we rewrite $M = \varepsilon\tilde{M}(w, m)$. The value of ε can be determined in such a way that the maximum absolute values of $\frac{dw}{dt}$ and $\tilde{M}(w, m)$ are the same.

$$\begin{cases} \frac{dw}{dt} = G(w) - ar - bm, \\ \frac{dm}{dt} = \varepsilon\tilde{M}(w, r, m), \end{cases} \tag{2}$$

System like (2) is usually called “slow-fast” system (Rinaldi, S., 2000, p507-521). Given the initial conditions, when ε is very small trying to integrate the system (1) is not easy, even with the most powerful simulation software. That is because it is almost impossible to keep numerical errors under control when dealing simultaneously with numbers differ by a few orders of magnitude. Fortunately Geometric Analysis provides an effective way to treat this kind of problem (Rinaldi, S., 2000, p507-521). The singular perturbation approach allows one to study the dynamics of the system in a very clear and appealing geometrical form. (1) is first studied in the singular case $\varepsilon = 0$ which is usually a bifurcation analysis. When ε is very small, the real solution differs less than ε from the singular solution.

In the limit case $\varepsilon = 0$, we get $\frac{dw}{dt} = G(w) - ar - bm$, then $m = \frac{G(w)-ar}{b}$. The relation between m and w is just a linear transformation. So we can get the state variable bifurcation graph from the consumption rate functions.

Trajectories can be obtained from bifurcation graph even if one does not know the exact function $\tilde{M}$ but knows only where $\tilde{M}$ is positive and where it is negative. If $\tilde{M}$ is positive, m increases and the singular trajectory develops from the left to the right, whereas it develops in the opposite direction if $\tilde{M}$ is negative.

3.1 Logistic case

Let w_1 and w_2 ($w_1 < w_2$) be the roots of $G(w) - ar - bm = 0$. They represent the smallest and highest water consumption respectively. Fig. 3 shows the bifurcation diagram when the consumption rate is Logistic. A stable equilibrium (upper solid line) collides with an unstable equilibrium (dashed line) at the critic value m^* of the parameter. The arrows in the figure indicate the time evolution of the water consumption. Longer arrow means faster evolution. Assume the smallest consumption $w = w_1$ be a stable equilibrium for all values of m. Thus, for $m < m^*$ there are two attractors, and their basins of attraction are separated by the unstable equilibrium (dashed line); whereas for $m > m^*$ there is only one global attractor (the smallest consumption).

Insert < Figure 3, Figure 4 > here

A tiny perturbation (Δ) of the critic value m^* induces a large shift of the state of the system. Suppose that m is slightly smaller than m^* and that the system is at its stable equilibrium (solid line)and that m is increased by a small amount Δ such that $m + \Delta > m^*$. After such a perturbation the consumption will start moving downward. At first, this motion is very slow. But after some time, the motion becomes fast and the consumption goes down to the smallest state. Such a shift in dynamic systems may have large economic or social consequences.

Suppose the management effort is strengthened very slowly, the consumption will vary accordingly. From the bifurcation diagram, we can see the consumption variable move slowly to the right along the solid line until the fold point is reached, then drop vertically (fast transient) to the lower stable branch, and finally, continue to move to the right along the lower branch. Thus, the time evolution of the water consumption can be obtained as in Fig. 4. The consumption remains higher for a long time, then it drops to the smallest in a relatively short time and finally remains at the smallest level forever. It is of no significance to strengthen the management any more when the management strength has already reached the critical value m^*.

3.2 Critical case: Vital rhythms

Fig. 5 shows the bifurcation diagram when the consumption rate is critical. The solid lines indicate stable equilibria, whereas the dashed line represents an unstable equilibrium. The system is bistable because it has two alternative stable equilibria for $m^* < m < m^{**}$.

Insert < Figure 5-8 > here

Consider the situation in which the managing authority adjusts its regulation adaptively in such a way that management is enhanced if water consumption exceeds a certain limit. As shown in Figs. 6 and 7, this regulation can be interpreted as the isocline of zero change in management ($\frac{dm}{dt} = 0$) separating the area of increasing from that of decreasing consumption.

In the tight case (Fig. 6), the management isocline intersects the bifurcation diagram at the upper branch, resulting in a unique stable equilibrium E with relatively large consumption from any initial state, the system converges to this water consumption equilibrium through a series of fast and slow transitions, as illustrated by two singular trajectories in the figure. Note that the first trajectory is composed of one fast $Q \rightarrow Q^*$ and one slow Q^*E phase, whereas the second trajectory is composed of four alternate fast (double arrows) and slow (single arrow) phases. On the other hand, a strict management, such that the dotted isocline intersects at the lower branch of the bifurcation diagram, would result in a similarly unique low consumption stable state.

In the loosen case, the isocline separates the two stable pieces of the bifurcation diagram (Fig. 7), the system converges to a cycle from any initial state (Fig. 8). Such cycles are characterized by periods of relatively little change, separated by rapid dramatic transitions in the consumption. For that reason such cycles are called slow-fast cycles. Such slow-fast cycles were observed in the form 10-day average water saving percentage in (Okada, N., 2006) and were called Vital rhythms.

4. Chaotic model

If $M = const.$ and $r = const$, the water consumption has the fast transition and the relaxation cycles. It is reasonable for both the disaster risk and the management to change. Climatic changes periodically. Assume the risk is periodic and the management changes according to the consumption and management itself. In slow-fast system, Van der Pol system is a famous equation, and for simplicity we assume the management differential function is linear. We obtain the following integrate system under periodic risk.

$$\begin{cases} \frac{dw}{dt} = w - w^3/3 - ar - bm, \\ \frac{dr}{dt} = fcos(\omega t), \\ \frac{dm}{dt} = cw + \alpha - \beta m, \end{cases} \tag{3}$$

Notice that (3) can be rewrite as a planar system which is chaotic for some special paremeters (Ramesh, M., 2001, p2395-2405). The integrate disaster system can be chaotic under some cases. Taken $f = 0.74, \alpha = 0.07, \beta = 0.08, c = 0.1$, we get a chaotic attractor as shown in Fig. 9. The consumption varies unpredictable with the management as shown in Fig. 10. There are many researches on chaotic system (Ramesh, M., 2001, p2395-2405).

Insert < Figure 9, Figure 10 > here

5. Conclusion

We gave a differential system as the model for integrated drought risk management. When the risk was taken to be constant, this model turned to be a slow-fast system. Geometric analysis was applied to it. Various kind of water consumption trends under different management were displayed. Under a period risk background, the differential model might be chaotic for some system parameters.

The mathematical models were our beginning study on quantifying the Vitae system. We took a short-cut approach to the problem by only researching the constant and periodic risk backgrounds. Further studies such as practical simulation, model explanation and applications are needed.

Acknowledgements

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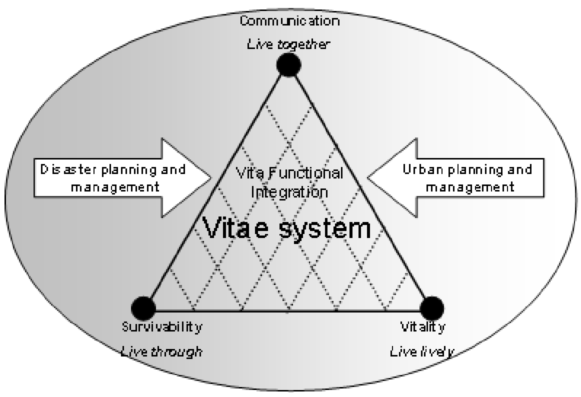

Figure 1. Integrated Urban (Reginal) Management Viewed as Vitae System [2]

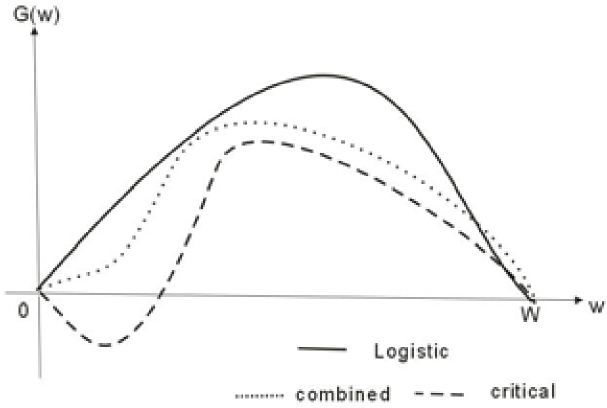

Figure 2. Different types of natural consumption rate functions

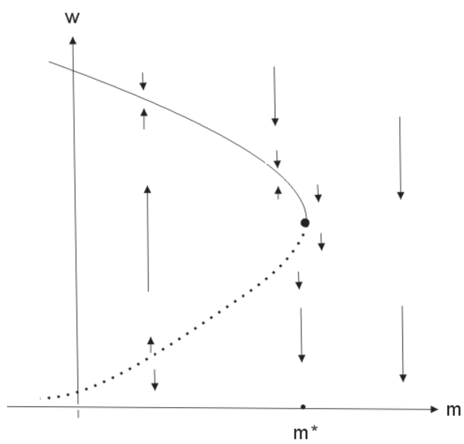

Figure 3. Logistic case

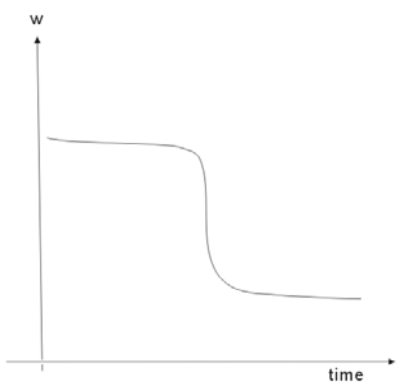

Figure 4. Water consumption to time

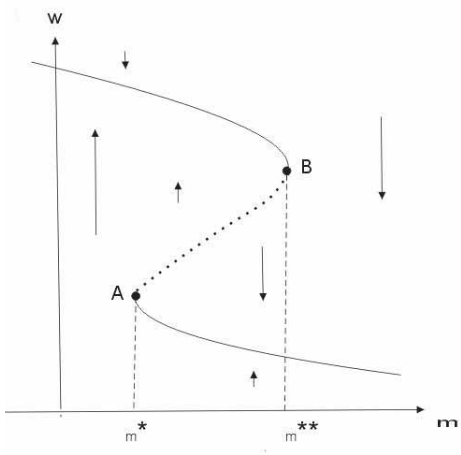

Figure 5. Bifurcation diagram for critical case

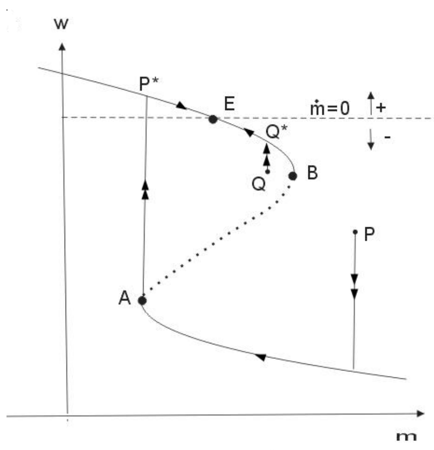

Figure 6. Tight management cause water consumption to a stable equilibrium

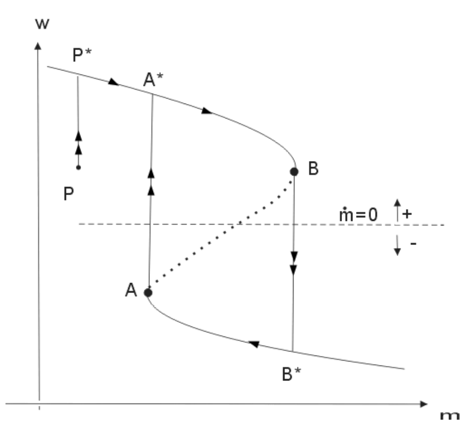

Figure 7. Loose management

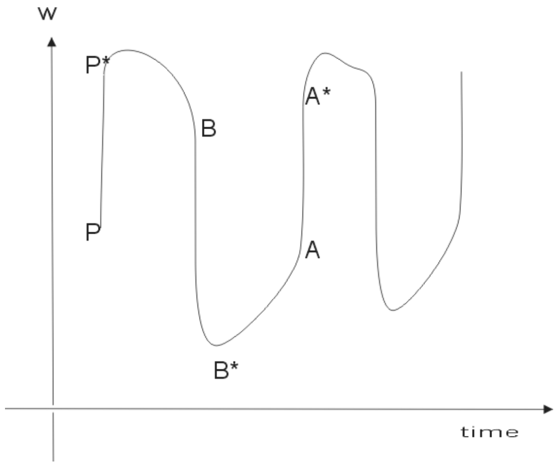

Figure 8. Vital rhythms

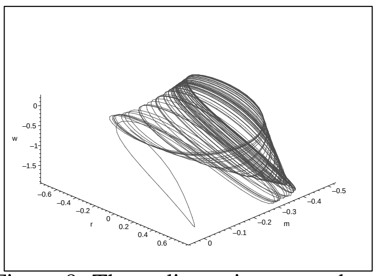

Figure 9. Three-dimension state phase

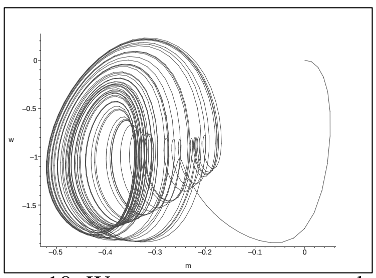

Figure 10. Water-management state phase

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Delay-dependent Robust Stability of Uncertain Discrete-Time Switched Systems

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Abstract

This paper address the problems of robust stability for uncertain discrete-time switched systems. The uncertainty is assumed to be of structured linear fractional from which includes the norm-bounded uncertainty as a special case. By introducing a novel difference inequality, new delay-dependent stability criteria are formulated in terms of linear matrix inequalities(LMIs) which are not contained in known literature. Numerical examples are given to demonstrate the effectiveness of the theoretical results.

Keywords: Difference inequality, Discrete-time switched system, Delay-dependent stability, Linear matrix inequality

1. Introduction

Switched systems are a class of hybrid dynamical systems consisting of a family of continuous- (or discrete-) time subsystems, and a rule that orchestrates the switching between them. It have gained a great deal of attention mainly because various real-world systems,such as chemical processing (S.Engell, 2000), communication networks, traffic control (R.Horowitz, 2000; C. Livadas, 2000; P.Varaiya, 1993), the control of manufacturing systems (D.Pepyne,2000; M.Song, 2000), and automotive engine control and aircraft control (P.Antsaklis, 2000) can be modeled as switched systems. In the last two decades, there has been increasing interest in stability analysis and controller design for switched systems, the reader is referred to the survey paper (Hai lin, 2009), and the references therein. Beside many researches on the continuous switched system (S.Pettersson, 1997)-(Kim. S, 2006), the discrete switched system has also been considered in many paper (Du D., 2006)-(Yuangong Sun, 2007) and see the references therein. It has been recognized that time delays, which are the inherent features of many physical process, are the big sources of instability and poor performances. For time delay systems, stability criteria are usually classified into two types: delay-independent criteria and delay-dependent ones. In general, delay-independent criteria are conservative since they can not handle the systems whose stability depends on the size of time delay. Recently, (Yuangong Sun, 2006) and (Yuangong Sun, 2007) obtained the delay dependent stability condition of uncertain discrete-time switched systems, However, the results of discrete delay is small to some extent. It may be improved significantly with some useful approaches, this has motivated our research.

In this paper, we are interested in establishing delay-dependent stability criteria in terms of linear matrix inequalities (LMIs) for the uncertain discrete-time switched delay systems under arbitrary switching sequences. The main idea of our method is inspired by Zhang's recent work (Xianming Zhang, 2006), where some novel integrate inequalities is introduced for stability analysis and controller synthesis of continuous deterministic delay systems. We extend this approach to uncertain discrete-time switched delay systems based on the constructed switched Lyapunov functionals (J.Daafouz, 2002). The advantage of the introduction of difference inequality lies in that it considerably reduces the conservatism entailed in the previously developed transformation methods since it isn't transform the systems which could introduce additional dynamics in the sense defined in (Gu. K., 2001). Another important idea of the proposed method is that some free weighting matrices are introduced properly to counteract the influence, bringing by the difference inequality, to the delays. Note that these advantages are not obtained at the cost of high computational complexity. Finally, numerical examples are given to illustrate the superiority of present result to those in the literature.

This paper is organized as follows. In Section2, we give the problem formulation and introduce an important lemmas to our later results. Section3 is dedicated to stability analysis of switched systems by mean of a switched quadratic Lyapunov function and our lemma. Two numerical evaluations are given in Section4.

2. Problem preliminaries

Nomenclature

R^n n-dimensional real space

$R^{n\times n}$ set of all real n by n matrices

x^T or A^T transpose of vector x (or matrix A)

$P > 0$ (respectively, $P < 0$) matrix P is symmetric positive (respectively, negative) definite

$P \geq 0$ (respectively, $P \leq 0$) matrix P is symmetric positive (respectively, negative) semi-definite

* the elements below the main diagonal of a symmetric block matrix

Consider linear switched system in the domain of discrete time:

$$\begin{array}{l} x(k+1) = A(k, r(k))x(k) + B(k, r(k))x(k-d), \\ x(s) = \phi(s), \quad s = -d, \ldots, -1, 0. \end{array} \tag{1}$$

where $x(k) \in R^n$ is the system state, $r(k) : Z^+ = \{0, 1, 2, \ldots\} \to \mathcal{N} = \{1, 2, \ldots, N\}$ is the control signal. d denote the delay of the system. $\phi : \{-d, -d+1, \ldots, 0\} \to R^n$ represents the initial condition. For each $i \in \mathcal{N}$, the system matrices are assumed to be uncertain and satisfy:

$$\begin{bmatrix} A(k,i) & B(k,i) \end{bmatrix} = \begin{bmatrix} A_i & B_i \end{bmatrix} + H_i \triangle \begin{bmatrix} E_{i1} & E_{i2} \end{bmatrix} \tag{2}$$

$$\triangle = [I - F(k)J]^{-1} F(k) \tag{3}$$

$$0 < I - JJ^T \tag{4}$$

where A_i, B_i are constant matrices that describe the ith nominal mode, H_i, E_{i1} and E_{i2} are given constant matrices which characterize the structure of the uncertainty, and the admissible uncertain matrix $F(k)$ satisfies

$$F^T(k)F(k) \leqslant I \tag{5}$$

for $k \in Z^+$.

The linear fractional parametric uncertainties have been investigated in the robust control setting as related in (Du D., 2006) It is easy to see that when $J = 0$, the linear fractional uncertainty reduces to norm bound one. Notice also that condition (4) guarantees that $I - FJ$ is invertible.

We are here interested to establish delay-dependent robust stability criteria for systems (1) by introducing a novel difference inequality and using linear matrix inequality technique. Before giving the main theorem of this paper, we firstly provide the following lemmas which plays an important role in our later development.

Lemma 2.1 (S.-S. Zhou, 2003) Suppose that $\triangle$ is given by (2)-(5), with matrices $M = M^T$, S and N of appropriate dimensions. Then the inequality

$$M + S\triangle N + N^T \triangle^T S^T < 0,$$

holds for any F such that $FF^T \leqslant I$, if and only if for some $\delta > 0$,

$$\begin{pmatrix} \delta M & S & \delta N^T \\ S^T & -I & J^T \\ \delta N & J & -I \end{pmatrix} < 0.$$

Lemma 2.2 (Finsler's lemma) For vector $x \in R^n$, matrix $P \in R^{n\times n}$ and $H \in R^{m\times n}$, satisfying $rank(H) = r < n$, the following statements are equivalent,
(i) $\forall\, x \neq 0$ and $Hx = 0$, satisfying $x^T Px < 0$;
(ii) $\exists X \in R^{n\times m}$, satisfying $P + XH + H^T X^T$.

Lemma 2.3 For any constant symmetric matrix $Q \in R^{n\times n}$, $Q = Q^T > 0$, and any appropriate dimensional matrices, $M_1 \in R^{n\times n}$, $M_2 \in R^{n\times n}$, $Z = \begin{pmatrix} Z_{11} & Z_{12} \\ * & Z_{22} \end{pmatrix} \in R^{2n\times 2n}$, $Y = \begin{bmatrix} M_1 & M_2 \end{bmatrix} \in R^{n\times 2n}$, if $\begin{pmatrix} Q & Y \\ * & Z \end{pmatrix} > 0$, we have

$$-2\sum_{l=k-d}^{k-1} x^T(l)Qx(l) \leq \xi^T(k) \begin{pmatrix} \Lambda_{11} & \Lambda_{11} \\ * & \Lambda_{22} \end{pmatrix} \xi(k)$$

with $\xi^T(k) = \begin{bmatrix} x^T(k) & x^T(k-d) \end{bmatrix}$, where,

$$\Lambda_{11} = M_1 + M_1^T + dZ_{11} + Q + dM_1^T Q^{-1} M_1,$$

$$\Lambda_{12} = -M_1^T + M_2 + dZ_{12} + dM_1^T Q^{-1} M_2,$$

$$\Lambda_{22} = -M_2 - M_2^T + dZ_{22} - Q + dM_2^T Q^{-1} M_2.$$

Proof. With the fact,

$$x(k) - x(k-d) - \sum_{l=k-d}^{k-1} (x(l+1) - x(l)) = 0,$$

$\forall N_1, N_2 \in R^{n\times n}$, we have,

$$\begin{aligned} 0 &= 2[x^T(k)N_1^T + x^T(k-d)N_2^T][x(k) - x(k-d) - \sum_{l=k-d}^{k-1}(x(l+1) - x(l))] \\ &= 2\xi^T(k)N^T \begin{bmatrix} I & -I \end{bmatrix} \xi(k) - 2\xi^T(k)N^T \sum_{l=k-d}^{k-1} x(l+1) + 2\xi^T(k)N^T \sum_{l=k-d}^{k-1} x(l) \end{aligned} \tag{6}$$

where $N = \begin{bmatrix} N_1 & N_2 \end{bmatrix}$, $\xi^T(k) = \begin{bmatrix} x^T(k) & x^T(k-d) \end{bmatrix}$, by using the Moon's inequality (Moon Y.S., 2001), we have,

$$\begin{aligned} -2\xi^T(k)N^T \sum_{l=k-d}^{k-1} x(l+1) &\leqslant \sum_{l=k-d}^{k-1} \begin{pmatrix} x(l+1) \\ \xi(k) \end{pmatrix}^T \begin{pmatrix} Q & Y-N \\ Y^T - N^T & Z \end{pmatrix} \begin{pmatrix} x(l+1) \\ \xi(k) \end{pmatrix} \\ &= \sum_{l=k-d}^{k-1} x^T(l+1)Qx(l+1) + d\xi^T(k)Z\xi^T(k) \\ &\quad +2\xi^T(k)(Y^T - N^T)\begin{bmatrix} I & -I \end{bmatrix}\xi(k) \\ &\quad +2\xi^T(k)(Y^T - N^T)\sum_{l=k-d}^{k-1} x(l) \end{aligned} \tag{7}$$

Substitute (7) into (6), and with the fundamental inequality, we get

$$\begin{aligned} 0 &\leqslant 2\xi^T(k)Y^T \begin{bmatrix} I & -I \end{bmatrix}\xi(k) + d\xi^T(k)Z\xi(k) + 2\xi^T(k)Y^T \sum_{l=k-d}^{k-1} x(l) \\ &\quad + \sum_{l=k-d}^{k-1} x^T(l+1)Qx(l+1) \\ &\leqslant 2\xi^T(k)Y^T \begin{bmatrix} I & -I \end{bmatrix}\xi(k) + d\xi^T(k)Z\xi(k) + d\xi^T(k)Y^T Q^{-1} Y\xi(k) \\ &\quad + \sum_{l=k-d}^{k-1} x^T(l)Qx(l) + \sum_{l=k-d}^{k-1} x^T(l+1)Qx(l+1) \\ &= 2\xi^T(k)Y^T \begin{bmatrix} I & -I \end{bmatrix}\xi(k) + d\xi^T(k)Z\xi(k) + d\xi^T(k)Y^T Q^{-1} Y\xi(k) \\ &\quad + 2\sum_{l=k-d}^{k-1} x^T(l)Qx(l) + \xi^T(k)\begin{pmatrix} Q & 0 \\ 0 & -Q \end{pmatrix}\xi(k) \end{aligned}$$

it can easy be seen from this that the conclusion is true.

3. Main results

In this section, we present asymptotically stability criteria dependent on delays for the uncertain discrete-time switched systems described by (1) and (2) with strict LMI approaches.

For system (1), we define the following switched Lyapunov function :

$$V(k, x(k)) = x^T(k)P_{r(k)}x(k) + 2\sum_{\theta=-d+1}^{0}\sum_{l=k-1+\theta}^{k-1} x^T(l)Qx(l) \tag{8}$$

with $P_1, P_2, \ldots, P_N, Q$ being symmetric positive definite matrices.

If such a Lyapunov function exists and its difference $\triangle V(k, x(k)) = V(k+1, x(k+1)) - V(k, x(k))$ is negative definite along the solution of (1), the the origin of the system(1) is globally asymptotically stable as shown by the following general lemma.

Lemma 3.1 (M. Vidyasagar, 1993) The equilibrium 0 of

$$x(k+1) = f(x(k)) \tag{9}$$

is globally uniformly asymptotically stable if there is a function $V : \mathbb{Z}^+ \times \mathbb{R}^{\ltimes} \to \mathbb{R}$ such that,
(i) V is a positive definite function, decrescent, and radially unbounded;
(ii) $\triangle V(k, x(k)) = V(k+1, x(k+1)) - V(k, x(k))$ is negative definite along the solution of (9).

For the asymptotically stability of systems described by (1), we have the following result.

Theorem 3.1 The systems (1) is asymptotically stability, if there exist symmetric matrices $P_1, P_2, \ldots, P_N, Q, Z_{11}, Z_{22} \in R^{n\times n}$ and any appropriate dimensional matrices G_i, T_i, U_i, M_1, $M_2 \in R^{n\times n}$, such that the following LMIs holds,

$$\begin{pmatrix} Q & M_1 & M_2 \\ * & Z_{11} & Z_{12} \\ * & * & Z_{22} \end{pmatrix} > 0, \tag{10}$$

$$\Gamma = \begin{pmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} & 0 \\ * & \Gamma_{22} & \Gamma_{23} & dM_1 \\ * & * & \Gamma_{33} & dM_2 \\ * & * & * & -dQ \end{pmatrix} < 0, \quad i, j \in \mathscr{N}, \tag{11}$$

where

$$\Gamma_{11} = P_j - G_i^T - G_i, \quad \Gamma_{12} = G_iA(k,i) - U_i^T, \quad \Gamma_{13} = G_iB(k,i) - W_i^T,$$
$$\Gamma_{22} = -P_i + M_1 + M_1^T + U_iA(k,i) + A^T(k,i)U_i^T + (2d+1)Q + dZ_{11},$$
$$\Gamma_{23} = -M_1^T + M_2 + U_iB(k,i) + A^T(k,i)W_i^T + dZ_{12},$$
$$\Gamma_{33} = -M_2 - M_2^T - Q + W_iB(k,i) + B^T(k,i)W_i^T + dZ_{22}.$$

Proof. Choose a switching Lyapunov functional candidate for systems (1) as following:

$$V(k, x(k)) = x^T(k)P_{r(k)}x(k) + 2\sum_{\theta=-d+1}^{0}\sum_{l=k-1+\theta}^{k-1} x^T(l)Qx(l)$$

Let the mode at time k and $k+1$ be i and j, respectively .That is,$r(k) = i$ and $r(k+1) = j$ for any $i, j \in \mathscr{N}$. Along the solution of (1), and using Lemma 2.3, we have

$$\begin{aligned} \triangle V(k, x(k)) &= V(k+1, x(k+1)) - V(k, x(k)) \\ &= x^T(k+1)P_jx(k+1) - x^T(k)P_ix(k) + 2dx^T(k)Qx(k) \\ &\quad -2\sum_{l=k-d}^{k-1} x(l)^TQx(l) + 2dx^T(k)Qx(k) - 2\sum_{l=k-d}^{k-1} x(l)^TQx(l) \\ &\leqslant \zeta^T\Phi(i,j)\zeta \end{aligned}$$

where $\zeta^T = \begin{pmatrix} x^T(k+1) & x^T(k) & x^T(k-d) \end{pmatrix} \neq 0$, and

$$\Phi(i,j) = \begin{pmatrix} P_j & 0 & 0 \\ * & \Phi_1 & \Phi_2 \\ * & * & \Phi_3 \end{pmatrix}, \tag{12}$$

with

$$\Phi_1 = -P_i + M_1 + M_1^T + (2d+1)Q + dZ_{11} + dM_1^TQ^{-1}M_1,$$
$$\Phi_2 = -M_1^T + M_2 + dZ_{12} + dM_1^TQ^{-1}M_2, \quad \Phi_3 = -Q - M_2 - M_2^T + dZ_{22} + dM_2^TQ^{-1}M_2,$$

applying Schur's complement (Boyd S., 1993), (11) is equivalent to the following,

$$\Phi(i,j) + \begin{pmatrix} G_i \\ U_i \\ W_i \end{pmatrix}\begin{pmatrix} -I & A_i & B_i \end{pmatrix} + \begin{pmatrix} -I \\ A_i^T \\ B_i^T \end{pmatrix}\begin{pmatrix} G_i^T & U_i^T & W_i^T \end{pmatrix} < 0$$

Therefore, we have $\triangle V(k, x(k)) \leqslant \zeta^T \Phi(i, j)\zeta < 0$ for all $k \geqslant 0$ from the Finsler's lemma. This completes the proof of Theorem1 according to Lemma 3.1.

Remark 3.1 As is well known, it could bring conservativeness inevitably if one use inequality analysis technique to analyze the stability of delay systems. In this paper, it may reduce the conservativeness of our results by introducing some free weighting matrices appropriately with Finsler's lemma.

Remark 3.2 It should be noted that Theorem 3.1 is obtained by using the Lyapunov functional $V(k, x(k))$ given by (8). It is clear that when $P_{r(k)} = P$ for any $i \in \{1, 2, \cdots, N\}$, $V(k, x(k))$ becomes $V(k, x(k)) = x^T(k)Px(k) + 2\sum_{\theta=-d+1}^{0}\sum_{l=k-1+\theta}^{k-1} x^T(l)Qx(l)$, which is called a single quadratic function and has been widely used in research work on this topic. Compared with their results, we can see that our results are more general and less conservative by Example 1.

Considered the uncertainty described by (2)-(5), similar to the proof of Theorem 3.1, we can obtain the following Corollary.

Corollary 3.1 The systems (1) with uncertainty as above mentioned is robust asymptotically stability, if there exist constants $\delta_i > 0$, symmetric matrices $P_1, P_2, \ldots, P_N, Q, Z_{11}, Z_{22} \in R^{n\times n}$ and any appropriate dimensional matrices $G_i, T_i, U_i, M_1, M_2 \in R^{n\times n}$, such that the following LMIs holds,

$$\begin{pmatrix} Q & M_1 & M_2 \\ * & Z_{11} & Z_{12} \\ * & * & Z_{22} \end{pmatrix} > 0 \tag{13}$$

$$\Psi = \begin{pmatrix} \delta_i\Psi_{11} & \delta_i\Psi_{12} & \delta_i\Psi_{13} & 0 & G_iH_i & 0 \\ * & \delta_i\Psi_{22} & \delta_i\Psi_{23} & \delta_i dM_1 & U_iH_i & \delta_i E_{i1}^T \\ * & * & \delta_i\Psi_{33} & \delta_i dM_2 & W_iH_i & \delta_i E_{i2}^T \\ * & * & * & -\delta_i dQ & 0 & 0 \\ * & * & * & * & -I & J^T \\ * & * & * & * & * & -I \end{pmatrix} < 0 \tag{14}$$

where

$$\Psi_{11} = P_j - G_i^T - G_i, \quad \Psi_{12} = G_iA_i - U_i^T, \quad \Psi_{13} = G_iB_i - W_i^T,$$
$$\Psi_{22} = -P_i + M_1 + M_1^T + U_iA_i + A_i^T U_i^T + (2d+1)Q + dZ_{11},$$
$$\Psi_{23} = -M_1^T + M_2 + U_iB_i + A_i^T W_i^T + dZ_{12},$$
$$\Psi_{33} = -M_2 - M_2^T - Q + W_iB_i + B_i^T W_i^T + dZ_{22}.$$

Proof. Using the uncertain condition (2), we have

$$\begin{aligned}\Gamma = &\begin{pmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} & 0 \\ * & \widetilde{\Gamma}_{22} & \widetilde{\Gamma}_{23} & dM_1 \\ * & * & \widetilde{\Gamma}_{33} & dM_2 \\ * & * & * & -dQ \end{pmatrix} + \begin{pmatrix} G_iH_i \\ U_iH_i \\ W_iH_i \\ 0 \end{pmatrix} \triangle \begin{pmatrix} 0 & E_{i1} & E_{i2} & 0 \end{pmatrix} \\ &+ \begin{pmatrix} 0 \\ E_{i1}^T \\ E_{i2}^T \\ 0 \end{pmatrix} \triangle^T \begin{pmatrix} H_i^TG_i^T & H_i^TU_i^T & H_i^TW_i^T & 0 \end{pmatrix} < 0\end{aligned} \tag{15}$$

where $\widetilde{\Gamma}_{22}, \widetilde{\Gamma}_{23}$ and $\widetilde{\Gamma}_{33}$ are taken from Γ_{22}, Γ_{23} and Γ_{33} in theorem 3.1 by replacing $A(k, i)$ and $B(k, i)$ with A_i and B_i respectively. Using lemma 1, a sufficient condition guaranteeing $\Gamma < 0$ is that there exists positive constants δ_i such that (13) and (14) are hold, which completes this proof.

4. Numerical examples

In order to show the effectiveness of the approaches presented in Section 3, in this section, two numerical examples are provided.

Example 1. Consider the uncertain systems described by (1) and (2) $\mathscr{N} = 1, 2$ and

$$A_1 = \begin{pmatrix} 0.8 & 0.2 \\ 0 & 0.91 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0.3 & a \\ b & 0.58 \end{pmatrix}, \qquad E_{11} = E_{12} = 0.01I, H_1 = cI$$

$$A_2 = \begin{pmatrix} -0.1 & 0 \\ -0.1 & -0.1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0.12 & 0 \\ 0.11 & 0.11 \end{pmatrix}, E_{21} = E_{22} = 0.01I, H_2 = cI.$$

It becomes nominal systems if we set $c = 0$. In this case, when $a = 0, b = 0$, using Theorem1, both the results in (Yuangong Sun, 2006) and our results are same, viz.$d \leqslant 1$. Whereas, when $a = 0.2, b = 0.1$, the delay d can be obtained as much as 21 by theorem1, while d in (Yuangong Sun, 2006) remain as $d \leqslant 1$. This comparison shows that our result is much less conservative than that in (Yuangong Sun, 2006).

For comparison, let $J = 0$, applying corollary 3.1 to this example shows that the system is robust stable for $d = 1$ as $a = b = 0, c = 0.1$, which is much less than (Yuangong Sun, 2007) whose results is $d \leqslant 5$. However, as $a \neq 0, b \neq 0$ our results is much less conservative than that in (Yuangong Sun, 2007). Take $a = 0.2, b = 0.1, c = 0.1$ for example, we can obtain the system is robust stable for $d \leqslant 11$ while its results in (Yuangong Sun, 2007) is $d \leqslant 5$ which has not changed again. And we can get

$$G_1 = \begin{pmatrix} 11.2108 & 5.9572 \\ 5.9967 & 39.8991 \end{pmatrix}, \quad G_2 = \begin{pmatrix} 157.8184 & -3.8026 \\ -5.8056 & 0.6278 \end{pmatrix}, \quad U_1 = \begin{pmatrix} -9.8902 & -10.5418 \\ -7.6314 & -37.0833 \end{pmatrix},$$

$$U_2 = \begin{pmatrix} 15.1804 & -0.3973 \\ -0.3151 & 0.0343 \end{pmatrix}, \quad W_1 = \begin{pmatrix} -3.9469 & -5.7376 \\ -5.7395 & -24.3965 \end{pmatrix}, \quad W_2 = \begin{pmatrix} -18.7054 & 0.3828 \\ 0.5412 & -0.0722 \end{pmatrix},$$

$$P_1 = \begin{pmatrix} 1.0008 & -0.1025 \\ -0.1025 & 0.0172 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0.0414 & -0.0264 \\ -0.0264 & 0.0169 \end{pmatrix}, \quad Q = \begin{pmatrix} 0.0027 & -0.0012 \\ -0.0012 & 0.0007 \end{pmatrix}.$$

This comparison shows that our result is also less conservative than that in (Yuangong Sun, 2007) when B_1 is not a diagonal matrix. This is also shows that our results and that in (Yuangong Sun, 2007) are not contain each other.

Example 2. Consider the systems described by (1) and assumed to have two modes,i.e., $\mathcal{N} = 1, 2$ with

$$A(k,1) = \begin{pmatrix} 0.62 & 0.27 \\ 0.13 & 0.91 \end{pmatrix}, \quad B(k,1) = \begin{pmatrix} 0.31 & 0.23 \\ 0.12 & 0.58 \end{pmatrix}, \quad A(k,2) = \begin{pmatrix} -0.25 & 0.36 \\ -0.18 & -0.71 \end{pmatrix}, \quad B(k,2) = \begin{pmatrix} 0.12 & 0.21 \\ 0.15 & 0.11 \end{pmatrix}.$$

Thus, we apply Theorem 3.1 to this example shows that the system is asymptotical stable for $d \leqslant 5$. It shows that our results are effective. On the other hand, by solving the inequalities(10) and (11),we get

$$G_1 = 1.0\times10^3 \begin{pmatrix} 2.2917 & -0.1749 \\ -1.1044 & 0.5517 \end{pmatrix}, \quad G_2 = 1.0\times10^3 \begin{pmatrix} 4.2759 & 2.7289 \\ 2.1015 & 4.2261 \end{pmatrix}, \quad U_1 = \begin{pmatrix} 270.6635 & -831.4655 \\ -396.6739 & 49.6896 \end{pmatrix},$$

$$U_2 = 1.0\times10^3 \begin{pmatrix} -0.1093 & 2.0249 \\ 1.6872 & 2.4878 \end{pmatrix}, \quad P_1 = 1.0\times10^3 \begin{pmatrix} 3.9212 & -0.8476 \\ -0.8476 & 1.0968 \end{pmatrix}, \quad Q = \begin{pmatrix} 236.7487 & 10.0861 \\ 10.0861 & 13.6549 \end{pmatrix},$$

$$P_2 = 1.0\times10^3 \begin{pmatrix} 2.5082 & -1.2497 \\ -1.2497 & 0.6600 \end{pmatrix}, \quad W_1 = \begin{pmatrix} -34.6963 & -26.6685 \\ -315.5136 & -151.8544 \end{pmatrix}, \quad W_2 = \begin{pmatrix} -420.9922 & -356.7214 \\ -871.1593 & -971.3711 \end{pmatrix},$$

This example also shows that our results obtained in this paper are effectiveness.

5. Conclusion

The robust stability for uncertain discrete-time switched systems with has been investigated. Based on the switched Lyapunov functional approach, combined with the introduced difference inequality and free matrix method, delay dependent stability criteria have been established in form of LMIs. Numerical examples have shown significant improvements over some existing results.

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Optimal Replacement Policies for a Multistate Degenerative System with Negligible or Nonnegligible Repair Times

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Abstract

In this paper, a repairable system of a monotone process model for a one-component multistate degenerative system with $(k + 1)$ states(k-failure states and one working state) is studied. Also an alternative model, called the *Negligible Or Non-Negligible* (NONN) repair times introduced by Thangaraj and Rizwan (2001) is incorporated in this model to develop some new repair models. A replacement policy T is adapted by which the system will be replaced whenever the working age of the system reaches T; another replacement policy T but with NONN repair times; the N policy, based on the number of failures of the system assuming NONN repair times and a bivariate replacement policy (T, N) under NONN repair times, where T is the working age and N is the number of failures of the system are studied. Furthermore, explicit expressions for the long-run average cost of the above policies are derived. Also the conditions for the existence of univariate optimal replacement policies are derived. Finally, we show that the optimal policy $(T, N)^*$ is better than the optimal policy N^* or the optimal policy T^*. We conclude with a conjecture that N-policy is the best replacement policy for any deteriorating system among the univariate policies.

Keywords: Geometric process, Replacement policy, Renewal reward process, Stochastic orders, NONN repair times

1. Introduction

The mathematical theory of reliability has applied itself to problems of life-testing, machine maintenance, replacement, order statistics, etc. Most of the systems are degenerative, because of the ageing effect and accumulated wearing. The successive operating times between failures tend to decrease, while the consecutive repair times after failures tend to increase. That is, the successive operating times are stochastically decreasing, while the consecutive repair times are stochastically increasing. In practice, the repair of a failure system will usually yield a functioning system, its successive operating times are decreasing and finally dying out. For deteriorating systems, this is often the case. On the other hand, the consecutive repair times will be increasing and finally, tend to infinity, that is, finally the system is non-repairable. Thus, it is reasonable to assume that the repair times are, in general, non-negligible. This is referred to as a *geometric process*, introduced and studied by Lam (1988) in his pioneering work.

In the above models, a system, in general, has only two states, the operating and the failure states; usually referred to as: *up* and *down* states. However, in real life situations, a system, in general, may have more than two states. For example, the system constituted by an electronic instrument may break down due to a short circuit or an open circuit, in which case the system will have two failure states and one working state. Lam, Zhang and Zheng (2002) have considered an equivalent geometric process model for a multistate degenerative system with k-failure states and one working state. Also, they have given an analytic expression for the long-run average cost per unit time using a replacement policy which depends only on the number of failures of the system. However Zhang, Yam and Zuo(2002) have derived the long-run expected profit for the model introduced and studied by Lam, Zhang and Zheng (2002). Further Zhang, Yam and Zuo (2007) have derived an explicit expression for the long-run expected profit for the same model, considering a bivariate replacement policy, which depends either on the working age of the system or the number of repairs of the system. There is always a speculation about the influence of working age of a system. It urges us to consider, in this paper, a replacement policy which depends on the working age of the system for a multistate degenerative system with k-failure states and one working state. Further, repair times are generally assumed to be non-decreasing in the case of geometric process. It is not true that the repair times always tend to increase. The repair times usually depend on the type of failure occurred at a specific moment in the system. To overcome this, Thangaraj and Rizwan (2001) have introduced probability for different types of failures of a repairable system and called it NONN repair times. Thus it is reasonable to assume this alternative repair model. Considering this alternative repair model, it is proposed to study some univariate and bivariate optimal replacement policies and desired to compare these policies using a numerical example.

Here a customary summary of known definitions of some basic concepts which are needed for our discussion is outlined below.

Definition 1.1. A random variable X is said to be *stochastically smaller than* another random variable Y, if $P(X > \alpha) \leq P(Y > \alpha)$, for all real α. It is denoted by $X \leq_{st} Y$. Further, a stochastic process $\{X_n, n = 1, 2, \ldots\}$ is said to be stochastically increasing, if $X_n \leq_{st} X_{n+1}$, for $n = 1, 2, \ldots$.

The definition of monotone concept for a stochastic process $\{X_n, n = 1, 2, \ldots\}$ is based on the distributions of X_n and X_{n+1}, for $n = 1, 2, \ldots$. Lam(1988) has introduced the geometric process for a simple monotone process.

Definition 1.2. A stochastic process $\{X_n, n = 1, 2, \ldots\}$ is a *geometric process* (GP), if there exist a real constant $a > 0$ such that $\left\{a^{n-1}X_n, n = 1, 2, \ldots\right\}$ forms a renewal process. The number a is called the ratio of the geometric process.

If $0 < a < 1$, the GP is stochastically increasing; if $a > 1$, the GP is stochastically decreasing and if $a = 1$, the GP will reduce to a renewal process.

Definition 1.3. An integer valued random variable N is said to be a *stopping time* for the sequence of independent random variables $X_1, X_2, \ldots,$ if the event $\{N = n\}$ is independent of $X_{n+1}, X_{n+2}, \ldots,$ for all $n = 1, 2, \ldots$.

Theorem 1.4. Wald's equation. *If $X_1, X_2, \ldots$ are independent and identically distributed random variables having finite expectations and if N is the stopping time for $X_1, X_2, \ldots$ such that $E[N] < \infty$, then*

$$E\left[\sum_{n=1}^{N} X_n\right] = E(N)E(X_1).$$

Definition 1.5. *The T-policy.*

It is a policy under which the system will be replaced whenever the working age of the system reaches T.

Definition1.6. *The N-policy.*

It is a policy under which the system will be replaced upon the N-th failure of the system, since the last replacement.

Definition 1.7. *The bivariate (T, N)-policy.*
It is a policy under which the system will be replaced at the working age T or at the time of N-th failure since the last replacement, whichever occurs earlier.

Definition 1.8. *NONN Repair times*
If a repair to a system after failure is done in negligible or non-negligible time, then it will be called a model with *NONN repair times*.

In this case, whenever the system fails, two possibilities may arise: either, the repair takes Negligible time with probability p; or Non-Negligible time with probability $1 - p$.

The main objective of this paper is to obtain explicit expressions for the long-run average cost per unit time for the maintenance model of a multistate degenerative system with k-failure states and one working state under the aforesaid preventive maintenance policies, assuming different repair models.

The rest of the paper is organized as follows: In Section 2, we present the model and derive an expression for the long-run average cost per unit time for this model under T-policy. We also derive the conditions for existence of the optimal replacement policy T^* (under this model) in this section. In Section 3, we present the alternative repair model and derive an explicit expression for the long-run average cost per unit time for this model. The existence of the optimal replacement policy is also derived in this section. In Section 4, we derive an expression for the long-run average cost for this model under N-policy. The necessary conditions for the existence of the optimal replacement policy N^* are also derived in this section. In Section 5, we derive an expression for the long-run average cost for this model under a bivariate replacement policy (T, N). Comparison between policies have been made in Section 6. A numeric example is given in Section 7, to illustrate the results, developed in this paper. Comparison of numerical result have been carried out in Section 8. Finally, conclusion is given in Section 9.

2. The Replacement Policy T

In this section, we introduce and study a T-policy for a multistate one component degenerative system. Under the replacement policy T, the problem is to determine an optimal replacement policy T^* such that the long-run average cost per unit time is minimized.

We consider a monotone process model for a multistate one-component degenerative system and following Lam, Zhang and Zheng (2002), we make the following assumptions. For the sake of readability, we state the same here.

Assumption 2.1. At time $t = 0$, a new system is put into field use. Whenever the system fails, it will be repaired. The system will be replaced by an identical new one, some time later.

Assumption 2.2. The system state at time t, denoted by $S(t)$ is

$$S(t) = \begin{cases} 0, & \text{if the system is working at time } t \\ i, & \text{if the system is in the } i\text{-th type of failure} \\ & \text{state at time } t,\ i = 1, 2, \ldots, k. \end{cases} \tag{2.1}$$

Thus the state space is $\Omega = \{0, 1, 2, \ldots, k\}$. If the system fails, then with probability p_i, the system will be in state i, $i = 1, 2, \ldots, k$ and $\sum_{i=1}^{k} p_i = 1$.

Assumption 2.3. Let X_1 be the first operating time. For $n \geq 2$, let X_n be the operating time of the system after $(n-1)$-st repair, let Y_n be the repair time after the n-th failure and Z be the replacement time. Now, denote the time of the n-th failure by t_n. Assume that

$$\begin{aligned} P(X_1 \leq t) &= U(t), \\ \text{and} \qquad P(X_2 \leq t | S(t_1) = i) &= U(a_i t), \quad i = 1, 2, \ldots, k. \end{aligned}$$

In general, for $j = 1, 2, \ldots, n-1;\ i_j = 1, 2, \ldots, k$,

$$P(X_n \leq t | S(t_1) = i_1, \ldots, S(t_{n-1}) = i_{n-1}) = U(a_{i_1} \cdots a_{i_{n-1}} t),$$

where $1 \leq a_1 \leq a_2 \leq \cdots \leq a_k$. Similarly, assume that

$$P(Y_1 \leq t | S(t_1) = i) = V(b_i t), \quad i = 1, 2, \ldots, k\ .$$

In general, for $j = 1, 2, \ldots, n;\ i_j = 1, 2, \ldots, k$,

$$P(Y_n \leq t | S(t_1) = i_1, \ldots, S(t_n) = i_n) = V(b_{i_1} \cdots b_{i_n} t),$$

where $1 \geq b_1 \geq b_2 \geq \cdots \geq b_k > 0$.

Assumption 2.4. The *working age* of the system at time T is the cumulative life-time given by

$$T(t) = \begin{cases} t - M_n, & L_n + M_n \leq t < L_{n+1} + M_n \\ L_{n+1}, & L_{n+1} + M_n \leq t < L_{n+1} + M_{n+1} \end{cases}$$

where $L_n = \sum_{i=1}^{n} X_i$ and $M_n = \sum_{i=1}^{n} Y_i$ and $M_0 = N_0 = 0$.

Assumption 2.5. Let r be the reward rate per unit time of the system when it is operating and c be the repair cost rate per unit time of the system. Assume further that the replacement cost comprises of two parts: one part is the basic replacement cost R and the other part is the cost proportional to the length of replacement time Z at rate c_p.

Assumption 2.6. The replacement policy T is adapted under which the system will be replaced whenever its working age reaches T.

Let $E(X_1) = \int_0^\infty t dU(t) = \lambda$ and $E(Y_1) = \int_0^\infty t dV(t) = \mu$. Following Lam, Zhang and Zheng (2002), we have $E(X_n) = \frac{\lambda}{a^{n-1}}$, where $a = \left(\sum_{i=1}^{k} \frac{p_i}{a_i}\right)^{-1}$ and $E(Y_n) = \frac{\mu}{b^n}$, where $b = \left(\sum_{i=1}^{k} \frac{p_i}{b_i}\right)^{-1}$.

Let T_1 be the first replacement time and in general for $n \geq 2$, let T_n be the time between $(n-1)$-st replacement and n-th replacement. Thus the sequence $T_1, T_2, \ldots$ constitutes a renewal process, while the interarrival times between two consecutive replacements is a renewal cycle. Further, a cycle is completed, if a replacement is done. By the theory of renewal reward process, the long-run average cost per unit time is given by

$$\begin{aligned}\mathscr{C}(T) &= \frac{\text{the expected cost incurred in a cycle}}{\text{the expected length of a cycle}}\\ &= \frac{cE\left(\sum_{i=1}^{\eta-1} Y_i\right) + R + c_p E(Z) - rE\left(\sum_{i=1}^{\eta} X_i\right)}{E\left(\sum_{i=1}^{\eta} X_i\right) + E\left(\sum_{i=1}^{\eta-1} Y_i\right) + E(Z)},\end{aligned} \tag{2.2}$$

where η is a random variable denoting the number of failures in time T. Since η is also a stopping time with respect to the σ-fields $\left\{\sigma < X_1, X_2, \ldots, X_\eta >, \quad \eta = 1, 2, \ldots\right\}$, by Wald's equation, we have

$$E\left(\sum_{i=1}^{\eta} X_i\right) = E\left(E\left[\sum_{i=1}^{\eta} X_i | \eta = n\right]\right) = \lambda \sum_{n=1}^{\infty} \frac{F_n(T)}{a^{n-1}}. \tag{2.3}$$

where $F_n(\cdot)$ is the n-fold convolution of $F(\cdot)$ with itself and

$$E\left(\sum_{i=1}^{\eta-1} Y_i\right) = E\left(E\left[\sum_{i=1}^{\eta-1} Y_i | \eta = n-1\right]\right) = \mu \sum_{n=1}^{\infty} \frac{G_n(T)}{b^n}, \tag{2.4}$$

where $G_n(\cdot)$ is the n-fold convolution of $G(\cdot)$ with itself.

Using equations (2.3) and (2.4), equation (2.2) becomes

$$\mathscr{C}(T) = \frac{c\mu \sum_{n=1}^{\infty} \frac{G_n(T)}{b^n} + R + c_p\tau - r\lambda \sum_{n=1}^{\infty} \frac{F_n(T)}{a^{n-1}}}{\lambda \sum_{n=1}^{\infty} \frac{F_n(T)}{a^{n-1}} + \mu \sum_{n=1}^{\infty} \frac{G_n(T)}{b^n} + \tau} \tag{2.5}$$

where $E(Z) = \tau$. Further

$$\mathscr{C}(T) = \frac{(c+r)\mu \sum_{n=1}^{\infty} \frac{G_n(T)}{b^n} + R_1}{\lambda \sum_{n=1}^{\infty} \frac{F_n(T)}{a^{n-1}} + \mu \sum_{n=1}^{\infty} \frac{G_n(T)}{b^n} + \tau} - r, \tag{2.6}$$

where $R_1 = R + (c_p + r)\tau$. Let $A_n = \frac{\mu}{b^n}$ and $B_n = \frac{\lambda}{a^{n-1}}$. Then $\mathscr{C}(T)$ can be rewritten as

$$\mathscr{C}(T) = \frac{(c+r) \sum_{n=1}^{\infty} A_n G_n(T) + R_1}{\sum_{n=1}^{\infty} B_n F_n(T) + \sum_{n=1}^{\infty} A_n G_n(T) + \tau} - r. \tag{2.7}$$

We observe here that $\sum_{n=1}^{\infty} A_n G_n(T)$ and $\sum_{n=1}^{\infty} B_n F_n(T)$ converges absolutely. It follows from Mertens (Rudin(1976)) Theorem that

$$\sum_{n=1}^{\infty} A_n G_n(T) \sum_{n=1}^{\infty} B_n F_n(T) = \sum_{n=1}^{\infty} \sum_{k=1}^{n} A_k B_{n-k} G_k(T) F_{n-k}(T).$$

Since the series are uniformly convergent, term-by-term differentiation is applicable. On equating $\mathscr{C}'(T)$ to zero, we obtain

$$\begin{aligned}&(c+r) \sum_{n=1}^{\infty} \sum_{k=1}^{n} A_k B_{n-k} \left\{F_{n-k}(T) G_k'(T) - G_k(T) F_{n-k}'(T)\right\}\\ &\qquad + \tau(c+r) \sum_{n=1}^{\infty} A_n G_n'(T) - R_1 \sum_{n=1}^{\infty} B_n F_n'(T) = 0.\end{aligned} \tag{2.8}$$

If $\mathscr{C}''(T) > 0$, then we must have

$$(c+r)\sum_{n=1}^{\infty}\sum_{k=1}^{n} A_k B_{n-k}\left\{G_k''(T)F_{n-k}(T) - G_k(T)F_{n-k}''(T)\right\} - R\sum_{n=1}^{\infty} A_n G_n''(T) - R_1\sum_{n=1}^{\infty} B_n F_n''(T) > 0. \tag{2.9}$$

Thus (2.8) gives T^* for which $\mathscr{C}(T^*)$ is minimum. Summarizing the above facts, we have the following theorem.

Theorem 2.1. *The long-run average cost per unit per unit time, $C(T)$ given by* (2.5) *for the monotone process model for a multistate one-component system under T-policy is minimum, if* (2.8) *and* (2.9) *hold.*

Moreover, the minimization procedure can also be done by numerical method.

3. The Replacement Policy T with NONN repair times

In this section, we introduce and study a T-policy for a multistate one-component degenerative system with NONN repair times. Under the replacement policy T, the problem is to determine an optimal replacement policy T^* such that the long-run average cost per unit time is minimized.

We consider a monotone process model for a multistate one-component degenerative system and make the following assumptions:

Assumptions 3.1, 3.2, 3.4, 3.5 are the same as Assumptions 2.1, 2.2, 2.4 and 2.5.

Assumption 3.3. Let X_1 be the first operating time. For $n \geq 2$, let X_n be the operating time of the system after $(n-1)$-st repair, let ξ_n be the repair time after the n-th failure and Z be the replacement time. Now, denote the time of the n-th failure by t_n.

Assumption 3.6. The survival times (X_i) and the NONN repair times (ξ_i) and the replacement time Z, for $i = 1, 2, ...$ are independent.

Assumption 3.7. The replacement policy T is adapted under which the system will be replaced whenever its working age reaches T.
Therefore

$$\begin{aligned} E(\xi_n) &= E(Y_n)\,P(Y_n > 0) + 1\;P(Y_n = 0) \\ &= \frac{\mu}{b^{n-1}}(1-p) + p\;. \end{aligned}$$

Let T_1 be the first replacement time and in general for $n \geq 2$, let T_n be the time between $(n-1)$-st replacement and n-th replacement. Thus the sequence $T_1, T_2, \ldots$ constitutes a renewal process. Further, a cycle is completed, if a replacement is done. By the theory of renewal reward process, the long-run average cost per ut time is given by

$$\begin{aligned} \mathscr{C}(T) &= \frac{\text{the expected cost incurred in a cycle}}{\text{the expected length of a cycle}} \\ &= \frac{cE\left(\sum_{n=1}^{\eta-1}\xi_n\right) + R + c_pE(Z) - rE\left(\sum_{n=1}^{\eta}X_n\right)}{E\left(\sum_{n=1}^{\eta}X_n\right) + E\left(\sum_{n=1}^{\eta-1}\xi_n\right) + E(Z)}\,, \end{aligned} \tag{3.1}$$

where η is a random variable denoting the number of failures in time T. Since η is also a stopping time with respect to the σ-fields $\left\{\sigma < X_1, X_2, \ldots, X_\eta >, \quad \eta = 1, 2, \ldots\right\}$, by Wald's equation,
we have

$$E\left(\sum_{n=1}^{\eta}X_n\right) = \lambda\sum_{n=1}^{\infty}\frac{F_n(T)}{a^{n-1}}, \tag{3.2}$$

where $F_n(\cdot)$ is the n-fold convolution of $F(\cdot)$ with itself.

Consider

$$\begin{aligned} E\left(\sum_{n=1}^{\eta-1}\xi_n\right) &= E\left[E\left(\sum_{n=1}^{\eta-1}E(\xi_n)|\eta\right)\right] \\ &= \sum_{n=1}^{\infty}\left(\frac{\mu(1-p)}{b^{n-1}} + p\right)G_n(T), \end{aligned} \tag{3.3}$$

where $G_n(\cdot)$ denotes the n-fold convolution of $G(\cdot)$.

Using equations (3.2) and (3.3), equation (3.1) becomes

$$\mathscr{C}(T) = \frac{c\sum_{n=1}^{\infty}\left(\frac{\mu(1-p)}{b^{n-1}} + p\right)G_n(T) + R + c_p\tau - r\lambda\sum_{n=1}^{\infty}\frac{F_n(T)}{a^{n-1}}}{\lambda\sum_{n=1}^{\infty}\frac{F_n(T)}{a^{n-1}} + \sum_{n=1}^{\infty}\left(\frac{\mu(1-p)}{b^{n-1}} + p\right)G_n(T) + \tau},$$

where $E(Z) = \tau$. Further

$$\mathscr{C}(T) = \frac{(c+r)\sum_{n=1}^{\infty}\left(\frac{\mu(1-p)}{b^{n-1}} + p\right)G_n(T) + R_1}{\lambda\sum_{n=1}^{\infty}\frac{F_n(T)}{a^{n-1}} + \sum_{n=1}^{\infty}\left(\frac{\mu(1-p)}{b^{n-1}} + p\right)G_n(T) + \tau} - r, \tag{3.5}$$

where $R_1 = R + (c_p + r)\tau$. Let

$$A_n = \frac{\lambda}{a^{n-1}} \quad \text{and} \quad B_n = \frac{\mu(1-p)}{b^{n-1}} + p.$$

Then $\mathscr{C}(T)$ can be rewritten as

$$\mathscr{C}(T) = \frac{(c+r)\sum_{n=1}^{\infty}B_nG_n(T) + R_1}{\sum_{n=1}^{\infty}A_nF_n(T) + \sum_{n=1}^{\infty}B_nG_n(T) + \tau} - r. \tag{3.6}$$

On equating $\mathscr{C}'(T)$ to zero, we obtain

$$(c+r)\sum_{n=1}^{\infty}\sum_{k=1}^{n}B_kA_{n-k}\left\{F_{n-k}(T)G_k'(T) - G_k(T)F_{n-k}'(T)\right\} + \tau(c+r)\sum_{n=1}^{\infty}B_nG_n'(T) - R_1\sum_{n=1}^{\infty}A_nF_n'(T) = 0. \tag{3.7}$$

If $\mathscr{C}''(T) > 0$, then we must have

$$(c+r)\sum_{n=1}^{\infty}\sum_{k=1}^{n}B_kA_{n-k}\left[G_k''(T)F_{n-k}(T) - G_k(T)F_{n-k}''(T)\right] - R_1\sum_{n=1}^{\infty}\left[A_nF_n''(T) + B_nG_n''(T)\right] > 0. \tag{3.8}$$

For $\mathscr{C}(T)$ to attain mimum, $\mathscr{C}'(T) = 0$ and $\mathscr{C}''(T) > 0$. Thus (3.7) gives T^* for which $\mathscr{C}(T^*)$ is mimum. Summarizing the above facts, we have the following theorem.

Theorem 1 *The long-run average cost per ut per ut time, $\mathscr{C}(T)$ given by* (3.4) *for the monotone process alternative repair model of a multistate one-component system under T-policy with NONN repair times is mimum, if* (3.7) *and* (3.8) *hold.*

Remarks.

(i) When $p = 0$, that is when the repair times non-negligible, equation (3.4) reduces to

$$\mathscr{C}(T) = \frac{c\mu\sum_{n=1}^{\infty}\frac{G_n(T)}{b^{n-1}} + R + c_p\tau - r\lambda\sum_{n=1}^{\infty}\frac{F_n(T)}{a^{n-1}}}{\lambda\sum_{n=1}^{\infty}\frac{F_n(T)}{a^{n-1}} + \mu\sum_{n=1}^{\infty}\frac{G_n(T)}{b^{n-1}} + \tau}.$$

(ii) When $p = 1$, that is, when the repair times are negligible, equation(3.4) reduces to

$$\mathscr{C}(T) = \frac{c\sum_{n=1}^{\infty}G_n(T) + R + c_p\tau - r\lambda\sum_{n=1}^{\infty}\frac{F_n(T)}{a^{n-1}}}{\lambda\sum_{n=1}^{\infty}\frac{F_n(T)}{a^{n-1}} + \sum_{n=1}^{\infty}G_n(T) + \tau}.$$

4. The Replacement Policy N with NONN repair times

In this section, we study the alternative repair model introduced in the previous section for the maintenance problem of a repairable system and we use the N policy with NONN repair times. Under the replacement policy N, the problem is to determine an optimal N^* such that the long-run average cost per unit time is minimized. We make the following assumptions :

Assumptions 4.1 to 4.6 are the same as assumptions 3.1 to 3.6.

Assumption 4.7. A replacement policy N with NONN repair times is adapted. By applying the replacement policy$N$, the system will be replaced by an identical new one at the time following the N-th failure. The replacement time is a random variable Z with $E(Z) = \tau$.

Then by the renewal reward theorem, the long-run average cost per unit time under the replacement policy N with NONN repair times is given by

$$\begin{aligned}\mathscr{C}(N) &= \frac{\text{the expected cost incurred in a cycle}}{\text{the expected length of a cycle}}\\ &= \frac{cE\left(\sum_{n=1}^{N-1}\xi_n\right) + R + c_pE(Z) - rE\left(\sum_{n=1}^{N}X_n\right)}{E\left(\sum_{n=1}^{N}X_n\right) + E\left(\sum_{n=1}^{N-1}\xi_n\right) + E(Z)}\\ &= \frac{c\sum_{n=1}^{N-1}E(\xi_n) - r\sum_{n=1}^{N}E(X_n) + R + c_pE(Z)}{\sum_{n=1}^{N}E(X_n) + \sum_{n=1}^{N-1}E(\xi_n) + E(Z)}. \qquad (4.1)\end{aligned}$$

Now

$$\begin{aligned}E\left(\sum_{n=1}^{N-1}\xi_n\right) &= \sum_{n=1}^{N-1}\left[E\left(Y_n\right)P(Y_n > 0) + 1P(Y_n = 0)\right]\\ &= \sum_{n=1}^{N-1}\left(\frac{\mu}{b^n}(1-p) + p\right) \qquad (4.2)\end{aligned}$$

and

$$E\left(\sum_{n=1}^{N}X_n\right) = \sum_{n=1}^{N}\frac{\lambda}{a^{n-1}}. \qquad (4.3)$$

On substituting (4.2) and (4.3), equation (4.1) becones

$$\mathscr{C}(N) = \frac{(c+r)\sum_{n=1}^{N-1}\left(\frac{\mu}{b^n}(1-p)+p\right) + R_1}{\lambda\sum_{n=1}^{N}\frac{1}{a^{n-1}} + \sum_{n=1}^{N-1}\left(\frac{\mu}{b^n}(1-p)+p\right) + \tau} - r, \qquad (4.4)$$

where $R_1 = R + (c_p + r)\tau$.

In order to minimize $\mathscr{C}(N)$, we define $\mathscr{A}(N)$ and note that minimizing $\mathscr{C}(N)$ is equivalent to minimizing $\mathscr{A}(N)$.

$$\mathscr{A}(N) = \frac{(c+r)\sum_{i=1}^{N-1}\left(\frac{\mu}{b^n}(1-p)+p\right) + R_1}{\lambda\sum_{n=1}^{N}\frac{1}{a^{n-1}} + \sum_{n=1}^{N-1}\left(\frac{\mu}{b^n}(1-p)+p\right) + \tau}.$$

We now find the difference between $\mathscr{A}(N+1)$ and $\mathscr{A}(N)$.

$$\mathscr{A}(N+1)-\mathscr{A}(N) = \frac{\left[\begin{array}{c}(c+r)\Big(\frac{\mu}{b^N}(1-p)+p\Big)\Big(\tau+\sum_{n=1}^{N}\frac{\lambda}{a^{n-1}}\Big)-R_1\left(\frac{\lambda}{a^{N-2}}+\Big(\frac{\mu}{b^N}(1-p)+p\Big)\right)\\ -(c+r)\frac{\lambda}{a^N}\sum_{n=1}^{N-1}\left(\frac{\mu}{b^n}(1-p)+p\right)\end{array}\right]}{\left[\sum_{n=1}^{N+1}\frac{\lambda}{a^{n-1}}+\sum_{n=1}^{N}\left(\frac{\mu}{b^n}(1-p)+p\right)+\tau\right]\left[\sum_{n=1}^{N}\frac{\lambda}{a^{n-1}}+\sum_{n=1}^{N-1}\left(\frac{\mu}{b^n}(1-p)+p\right)+\tau\right]}.$$

Here the denominator of the right side above term is positive. Define

$$\mathscr{B}(N) = \frac{R_1\left[\frac{\lambda}{a^{N-2}}+\left(\frac{\mu}{b^N}(1-p)+p\right)\right]}{(c+r)\left[\left(\frac{\mu}{b^n}(1-p)+p\right)\sum_{n=1}^{N}\frac{\lambda}{a^{n-1}}-\frac{\lambda}{a^N}\left(\frac{\mu}{b^N}(1-p)+p\right)+\left(\frac{\mu}{b^n}(1-p)+p\right)\tau\right]}$$

Since the denominator of $\mathscr{A}(N+1)-\mathscr{A}(N)$ is always positive, the sign of $\mathscr{A}(N+1)-\mathscr{A}(N)$ is the same as the sign of its numerator. Concomitantly, we have the following.

Lemma 1 *$\mathscr{A}(N)$ is either non-decreasing (or non-increasing) in N if and only if $\mathscr{B}(N)\geq 1$ (≤ 1)*

Lemma 2 *For the model described in this section, under assumptions* 4.1 *to* 4.7, *we have,*

$$\mathscr{B}(N) \text{is non-decreasing in } N \text{ if and only if } b^{N+1}p(1-a)\geq \mu(1-p)(a-b) \tag{4.5}$$

Proof. Here

$$\begin{aligned}\mathscr{B}(N+1)-\mathscr{B}(N) &= \left[\frac{\lambda}{a^{N-1}}+\frac{\mu}{b^{N+1}}(1-p)+p\right]\Bigg[\left(\frac{\mu}{b^N}(1-p)+p\right)\sum_{n=1}^{N}\frac{\lambda}{a^{n-1}}\\ &\quad -\frac{\lambda}{b^N}\sum_{n=1}^{N-1}\left(\frac{\mu}{b^n}(1-p)+p\right)+\left(\frac{\mu}{b^N}(1-p)+p\right)\tau\Bigg]-\frac{\lambda}{a^{N-2}}\\ &\quad +\left(\frac{\mu}{b^N}(1-p)+p\right)\left(\frac{\mu}{b^{N+1}}(1-p)+p\right)\sum_{n=1}^{N+1}\frac{\lambda}{a^{n-1}}\\ &\quad -\frac{\lambda}{b^{N+1}}\sum_{n=1}^{N}\left(\frac{\mu}{b^n}(1-p)+p\right)+\left(\frac{\mu}{b^{N+1}}(1-p)+p\right)\tau\end{aligned}$$

Note here that $\mathscr{B}(N+1)-\mathscr{B}(N)\geq 0$, that is, $\mathscr{B}(N)$ is non-decreasing in N if and only if

$$\begin{aligned}\left(\frac{\mu}{b^N}(1-p)+p\right)-a\left(\frac{\mu}{b^{N+1}}(1-p)+p\right) &\geq 0\\ \mu b(1-p)+b^{N+1}p-a\mu(1-p)-ab^{N+1}p &\geq 0,\end{aligned}$$

which on simplification yields (4.5) ■

Moreover the minimization procedure can be done by analytical or numerical methods.

Theorem 2 *For the model described in this section under assumptions* 4.1 *to* 4.7, *an optimal replacement policy N^* can be determined by*

$$N^* = \min\{N:\mathscr{B}(N)\geq 1\}$$

Remark. If $N=1$, then

$$\mathscr{B}(1) = \frac{(c+r)\left[\mu(1-p)+p\right](\lambda_1+\tau)}{(R+r\tau)(\lambda_2+\mu(1-p))}$$

If $\mathscr{B}(1)\geq 1$, then N^*=1. This means that an optimal replacement policy is to be replace the system immediately whenever it fails. If $\mathscr{B}(\infty)$ exist and $\mathscr{B}(\infty)\leq 1$, then $N^*=\infty$.This means the optimal policy is to continually repair the system as it ages without ever replacing it.

We now give the expression for the long-run average cost per unit time derived for a multistate one-component system under N-policy by Lam, Zhang and Zheng (2002) which is needed for comparison of policies.

$$\mathscr{C}(N) = \frac{c\mu \sum_{i=1}^{N-1} \frac{1}{b^i} + R + c_p\tau - r\lambda \sum_{i=1}^{N} \frac{1}{a^{i-1}}}{\lambda \sum_{i=1}^{N} \frac{1}{a^{i-1}} + \mu \sum_{i=1}^{N-1} \frac{1}{b^i} + \tau}.$$

5. The Bivaritae Policy (T, N) with NONN repair times

In this section, we study a bivariate policy (T, N) with NONN repair times under which the system is replaced at working age T or at the time following the N-th failure, whichever occurs first. The problem is to choose an optimal replacement policy $(T, N)^*$ such that the long-run average cost per unit time is minimized. We make the following assumptions. Assumptions 5.1 to 5.5 are the same as Assumptions 2.1 to 2.5.

Assumption 5.6. The replacement policy (T, N) is used. Let T_1 be the first replacement time and in general for $n \geq 2$, let T_n be the time between the $(n-1)$-st replacement and the n-th replacement. Then the sequence $\{T_n,\ n = 1, 2, \ldots\}$ forms a renewal process. Therefore, the inter arrival times between two consecutive replacements is a renewal cycle.

Let $\mathscr{C}(T, N)$ be the long-run average cost per unit time under the bivariate replacement policy (T, N). Then according to the renewal reward theorem, the long-run average cost per unit time is given by

$$\begin{aligned}\mathscr{C}(T, N) &= \frac{\text{the expected cost incurred in a cycle}}{\text{the expected length of a cycle}}\\ &= \frac{E\left\{\left(c\sum_{n=1}^{\eta}\xi_n - rT\right)\chi_{(L_N>T)}\right\} + c_pE(Z) + E\left\{\left(c\sum_{n=1}^{N-1}\xi_n - r\sum_{n=1}^{N}X_n\right)\chi_{(L_N\leq T)}\right\} + R}{E(W)}, \end{aligned} \tag{5.1}$$

where η is a random variable denoting the number of failures before the working age of the system reaches T, W is the length of a cycle and

$$\chi_A = \begin{cases} 1, & \text{if event } A \text{ occurs} \\ 0, & \text{if event } A \text{ does not occur} \end{cases}$$

denotes the indicator function. Therefore $\eta = 0, 1, \ldots, N-1$.

The length of the cycle under the replacement policy (T, N) is

$$W = \left(T + \sum_{n=1}^{\eta}\xi_n\right)\chi_{(L_N>T)} + \left(\sum_{n=1}^{N}X_n + \sum_{n=1}^{N-1}\xi_n\right)\chi_{(L_N\leq T)} + Z,$$

where $\eta = 0, 1, 2, \ldots, N-1$ is the number of failures before the working age of the system exceeds T. Now

$$\begin{aligned} E\left[\left(\sum_{n=1}^{N}X_n\right)\chi_{(L_N\leq T)}\right] &= E\left\{E\left[\left(\sum_{n=1}^{N}X_n\right)\chi_{(L_N\leq T)}|L_N\right]\right\}\\ &= \int_0^T E\left(\sum_{n=1}^{N}X_n|L_N = u\right)dF_N(u)\\ &= \int_0^T u\,dF_N(u) \end{aligned} \tag{5.2}$$

and

$$
\begin{aligned}
E\left[\left(\sum_{n=1}^{N-1}\xi_n\right)\chi_{(L_N\le T)}\right] &= E\left\{E\left[\left(\sum_{n=1}^{N-1}\xi_n\right)\chi_{(L_N\le T)}|L_N\right]\right\}\\
&= \int_0^T E\left(\sum_{n=1}^{N-1}\xi_n|L_N=u\right)dF_N(u)\\
&= \int_0^T \left(\sum_{n=1}^{N-1}E(\xi_n)\right)dF_N(u)\\
&= \int_0^T \sum_{n=1}^{N-1}\left(\frac{\mu}{b^{n-1}}(1-p)+p\right)\ dF_N(u)\\
&= \sum_{n=1}^{N-1}\left(\frac{\mu}{b^{n-1}}(1-p)+p\right)\int_0^T dF_N(u)\\
&= \sum_{n=1}^{N-1}\left(\frac{\mu}{b^{n-1}}(1-p)+p\right)\ F_N(T)\,. \qquad (5.3)
\end{aligned}
$$

Further

$$
\begin{aligned}
E(W) &= E\left[\left(T+\sum_{n=1}^{\eta}\xi_n\right)\chi_{(L_N>T)}\right]\\
&\quad +E\left[\left(\sum_{n=1}^{N}X_n+\sum_{n=1}^{N-1}\xi_n\right)\chi_{(L_N\le T)}\right]+E(Z)\\
&= E\left[T\chi_{(L_N>T)}\right]+E\left[\left(\sum_{n=1}^{\eta}\xi_n\right)\chi_{(L_N>T)}\right]\\
&\quad +E\left\{E\left[\left(\sum_{n=1}^{N}X_n+\sum_{n=1}^{N-1}\xi_n\right)\chi_{(L_N\le\ T)}|L_N\right]\right\}+E(Z)\\
&= T\overline{F}_N(T)+E\left[\sum_{n=1}^{N-1}\xi_n\,\chi_{(L_i\,<\,T\,<\,L_N)}\right]\\
&\quad +\int_0^T u dF_N(u)+\sum_{n=1}^{N-1}\left(\frac{\mu}{b^{n-1}}(1-p)+p\right)\ F_N(T)+\tau\,, \qquad (5.4)
\end{aligned}
$$

where b is as given in section 2. Let $W_{N-n}=\sum_{j=n+1}^{N}X_j$. Then $L_N=L_n+W_{N-n}$. Moreover L_n and W_{N-n} are independent and

$$
H_{N-n}(t)=\int_0^\infty H_{N-1-n}\left(a(t-y)\right)\ dH(y), \qquad (5.5)
$$

where $H_{N-1-n}(t)$ is the distribution of $\sum_{j=n+1}^{N}X_j$ and a is as given in section 2. Since the distribution function of X_{n+1} is $H(t)=F(a^n t)$, equation (5.5) can be written, by induction, as $H_{N-n}(t)=F_{N-n}(a^n t)$. Now

$$
\begin{aligned}
E\left[\chi_{(L_n\,<\,T\,<\,L_N)}\right] &= P\left(L_n<T<L_n+W_{N-n}\right)\\
&= \int_0^T\int_{T-u}^\infty dH_{N-i}(t)\,dF_n(u)\\
&= \int_0^T \overline{F}_{N-n}(a^n(T-u))\,dF_n(u),
\end{aligned}
$$

so that equation (5.4) becomes

$$\begin{aligned} E(W) &= T\overline{F}_N(T) + \sum_{n=1}^{N-1}\left(\frac{\mu}{b^{n-1}}(1-p)+p\right)\int_0^T \overline{F}_{N-n}(a^n(T-u))\,dF_n(u) \\ &\qquad + \int_0^T u dF_N(u) + \sum_{n=1}^{N-1}\left(\frac{\mu}{b^{n-1}}(1-p)+p\right)F_N(t) + \tau \\ &= \int_0^T \overline{F}_N(u)\,du + \sum_{n=1}^{N-1}\left(\frac{\mu}{b^{n-1}}(1-p)+p\right)F_N(T) \\ &\qquad + \sum_{n=1}^{N-1}\left(\frac{\mu}{b^{n-1}}(1-p)+p\right)\int_0^T \overline{F}_{N-n}(a^n(T-u))\,dF_n(u) + \tau. \end{aligned}$$

Equation (5.1) now becomes

$$\mathscr{C}(T,N) = \frac{\left[\begin{array}{l} c\sum_{n=1}^{N-1}\left(\frac{\mu}{b^{n-1}}(1-p)+p\right)\int\limits_0^T \overline{F}_{N-n}(a^n(T-u))dF_n(u) - rT\overline{F}_N(T) \\ +\int\limits_0^T c\sum_{n=1}^{N-1}\left(\frac{\mu}{b^{n-1}}(1-p)+p\right)dF_N(u) - r\int_0^T u dF_N(u) + R + c_p\tau \end{array}\right]}{\left[\begin{array}{l} \int\limits_0^T \overline{F}_N(u)\,du + \sum_{n=1}^{N-1}\left(\frac{\mu}{b^{n-1}}(1-p)+p\right)F_N(T) \\ +\sum_{n=1}^{N-1}\left(\frac{\mu}{b^{n-1}}(1-p)+p\right)\int_0^T \overline{F}_{N-n}(a^n(T-u))\,dF_n(u) + \tau \end{array}\right]}$$

$$= \frac{\left[\begin{array}{l} c\left\{\sum_{n=1}^{N-1}\left(\frac{\mu}{b^{n-1}}(1-p)+p\right)F_N(T) \right. \\ \left. +\sum_{n=1}^{N-1}\left(\frac{\mu}{b^{n-1}}(1-p)+p\right)\int_0^T \overline{F}_{N-n}(a^n(T-u))dF_n(u)\right\} \\ +R + c_p\tau - r\int_0^T \overline{F}_N(u)du \end{array}\right]}{\left[\begin{array}{l} \int_0^T \overline{F}_N(u)\,du + \sum_{n=1}^{N-1}\left(\frac{\mu}{b^{n-1}}(1-p)+p\right)F_N(T) \\ +\sum_{n=1}^{N-1}\left(\frac{\mu}{b^{n-1}}(1-p)+p\right)\int_0^T \overline{F}_{N-n}(a^n(T-u))\,dF_n(u) + \tau \end{array}\right]}, \tag{5.6}$$

This is a bivariate function. Obviously, when N is fixed, $\mathscr{C}(T,N)$ is a function of T. For fixed $N = m$, it can be written as

$$\mathscr{C}(T,N) = \mathscr{C}_m(T), \quad m = 1,2,\ldots .$$

Thus, for a fixed m, we can find T_m^* by analytical or numerical methods such that $\mathscr{C}_m(T_m^*)$ is minimized. That is, when $N = 1,2,\ldots,m,\ldots$, we can find $T_1^*, T_2^*, \ldots, T_m^*, \ldots$, respectively, such that the corresponding $\mathscr{C}_1(T_1^*), \mathscr{C}_2(T_2^*), \ldots, \mathscr{C}_m(T_m^*), \ldots$ are minimized.

Because the total lifetime of a multistate degenerative system is limited, the minimum of the long-run average cost per unit time exists. So we can determine the minimum of the long-run average cost per unit time based on $\mathscr{C}_1(T_1^*)$, $\mathscr{C}_2(T_2^*), \ldots, \mathscr{C}_m(T_m^*), \ldots$.

For example, if the minimum is denoted by $\mathscr{C}_n(T_n^*)$, we obtain the bivariate optimal replacement policy $(T,N)^*$ such that

$$\mathscr{C}((T,N)^*) = \min_N \mathscr{C}_N(T_N^*).$$

Remarks.

(i) When $p = 0$, equation (5.6) reduces to

$$\mathscr{C}(T,N) = \frac{\left[\begin{array}{l} c\sum_{n=1}^{N-1}\left(\frac{\mu}{b^{n-1}}\right)F_N(T) + \sum_{n=1}^{N-1}\left(\frac{\mu}{b^{n-1}}\right)\int_0^T \overline{F}_{N-n}(a^n(T-u))dF_n(u) \\ +R + c_p\tau - r\int_0^T \overline{F}_N(u)du \end{array}\right]}{\int_0^T \overline{F}_N(u)\,du + \sum_{n=1}^{N-1}\left(\frac{\mu}{b^{n-1}}\right)F_N(T)\sum_{n=1}^{N-1}\left(\frac{\mu}{b^{n-1}}\right)\int_0^T \overline{F}_{N-n}(a^n(T-u))\,dF_n(u) + \tau},$$

and in this case (5.6) reduces to $\mathscr{C}(T,N)$ given by (10) of Zhang, Yam and Zuo (2007).

(ii) When $p = 1$, equation (5.6) reduces to

$$\mathscr{C}(T,N) = \frac{c(N-1)F_N(T) + \sum_{n=1}^{N-1}(N-1)\int_0^T \overline{F}_{N-n}(a^n(T-u))dF_n(u) + R + c_p\tau - r\int_0^T \overline{F}_N(u)du}{\int_0^T \overline{F}_N(u)\,du + (N-1)F_N(T)(N-1)\int_0^T \overline{F}_{N-n}(a^n(T-u))\,dF_n(u) + \tau},$$

6. Comparison of Policies

6.1 Comparison of T-policy with N-policy.
If for any $T > 0$,

$$\sum_{n=1}^{\infty}\frac{F_n(T)}{a^{n-2}} = \infty,$$

then $\mathscr{C}(T) = c$. Therefore

$$\mathscr{C}(N^*) = \min_N \mathscr{C}(N) \le \mathscr{C}(\infty) \equiv c.$$

It follows that N-policy is better than T-policy.

6.2 Comparison of the Bivariate policy (T,N) with the univariate T-policy and N-policy.
When $T \to \infty$,

$$\lim_{T\to\infty}\mathscr{C}(T,N) = \frac{c\mu\sum_{i=1}^{N-1}\frac{1}{b^i} - \lambda r\sum_{i=1}^{N}\frac{1}{a^{i-1}} + R + c_p\tau}{\lambda\sum_{i=1}^{N}\frac{1}{a^{i-1}} + \mu\sum_{i=1}^{N-1}\frac{1}{b^i} + \tau}.$$

If the constants c, r, R, c_p, and the expected values λ, μ, and τ are the same as in Lam, Zhang and Zheng (2002)'s model, then

$$\begin{aligned}\mathscr{C}((T,N)^*) &= \min_N \mathscr{C}_N(T_N^*) = \min_N\left[\min_T \mathscr{C}(T,N)\right] \\ &\le \min_N [\mathscr{C}(\infty,N)] = \mathscr{C}(N^*).\end{aligned}$$

Hence the bivariate optimal replacement policy $(T,N)^*$ is better than the univariate optimal replacement policies T^* and N^*.

The above analysis leads us to conclude that a bivariate optimal replacement policy for a multistate degenerative system is still better than the univariate optimal replacement policies for the same system.

7. Numerical Examples

We consider a numerical example to demonstrate the models and methodology developed in this paper. Consider a degenerative simple repairable system with three states including two failure states and one working state, that is, $k = 2$. Assume that $p_1 = 0.495$, $p_2 = 0.505$; $a_1 = 1.005$, $a_2 = 1.1$; $b_1 = 0.95$ and $b_2 = 0.85$ so that $p_1 + p_2 = 1$, $1 < a_1 < a_2$ and $1 > b_1 > b_2$. Then $a = \left(\frac{p_1}{a_1} + \frac{p_2}{a_2}\right)^{-1} = 1.048$ and $b = \left(\frac{p_1}{b_1} + \frac{p_2}{b_2}\right)^{-1} = 0.897$. Let $\lambda = 400$, $\mu = 24$, $c = 8$ $r = 600$, $R = 50000$, $c_p = 3$, $\tau = 12$. Here λ and μ are in time units and $c,\ r,\ R,\ c_p$ and τ are in monetary units. Assuming these values, we now calculate the long run average cost for the two distinct replacement models developed in this paper in the following cases.

Case(i): The N-Policy with NNON repair times

In this case, using equation (4.4), overpassing numerical calculations, we arrive at $N^* = 10$ when $p = 0$ and $p = 0.3$. However, the long-run average cost decreases from 12.9169 to 10.8856 monetary units, as p increase from 0 to 0.3. For other values of p, the corresponding values of $\mathscr{C}(N)$ in equation (4.4) are given in Table 1.

Note here that as the probability p of the NONN repair times increases from 0 to 0.9, the optimal value N^* that minimizes $\mathscr{C}(N)$ given in (4.4) increases from $N^* = 10$ to $N^* = 14$, whereas the long-run average cost $\mathscr{C}(N)$ decreases from 12.9169 monetary units to -3.4846 monetary units. However, if p is taken as 1, that is, when the repair times are non-negligible, then the value of N^* is found to be 44 with the corresponding long-run average cost -22.4521 monetary units. Further, for each value of p, there is a unique optimal replacement policy N^* which minimizes $\mathscr{C}(N)$ given in (4.4). Here $\mathscr{C}(N^*)$ is the unique minimum of the average cost. These are plotted in Fig. 1.

Case(ii): The bivariate (T, N)-policy with NNON repair times

Assume that the distribution function of X_n is exponential, that is,

$$\begin{aligned} F_n(t) &= F_n(a^{n-1}t) \\ &= 1 - \exp\left(-\frac{a^{n-1}t}{\lambda}\right);\ \ t \geq 0;\ \ \frac{1}{\lambda} > 0;\ \ a \geq 1 \ \text{ and } \ n = 1, 2, \ldots. \end{aligned}$$

The density function of the sum $\sum_{i=1}^{n} X_i$ of the random variables $X_1, X_2, \ldots, X_n$ having exponential distribution and following an increasing geometric process is given by

$$h_n(t) = (-1)^{n-1}\ \frac{a^{\frac{n(n-1)}{2}}}{\lambda}\ \sum_{i=1}^{n} \frac{\exp\left(-\frac{a^{n-1}t}{\lambda}\right)}{\prod_{\substack{j=1\\ i\neq j}}^{n}(a^{i-1} - a^{j-1})}.$$

The distribution function of $\sum_{i=1}^{n} X_i$, in this case, is given by

$$H_n(t) = 1 - \sum_{i=1}^{n}\left[\prod_{\substack{j=1\\ i\neq j}}^{n} \frac{a^{j-1}}{a^{i-1} - a^{j-1}}\right]\exp\left(-\frac{a^{n-1}t}{\lambda}\right), \ \text{ for } \ t \geq 0.$$

Equation (5.6) becomes

$$\mathscr{C}(T, N) = \frac{c\sum_{n=1}^{N-1}\left(\frac{\mu}{b^{n-1}}(1-p) + p\right)H_n(T) + R + c_p\tau - r\int_0^T \overline{F}_N(u)du}{\int_0^T \overline{F}_N(u)\,du + \sum_{n=1}^{N-1}\left(\frac{\mu}{b^{n-1}}(1-p) + p\right)H_n(T) + \tau}, \tag{7.1}$$

where

$$\int_0^T \overline{F}_N(u)\,du = \sum_{i=1}^{N}\left[\prod_{\substack{j=1\\ i\neq j}}^{n} \frac{a^{j-1}}{a^{i-1} - a^{j-1}}\right]\left(\frac{\lambda}{a^{i-1}}\right)\left[1 - \exp\left(-\frac{a^{n-1}t}{\lambda}\right)\right].$$

In the above equation of the objective function, T is a continuous variable and N is a discrete variable. In this case, assuming $p = 0.5$ and using equation (7.1) overpassing numerical calculations, we arrive at $(T, N)^* = (301, 31)$, such that$\mathscr{C}(T, N)$is minimum at $(T, N)^*$ and the long-run average cost is $\mathscr{C}(T, N) = \mathscr{C}(301, 31) = -20792$ monetary units. The values of $\mathscr{C}(T, N)$ for T ranging from 0 to 400 time units and N ranging from 0 to 50, for different values of p are evaluated using Maple 6 and the results are plotted in the figures numbered 2 to 7.

When $p = 0$, then we obtain $(T, N)^* = (41, 11)$ with the corresponding $\mathscr{C}(T, N)^* = -1623.5022$. When $p = 0.3$, then we obtain $(T, N)^* = (41, 11)$ with the corresponding $\mathscr{C}(T, N)^* = -6610.03$. When $p = 0.5$, then we obtain $(T, N)^* = (261, 16)$ with the corresponding $\mathscr{C}(T, N)^* = -5276.50$. When $p = 0.7$, then we obtain $(T, N)^* = (1, 11)$ with the corresponding $\mathscr{C}(T, N)^* = -1216.36$. When $p = 0.9$, then we obtain $(T, N)^* = (121, 11)$ with the corresponding $\mathscr{C}(T, N)^* = -5584.03$. When $p = 1.0$, then we obtain $(T, N)^* = (21, 36)$ with the corresponding $\mathscr{C}(T, N)^* = -308.99$.

8. Comparison of Numerical Results

In this section, we compare the numerical results evaluated in the previous section.

We concluded that in Section 6 that a bivariate optimal replacement policy $(T, N)^*$ for a multistate degenerative system is always better than the univariate replacement polices for the same system.

When $p = 0$, that is, when the repair times are non-negligible, then under the univariate N policy, using equation (4.4), we obtain $N^* = 10$ and $\mathscr{C}(N^*) = 12.9169$ monetary units, where as under the bivariate (T, N) policy using equation (7.1), we obtain $(T, N)^* = (41, 11)$ with the corresponding $\mathscr{C}(T, N)^* = -1623.5022$. Therefore it is confirmed that the bivariate (T, N) policy is better than the univariate optimal replacement policy. For other values of p ranging from $p = 0$ to $p = 1$, the corresponding optimal values of N^*, $(T, N)^*$ and their respective costs are presented in Table 2.

9. Conclusion

By considering a repairable system for a monotone process model for a one component multistate degenerative system, explicit expressions for the long-run average cost per unit time under the univariate T-policy policy is derived.The conditions for the existence of the optimal replacement policy T^* are also derived. Assuming an alternate repair model, for the same system, long-run average cost per unit time under T policy with NNON repair times, N policy with NNON repair times are derived. Also the existence of the optimal replacement polices have been derived. Furthermore we have derived the long run average cost per unit time under a bivariate (T, N) policy with NNON repair times. Also, comparison between the existing N-policy and the above polices have been carried out. It is found that the bivariate optimal replacement policy for a multistate degenerative system is better than the univariate policies for the same system. Thangaraj and Rizwan (2001) have established the same result for a two state system with burn-in. Numerical examples are given to illustrate the models and methodology developed in this paper. Also comparison with the existing models have also been carried out. It is concluded here that the bivariate (T, N) policy is the best replacement policy for any multivariate deteriorating system. It is confirmed in this study that the N-policy is better than T-policy for a monotone process model for a one component multistate degenerative system. This situation triggers us to conjecture that N-policy is the best replacement policy among univariate policies for any deteriorating system.

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Table 1. The values of $\mathscr{C}(N)$ for different values of p

N	$p = 0.0$	$p = 0.3$	$p = 0.5$	$p = 0.7$	$p = 0.9$
5	16.1661	15.3032	14.4740	13.2894	11.4581
6	14.6282	13.3388	12.0699	10.2038	7.1891
7	13.7135	12.1213	10.5207	8.1039	4.0333
8	13.2036	11.4003	9.5523	6.6927	1.6778
9	12.9663	11.0219	8.9936	5.7813	-0.0788
10	**12.9169**	**10.8856**	8.7312	5.2430	-1.3729
11	12.9987	10.9229	**8.6869**	4.9891	-2.2998
12	13.1701	11.0854	8.8046	**4.9552**	-2.9284
13	13.4088	11.3379	9.0427	5.0928	-3.3102
14	13.6883	11.6540	9.3699	5.3645	**-3.4846**
15	13.9953	12.0140	9.7619	5.7407	-3.4827

Table 2. Comparison of numerical results between optimal univariate and bivariate polices for different values of p

N	$p = 0.0$	$p = 0.3$	$p = 0.5$	$p = 0.7$	$p = 0.9$	$p = 1.0$
N^*	10	10	11	12	14	44
$\mathscr{C}(N)^*$	12.92	10.89	8.69	4.69	-3.48	-22.45
$(T, N)^*$	(41,11)	(41,11)	(261,16)	(11,1)	(121,11)	(36,21)
$\mathscr{C}(T, N)^*$	-1623.50	-6610.30	-5276.50	-1216.36	-5584.03	-308.99

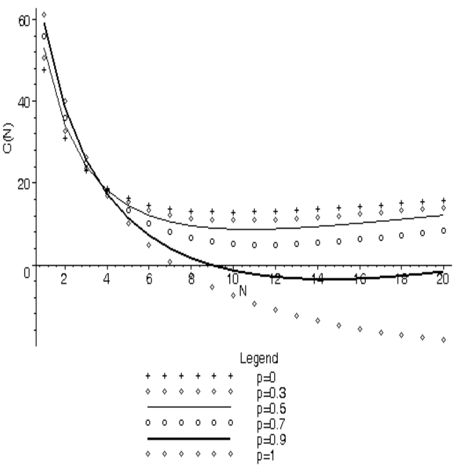

Figure 1. The plots of expected cost $\mathscr{C}(N)$ against N

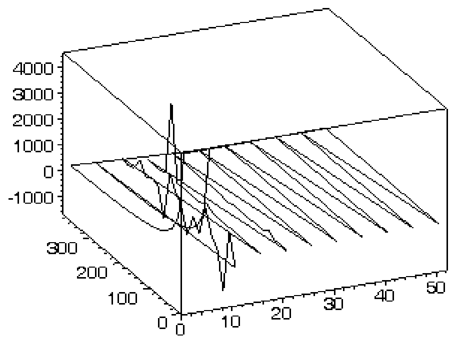

Figure 2. The plot of $\mathscr{C}(T, N)$,when $p = 0$

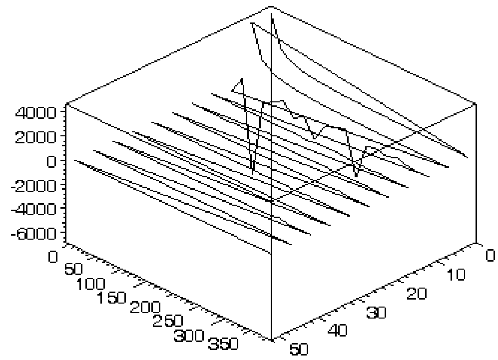

Figure 3. The plot of $\mathscr{C}(T, N)$,when $p = 0.3$

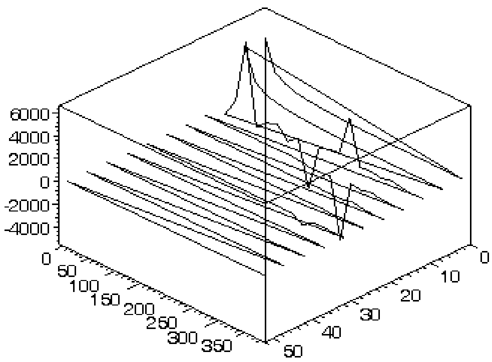

Figure 4. The plot of $\mathscr{C}(T, N)$, when $p = 0.5$

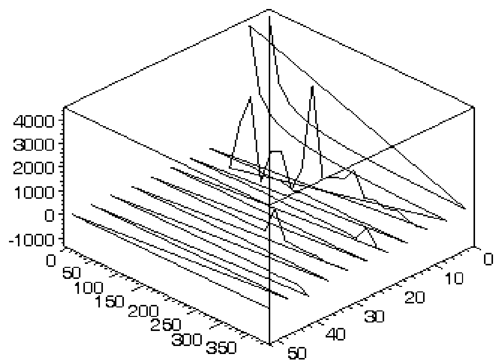

Figure 5. The plot of $\mathscr{C}(T, N)$,when $p = 0.7$

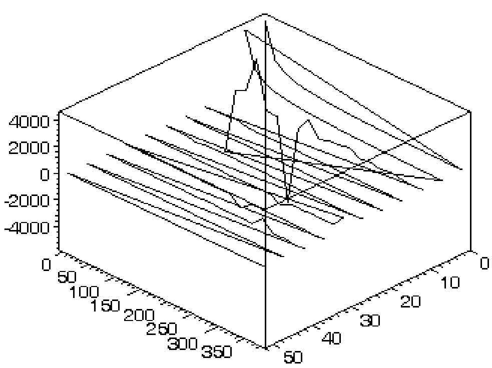

Figure 6. The plot of expected cost $\mathscr{C}(T, N)$, when $p = 0.9$

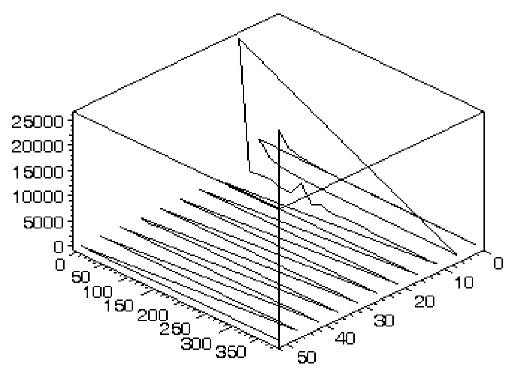

Figure 7. The plot of $\mathscr{C}(T, N)$,when $p = 1.0$

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