On Different Types of Chaos for $\mathbb{Z}^d$-Actions

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Abstract

In this paper we obtain a characterization of k-type transitivity for a $\mathbb{Z}^d$-action on certain spaces and then prove that k-type SDIC is redundant in the definition of k-type Devaney chaos for $\mathbb{Z}^d$-actions on infinite metric spaces. We define different types of chaos for $\mathbb{Z}^d$-actions and prove results related to their preservations under conjugacy and uniform conjugacy. Finally we discuss k-type properties on product spaces.

Keywords: k-type transitivity, Devaney chaos, mixing, weak mixing, conjugacy

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1. Introduction

Poincaré's work on celestial mechanics led to the discovery that ordinary differential equations can exhibit sensitive dependence on initial conditions, characterized by the possibility that two trajectories starting close together could diverge and eventually become uncorrelated. The new paradigm of chaos adopts the position that unpredictable and seemingly lawless disorder can be ascribed to simple deterministic rules. It might have been supposed that, with the advent of computers, the mathematical theory of chaotic dynamical systems would simply come to an end. In fact, the opposite is true, namely that nonlinear dynamics is one of the fastest growing fields of mathematics. The Mathematical term chaos was first introduced by Li and Yorke in 1975, where the authors gave a criterion on the existence of chaos for interval maps, known as "period three implies chaos". Since then chaos theory has played very important role in the study of dynamical systems even in nonlinear science. Lots of work have been done by now in the area of dynamical systems, making possible a well established theory of chaos. Chaos theory has applications in physical, biological, social sciences and technology.

Mathematically it is a natural question after rephrasing the idea of dynamical systems in terms of $\mathbb{Z}$-actions or $\mathbb{R}$-actions i.e., either discrete or continuous time evolutions to look at G-actions. In recent years there has been considerable progress in the study of higher dimensional actions i.e., actions of $\mathbb{Z}^d$ and $\mathbb{R}^d$ with $d > 1$. Historically much of the interest in $\mathbb{Z}^d$-actions arised from the study of classical lattice gas models. The simplest case where $d = 1$ led to the development of the thermodynamic approach to the ergodic theory of Anosov and Axiom A systems during 1960's and 1970's. The theory of edge shifts is used in computer science to adapt (by a recoding) data to the technical restrictions of a data storage media. Although many media are in fact two-dimensional (like a compact disc), they are used in a one-dimensional way (the data is arranged in many circles on the disc) and a one-dimensional recoding is used. However, some experimental media are now used as two-dimensional media and one reason to study $\mathbb{Z}^2$-actions is to develop a theory of coding in that context. In [Einsiedler, Lind, Miles and Ward, 2001], authors have studied expansive $\mathbb{Z}^d$-action and determine whether subgroups of $\mathbb{Z}^d$ also act expansive. In [Einsiedler and Schmidt, 1997], authors study various aspects of symbolic $\mathbb{Z}^d$-actions, where $X \subseteq \mathcal{A}^{\mathbb{Z}^d}$ is a (closed and shift invariant) subset of the full shift and use the restriction of the shift action to the space X.

In [Afraimovich and Chow, 1995], authors study a $\mathbb{Z}^d$-action on a finite dimensional subset of a Banach space representing a set of equilibrium solutions of a lattice dynamical systems. In [Schmidt, 1998], author study symbolic $\mathbb{Z}^d$-action. In [Oprocha, 2008], introducing the concept of k-type transitivity, author has proved the spectral decomposition theorem for multidimensional discrete-time dynamical systems which is a generalized version of similar results obtained by S. Smale and R. Bowen for certain maps and flows. In [Kim and Lee, 2014] authors have proved general version of the spectral decomposition theorem for the set of k-type nonwandering points of $\mathbb{Z}^2$-actions.

Different types of chaos have been defined and studied by various authors. Most definitions of chaos are based on one of the following aspects: complex trajectory behaviour of points, sensitive dependence on initial conditions, fast growth of different orbits of length n, strong recurrence property. For more details one can refer the survey article on chaos by [Li and Ye, 2015]. In [Shah and Das, 2015], we have defined and studied k-type Devaney chaos for $\mathbb{Z}^d$-actions. In [Lu, Zhu and Wu, 2013], authors have studied retentivity of different types of chaos under topological conjugacy and uniform conjugacy.

In section 2, we obtain a characterization of k-type transitivity on certain spaces and then prove that k-type SDIC is redundant in the definition of k-type Devaney chaos for $\mathbb{Z}^d$-actions on infinite metric spaces. In section 3, we define different types of chaos for $\mathbb{Z}^d$-actions and prove that they are preserved under conjugacy. In section 4, we define notions of k-type dense chaos, k-type dense δ-chaos and k-type Auslander-Yorke chaos and prove their preservation under uniform conjugacy. We finally discuss k-type properties on product spaces in section 5.

We first discuss some basic notions related to $\mathbb{Z}^d$-actions. Let (X, ρ) be a metric space. A $\mathbb{Z}^d$*-action* on X is a continuous map $T : \mathbb{Z}^d \times X \to X$ such that

(i) $T(0, x) = x$, for every $x \in X$,

(ii) $T(n, T(m, x)) = T(n + m, x)$, for all $n, m \in \mathbb{Z}^d$ and for every $x \in X$.

For a $\mathbb{Z}^d$-action $T : \mathbb{Z}^d \times X \to X$, $T^n : X \to X$ is defined by $T^n(x) = T(n, x)$ for all $n \in \mathbb{Z}^d$ and $x \in X$. The map T^n is a homeomorphism on X, for every $n \in \mathbb{Z}^d$. Let $e_1, e_2, ..., e_d$ denote the standard canonical basis vectors of $\mathbb{R}^d$. For $d \in \mathbb{N}$, let $k \in \{1, 2, ..., 2^d\}$ and let k^b represent $k - 1$ in the d-positional binary system, i.e., $k^b \in \{0, 1\}^d, k = 1 + \sum_{i=1}^{d} k_i^b 2^{i-1}$. Let $k \in \{1, 2, ..., 2^d\}$ and let $x = (x_1, x_2, ..., x_d), y = (y_1, y_2, ..., y_d) \in \mathbb{Z}^d$. We say that $x \geq^k y$ if $(-1)^{k_i^b} x_i \geq (-1)^{k_i^b} y_i$, for $i = 1, 2, ..., d$ [Oprocha, 2008]. For any $j = (j_1, j_2, ..., j_d) \in \mathbb{Z}^d$ we define $\| j \| = max\{|j_i| : i = 1, 2, ..., d\}$.

Let $T : \mathbb{Z}^d \times X \to X$ be a $\mathbb{Z}^d$-action on X. T is *k-type transitive* if for any two nonempty open sets G and H, there exists $n >^k 0$ such that $T^n(G) \cap H \neq \phi$ [Oprocha, 2008]. A point $x \in X$ is called a *k-type periodic point* if there is an $n \in \mathbb{Z}^d$, $n >^k 0$ satisfying $T^n(x) = x$ [Shah and Das, 2015]. For $x \in X$, $O_T^k(x) = \{T^n(x) | n \geq^k 0\}$ is called the k-type orbit of x. T has *k-type sensitive dependence on initial conditions* (k-type SDIC) if there exists $\delta > 0$ such that for every $x \in X$ and for every $\epsilon > 0$, there exist $y \in B_\rho(x, \epsilon)$ and $n >^k 0$ such that $\rho(T^n(x), T^n(y)) > \delta$ [Shah and Das, 2015].

2. k-type Devaney Chaos Using a Characterization of k-type Transitivity

In [Shah and Das, 2015], authors have introduced the notion of k-type Devaney chaos for a $\mathbb{Z}^d$-action $T : \mathbb{Z}^d \times X \to X$ on a compact metric space X. A map T is said to be *k-type Devaney Chaotic* if T is k-type transitive, T has dense set of k-type periodic points and T has k-type SDIC. We prove here that under certain conditions on space X, k-type transitivity is equivalent to existence of a dense k-type orbit analogous to Proposition 1.1 in [Silverman, 1992]. We also prove that k-type SDIC is redundant in the definition of k-type Devaney chaos for infinite metric spaces.

Theorem 2.1. *Let X be a perfect space and let* $T : \mathbb{Z}^d \times X \to X$ *be a* $\mathbb{Z}^d$*-action on X. Then T has a dense k-type orbit implies T has k-type transitivity. If X is separable and of second category then T has k-type transitivity implies T has a dense k-type orbit.*

Proof. Suppose T has dense k-type orbit say $O_T^k(x) = \{T^n(x) | n \geq^k 0\}$. Let U and V be nonempty open sets in X then by denseness of k-type orbit, there exists $m \geq^k 0$ such that $T^m(x) \in U$. Consider the set $W = V - \{T^i(x) | i \geq^k 0$ and $\| i \| \leq \| m \|\}$, which is nonempty and open. Since $O_T^k(x)$ is dense in X, there exists $l \geq^k 0$ such that $T^l(x) \in W$. Clearly $l >^k m$ and hence $l - m >^k 0$ and $T^{l-m}(U) \cap V \neq \phi$. Hence T is k-type transitive.

Next, suppose X is separable and of second category then X is second countable and hence has a countable basis, say $\{V_\lambda\}_{\lambda=1}^{\infty}$. Suppose T has no dense k-type orbit then for any $x \in X$, $\{T^n(x) | n \geq^k 0\}$ is not dense in X, which implies that for any $x \in X$, there exists $V_{\lambda(x)}$ such that $T^n(x) \notin V_{\lambda(x)}$ for all $n \geq^k 0$. Let $V = \cup_{n \geq^k 0} T^{-n}(V_{\lambda(x)})$ then V is open and intersects every open set as T is k-type transitive. Let $A_{\lambda(x)} = X - \cup_{n \geq^k 0} T^{-n}(V_{\lambda(x)})$ then $A_{\lambda(x)}$ is closed and $x \in A_{\lambda(x)}$. By definition of $A_{\lambda(x)}$, it follows that $A_{\lambda(x)}$ is nowhere dense and $X = \cup_{x\in X} A_{\lambda(x)}$. Thus X is a countable union of nowhere dense sets which is a contradiction. Hence the proof.

We now prove that k-type SDIC is redundant in the definition of k-type Devaney chaos for infinite metric spaces.

Theorem 2.2. *Let X be an infinite metric space and let $T : \mathbb{Z}^d \times X \to X$ be a $\mathbb{Z}^d$-action on X. If T has a dense k-type orbit and has dense set of k-type periodic points then T has k-type sensitive dependence on initial conditions.*

Proof. Note that since X is infinite therefore T has dense k-type orbit and has dense set of k-type periodic points implying that the space X is perfect. Therefore by Theorem 2.1 T is k-type transitive. Let p be a periodic point in X and let $q \in X$ be such that $q \notin O^k_T(p)$. Let $p' \in O^k_T(p)$ be such that $d = \rho(p', q) = \rho(O^k_T(p), q)$. Suppose T does not have k-type SDIC then there exist $x \in X$ and a neighborhood $N(x)$ of x such that $diam(T^n(N(x))) < \frac{d}{4}$, for every $n \geq^k 0$. Since T has dense set of k-type periodic points, $N(x)$ contains a k-type periodic point, say y of period l. By continuity of T, there exists a neighborhood $N(p)$ of p such that if $z \in N(p)$ then $\rho(T^\alpha(z), T^\alpha(p)) < \frac{d}{4}$, for $0 \leq\parallel \alpha \parallel<\parallel l \parallel$. Let $N(q)$ be a neighborhood of q with $diam(N(q)) < \frac{d}{4}$. By k-type transitivity of T, there exists $m >^k 0$ such that $T^m(N(x)) \cap N(p) \neq \phi$ and there exists $j >^k 0$ such that $T^j(N(x)) \cap N(q) \neq \phi$. Observe that $\rho(T^{m+\alpha}(y), T^\alpha(p)) < \frac{d}{2}$, for $0 \leq\parallel \alpha \parallel<\parallel l \parallel$ and $\rho(T^j(y), q) < \frac{d}{2}$. Since $\{T^m(T^i(y))| i \geq^k 0\}$ exhausts all the points of the orbit of y therefore $T^j(y) = T^{m+\alpha}(y)$, for some α such that $\parallel \alpha \parallel<\parallel l \parallel$. By triangle inequality it follows that $\rho(T^\alpha(p), q) < d$, which is a contradiction by definition of d.

3. Different Types of Chaos for $\mathbb{Z}^d$-actions Preserved under Conjugacy

In this section, we define different types of chaos for $\mathbb{Z}^d$-actions on an infinite metric space and prove that they are preserved under conjugacy.

Definition 3.1. *Let $T : \mathbb{Z}^d \times X \to X$ be a $\mathbb{Z}^d$-action on X then T is said to be k-type exact if for any nonempty open set $U \subset X$ there exists $m \in \mathbb{Z}^d$ such that $m >^k 0$ and $T^m(U) = X$.*

Definition 3.2. *Let $T : \mathbb{Z}^d \times X \to X$ be a $\mathbb{Z}^d$-action on X then T is said to be k-type exact Devaney chaotic (k-EDevC) if T is k-type exact and has dense set of k-type periodic points.*

Let $T : \mathbb{Z}^d \times X \to X$ be a $\mathbb{Z}^d$-action on X. Then T is *k-type mixing* if for any two nonempty open sets U and V of X, there exists $N \in \mathbb{Z}^d$ such that $T^n(U) \cap V \neq \phi$, for all $n >^k N$ [Shah and Das, 2015].

Definition 3.3. *Let $T : \mathbb{Z}^d \times X \to X$ be a $\mathbb{Z}^d$-action on X then T is said to be k-type mixing Devaney chaotic (k-MDevC) if T is k-type mixing and has dense set of k-type periodic points.*

Let $T : \mathbb{Z}^d \times X \to X$ be a $\mathbb{Z}^d$-action on X then T is *k-type weak mixing* if for every pair of nonempty open sets $(G_1, G_2), (H_1, H_2)$ in $X \times X$ there exists $n >^k 0$ in $\mathbb{Z}^d$ such that $T^n(G_i) \cap H_i \neq \emptyset$, for $i = 1, 2$ [Shah and Das, 2015].

Definition 3.4. *Let $T : \mathbb{Z}^d \times X \to X$ be a $\mathbb{Z}^d$-action on X then T is said to be k-type weak mixing Devaney chaotic (k-WMDevC) if T is k-type weak mixing and has dense set of k-type periodic points.*

Remark 3.5. By definitions we have:

$$k\text{-EDevC} \Longrightarrow k\text{-MDevC} \Longrightarrow k\text{-WMDevC} \Longrightarrow k\text{-DevC}$$

Definition 3.6. (Shah and Das, 2015) *Let $T_1 : \mathbb{Z}^d \times X \to X$ and $T_2 : \mathbb{Z}^d \times Y \to Y$ be $\mathbb{Z}^d$-actions on X and Y respectively then T_1 and T_2 are said to be conjugate if there exists a homeomorphism $h : X \to Y$ such that $h \circ T_1^n = T_2^n \circ h$, for every $n \in \mathbb{Z}^d$.*

Lemma 3.7. *Let $T_1 : \mathbb{Z}^d \times X \to X$ and $T_2 : \mathbb{Z}^d \times Y \to Y$ be conjugate $\mathbb{Z}^d$-actions then T_1 is k-type exact if and only if T_2 is k-type exact.*

Proof. By conjugacy, there exists a homeomorphism $h : X \to Y$ such that $h \circ T_1^n = T_2^n \circ h$, for every $n \in \mathbb{Z}^d$.

Suppose T_1 is k-type exact. Let V be a nonempty open set in Y then $h^{-1}(V)$ is a nonempty open set in X. By k-type exactness of T_1, there exists $n >^k 0$ such that $T_1^n(h^{-1}(V)) = X$. Therefore $X = T_1^n(h^{-1}(V)) = h^{-1}(T_2^n(V))$ which implies $T_2^n(V) = Y$. Hence T_2 is k-type exact.

Converse follows using openness of h and conjugacy condition.

Lemma 3.8. *Let $T_1 : \mathbb{Z}^d \times X \to X$ and $T_2 : \mathbb{Z}^d \times Y \to Y$ be conjugate $\mathbb{Z}^d$-actions on X and Y respectively then T_1 is k-type mixing if and only if T_2 is k-type mixing.*

Proof. By conjugacy, there exists a homeomorphism $h : X \to Y$ such that $h \circ T_1^n = T_2^n \circ h$, for every $n \in \mathbb{Z}^d$.

Suppose T_1 is k-type mixing. Let V_1 and V_2 be two nonempty open sets in Y then $h^{-1}(V_1)$ and $h^{-1}(V_2)$ are nonempty open sets in X. By k-type mixing property of T_1, there exists $N >^k 0$ such that $T_1^n(h^{-1}(V_1)) \cap h^{-1}(V_2) \neq \phi$, for every $n >^k N$. We therefore have $h^{-1}(T_2^n(V_1)) \cap h^{-1}(V_2) \neq \phi$, for every $n >^k N$. Hence $T_2^n(V_1) \cap V_2 \neq \phi$, for every $n >^k N$ proving T_2 is k-type mixing.

Conversely suppose T_2 is k-type mixing. Let U_1 and U_2 be two nonempty open sets in X then $h(U_1)$ and $h(U_2)$ are nonempty open sets in Y. By k-type mixing property of T_2, there exists $M >^k 0$ such that $T_2^m(h(U_1)) \cap h(U_2) \neq \phi$, for every $m >^k M$. Using conjugacy, we have $h(T_1^m(U_1)) \cap h(U_2) \neq \phi$, for every $m >^k M$. Hence $T_1^m(U_1) \cap U_2 \neq \phi$, for every $m >^k M$ proving T_1 is k-type mixing.

Lemma 3.9. *Let $T_1 : \mathbb{Z}^d \times X \to X$ and $T_2 : \mathbb{Z}^d \times Y \to Y$ be conjugate $\mathbb{Z}^d$-actions on X and Y respectively then T_1 is k-type weak mixing if and only if T_2 is k-type weak mixing.*

Proof. By conjugacy, there exists a homeomorphism $h : X \to Y$ such that $h \circ T_1^n = T_2^n \circ h$, for every $n \in \mathbb{Z}^d$.

Suppose T_1 is k-type weak mixing. Let (V_1, V_2) and (W_1, W_2) be nonempty open sets in $Y \times Y$. For the nonempty open sets $(h^{-1}(V_1), h^{-1}(V_2))$ and $(h^{-1}(W_1), h^{-1}(W_2))$ of $X \times X$, by k-type weak mixing property of T_1, there exists $n >^k 0$ such that $T_1^n(h^{-1}(V_i)) \cap h^{-1}(W_i) \neq \phi$, for $i = 1, 2$. This implies that $h^{-1}(T_2^n(V_i)) \cap h^{-1}(W_i) \neq \phi$, for $i = 1, 2$. Thus $T_2^n(V_i) \cap W_i \neq \phi$, for $i = 1, 2$ proving that T_2 is k-type weak mixing.

Conversely suppose T_2 is k-type weak mixing. Let (G_1, G_2) and (H_1, H_2) be nonempty open sets in $X \times X$. Then $(h(G_1), h(G_2))$ and $(h(H_1), h(H_2))$ are nonempty open sets in $Y \times Y$. By k-type weak mixing property of T_2, there exists $m >^k 0$ such that $T_2^m(h(G_i)) \cap h(H_i) \neq \phi$, for $i = 1, 2$ which implies that $h(T_1^m(G_i)) \cap h(H_i) \neq \phi$, for $i = 1, 2$. Thus $T_1^m(G_i) \cap H_i \neq \phi$, for $i = 1, 2$ proving that T_1 is k-type weak mixing.

Theorem 3.10. *Let X and Y be infinite metric spaces and let $T_1 : \mathbb{Z}^d \times X \to X$ and $T_2 : \mathbb{Z}^d \times Y \to Y$ be conjugate $\mathbb{Z}^d$-actions on X and Y respectively then T_1 is k-type Devaney chaotic if and only if T_2 is k-type Devaney chaotic.*

Proof. By conjugacy, there exists a homeomorphism $h : X \to Y$ such that $h \circ T_1^n = T_2^n \circ h$, for every $n \in \mathbb{Z}^d$.

Suppose T_1 is k-type Devaney chaotic. Then T_1 is k-type transitive. Let V_1 and V_2 be two nonempty open sets in Y then $h^{-1}(V_1)$ and $h^{-1}(V_2)$ are nonempty open sets in X. By k-type transitivity of T_1, there exists $n >^k 0$ such that $T_1^n(h^{-1}(V_1)) \cap h^{-1}(V_2) \neq \phi$. We therefore have $h^{-1}(T_2^n(V_1)) \cap h^{-1}(V_2) \neq \phi$. Hence $T_2^n(V_1) \cap V_2 \neq \phi$ proving T_2 is k-type transitive. We now show that k-type periodic density of T_1 implies k-type periodic density of T_2. Let $Per_k(T_1)$ denote the set of all k-type periodic points of T_1 then $h(Per_k(T_1)) \subset Per_k(T_2)$. Further $h(\overline{Per_k(T_1)}) = h(X) = Y$ and $h(\overline{Per_k(T_1)}) \subset \overline{h(Per_k(T_1))} \subset \overline{Per_k(T_2)}$ implies that $\overline{Per_k(T_2)} = Y$, where $\overline{S}$ denotes closure of $S \subset X$ in X. Hence using Theorem 2.2, we get T_2 is k-type Devaney chaotic. Proof for k-type Devaney chaoticity of T_1 when T_2 is k-type Devaney chaotic follows similarly.

By similar arguments, we have following results.

Theorem 3.11. *Let $T_1 : \mathbb{Z}^d \times X \to X$ and $T_2 : \mathbb{Z}^d \times Y \to Y$ be conjugate $\mathbb{Z}^d$-actions on X and Y respectively then T_1 is k-EDevC if and only if T_2 is k-EDevC.*

Theorem 3.12. *Let $T_1 : \mathbb{Z}^d \times X \to X$ and $T_2 : \mathbb{Z}^d \times Y \to Y$ be conjugate $\mathbb{Z}^d$-actions on X and Y respectively then T_1 is k-MDevC if and only if T_2 is k-MDevC.*

Theorem 3.13. *Let $T_1 : \mathbb{Z}^d \times X \to X$ and $T_2 : \mathbb{Z}^d \times Y \to Y$ be conjugate $\mathbb{Z}^d$-actions on X and Y respectively then T_1 is k-WMDevC if and only if T_2 is k-WMDevC.*

4. Different Types of Chaos for $\mathbb{Z}^d$-actions Preserved under Uniform Conjugacy

In this section, we define different types of chaos for $\mathbb{Z}^d$-actions and prove that they are preserved under uniform conjugacy.

Let X and Y be metric space. A homeomorphism $h : X \to Y$ is said to be *uniform homeomorphism* if both h and h^{-1} are uniformly continuous.

Definition 4.1. *Let $T_1 : \mathbb{Z}^d \times X \to X$ and $T_2 : \mathbb{Z}^d \times Y \to Y$ be $\mathbb{Z}^d$-actions on X and Y respectively then T_1 and T_2 are said to be uniformly conjugate if there exists a uniform homeomorphism $h : X \to Y$ such that $h \circ T_1^n = T_2^n \circ h$, for every $n \in \mathbb{Z}^d$.*

Definition 4.2. *Let $T : \mathbb{Z}^d \times X \to X$ be a $\mathbb{Z}^d$-action on X. A pair $\{x, y\}$ in X is called a k-type Li-Yorke pair if there exists a sequence $\{t_s\}$, $t_{s+1} >^k t_s$ satisfying*

$$\limsup_{s\to\infty} \rho(T^{t_s}(x), T^{t_s}(y)) > 0,$$
$$\liminf_{s\to\infty} \rho(T^{t_s}(x), T^{t_s}(y)) = 0.$$

The set of k-type Li-Yorke pairs of a $\mathbb{Z}^d$-action T on X is denoted by $LY_k(T)$, i.e.,

$$LY_k(T) = \{(x, y) \in X \times X | \limsup_{s\to\infty} \rho(T^{t_s}(x), T^{t_s}(y)) > 0, \liminf_{s\to\infty} \rho(T^{t_s}(x), T^{t_s}(y)) = 0\}.$$

The set of k-type Li-Yorke pairs with modulus δ of a $\mathbb{Z}^d$-action T on X is denoted by $LY_k(T, \delta)$, i.e.,

$$LY_k(T, \delta) = \{(x, y) \in X \times X | \limsup_{s\to\infty} \rho(T^{t_s}(x), T^{t_s}(y)) > \delta, \liminf_{s\to\infty} \rho(T^{t_s}(x), T^{t_s}(y)) = 0\}.$$

Definition 4.3. *Let $T : \mathbb{Z}^d \times X \to X$ be a $\mathbb{Z}^d$-action on X. Then T is said to be k-type densely chaotic if $\overline{LY_k(T)} = X \times X$.*

Definition 4.4. *Let $T : \mathbb{Z}^d \times X \to X$ be a $\mathbb{Z}^d$-action on X. Then T is said to be k-type densely δ-chaotic if $\overline{LY_k(T, \delta)} = X \times X$.*

Example 4.5. Consider $T : \mathbb{Z}^2 \times S^1 \to S^1$ defined by $T((n_1, n_2), \theta) = 2^{n_1+n_2}\theta$. For $0 < \delta < 1$ and $k \in \{2, 3\}$, $LY_k(T, \delta) = S^1 \times S^1$. Thus T is k-type densely δ-chaotic for $k \in \{2, 3\}$.

Theorem 4.6. *Let $T_1 : \mathbb{Z}^d \times X \to X$ and $T_2 : \mathbb{Z}^d \times Y \to Y$ be uniformly conjugate $\mathbb{Z}^d$-actions on metric spaces (X, ρ_1) and (Y, ρ_2) respectively then T_1 is k-type densely chaotic if and only if T_2 is k-type densely chaotic.*

Proof. By uniform conjugacy, there exists a uniform homeomorphism $h : X \to Y$ such that $h \circ T_1^n = T_2^n \circ h$, for every $n \in \mathbb{Z}^d$.

For $(y_1, y_2) \in Y \times Y$, $y_1 \neq y_2$, there exists $(x_1, x_2) \in X \times X$, $x_1 \neq x_2$, such that $h(x_1) = y_1$ and $h(x_2) = y_2$. By continuity of map h, for every $\epsilon > 0$, there exists $\delta > 0$ such that $h(B_{\rho_1}(x, \delta)) \subset B_{\rho_2}(y, \epsilon)$. Since T_1 is densely k-type chaotic, $\overline{LY_k(T_1)} = X \times X$. Thus there exists $x^* = (x_1, x_2) \in B_{\rho_1}(x, \delta) \cap LY_k(T_1)$. By definition of $LY_k(T_1)$, it follows that $\limsup_{s\to\infty} \rho_1(T_1^{t_s}(x_1), T_1^{t_s}(x_2)) > 0$ and $\liminf_{s\to\infty} \rho_1(T_1^{t_s}(x_1), T_1^{t_s}(x_2)) = 0$. Let $y^* = (h(x_1), h(x_2))$. Then $y^* \in B_{\rho_2}(y, \epsilon)$ and

$$\begin{aligned} &\limsup_{s\to\infty} \rho_2(T_2^{t_s}(h(x_1)), T_2^{t_s}(h(x_2))) \\ = &\limsup_{s\to\infty} \rho_2(h(T_1^{t_s}(x_1)), h(T_1^{t_s}(x_2))) > 0, \end{aligned}$$

as h^{-1} is uniformly continuous.

Otherwise we can find a subsequence of the convergent sequence $\{\rho_2(h(T_1^{t_s}(x_1)), h(T_1^{t_s}(x_2)))\}_{s=0}^{\infty}$ whose limit is 0. We can continue to write the subsequence as $\{\rho_2(h(T_1^{t_s}(x_1)), h(T_1^{t_s}(x_2)))\}_{s=0}^{\infty}$. By definition of convergence, for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\rho_2(h(T^{n_s}(x_1)), h(T^{n_s}(x_2))) < \epsilon$ for every $s > N$. By uniform continuity of h^{-1}, $\rho_1(T^{n_s}(x_1), T^{n_s}(x_2)) < \delta$, for every $\delta > 0$. Therefore limit of arbitrary convergent subsequence of $\{\rho_1(T^{n_s}(x_1), T^{n_s}(x_2))\}_{s=0}^{\infty}$ is 0 and hence $\limsup_{s\to\infty} \rho_1(T_1^{t_s}(x_1), T_1^{t_s}(x_2)) = 0$ which is a contradiction.

Similarly using uniform continuity of h, one can prove that

$$\liminf_{s\to\infty} \rho_2(T_2^{t_s}(h(x_1)), T_2^{t_s}(h(x_2))) = \liminf_{s\to\infty} \rho_2(h(T_1^{t_s}(x_1)), h(T_2^{t_s}(x_2))) = 0.$$

This implies that $y^* = (h(x_1), h(x_2)) \in B_{\rho_2}(y, \epsilon) \cap LY_k(T_2)$. Hence $\overline{LY_k(T_2)} = Y \times Y$ implying T_2 is densely k-type chaotic. By similar arguments converse follows.

Following Theorem can be proved along the lines of the proof of Theorem 4.6.

Theorem 4.7. *Let $T_1 : \mathbb{Z}^d \times X \to X$ and $T_2 : \mathbb{Z}^d \times Y \to Y$ be uniformly conjugate $\mathbb{Z}^d$-actions on metric spaces (X, ρ_1) and (Y, ρ_2) respectively then T_1 is k-type densely δ-chaotic if and only if T_2 is k-type densely δ-chaotic.*

Definition 4.8. *Let $T : \mathbb{Z}^d \times X \to X$ be a $\mathbb{Z}^d$-action on X. Then T is said to be k-type Auslander - Yorke chaotic if T is k-type transitive and T has k-type sensitive dependence on initial conditions.*

Theorem 4.9. *Let $T_1 : \mathbb{Z}^d \times X \to X$ and $T_2 : \mathbb{Z}^d \times Y \to Y$ be uniformly conjugate $\mathbb{Z}^d$-actions on metric spaces (X, ρ_1) and (Y, ρ_2) respectively then T_1 is k-type Auslander-Yorke chaotic if and only if T_2 is k-type Auslander-Yorke chaotic.*

Proof. By uniform conjugacy, there exists a uniform homeomorphism $h : X \to Y$ such that $h \circ T_1^n = T_2^n \circ h$, for every $n \in \mathbb{Z}^d$.

As proved in Theorem 3.10, T_1 is k-type transitive if and only if T_2 is k-type transitive. We now show that k-type SDIC of T_1 implies k-type SDIC of T_2. Suppose T_1 has k-type SDIC. Then by definition of k-type SDIC, there exists $\epsilon > 0$ such that for any $x_1 \in X$ and for any $\delta > 0$ there exists $x_2 \in X$ such that $\rho_1(x_1, x_2) < \delta$ and $\rho_1(T_1^n(x_1), T_1^n(x_2)) > \epsilon$, for some $n >^k 0$.

By uniform continuity of h^{-1}, for above $\epsilon > 0$ there exists $\eta > 0$ such that $\rho_2(h(x_1), h(x_2)) < \eta$ implies $\rho_1(x_1, x_2) < \epsilon$. Let $y_1 = h(x_1) \in Y$ and $\delta_1 > 0$ be given. By uniform continuity of h, there exists $\delta > 0$ such that $\rho_1(x_1, x_2) < \delta$ implies $\rho_2(h(x_1), h(x_2)) < \delta_1$. Choose $x_2 \in X$ such that $\rho_1(x_1, x_2) < \delta$ then $\rho_2(h(x_1), h(x_2)) < \delta_1$. Note that

$$\rho_2(T_2^n(h(x_1)), T_2^n(h(x_2))) = \rho_2(h(T_1^n(x_1)), h(T_1^n(x_2))) > \eta.$$

For if $\rho_2(h(T_1^n(x_1)), h(T_1^n(x_2))) < \eta$ then $\rho_1(T_1^n(x_1), T_1^n(x_2)) < \epsilon$ which is a contradiction. Reverse implication can be proved similarly.

5. k-type Properties on Product Spaces

In this section, we study k-type properties for product of two spaces. Let $T_1 : \mathbb{Z}^d \times X \to X$ and $T_2 : \mathbb{Z}^d \times Y \to Y$ be $\mathbb{Z}^d$-actions on metric spaces (X, ρ_1) and (Y, ρ_2) respectively. We define a $\mathbb{Z}^d$-action on product space $X \times Y$ by $T_1 \times T_2 : \mathbb{Z}^d \times X \times Y \to X \times Y$ by $(T_1 \times T_2)(n, (x, y)) = (T_1^n(x), T_2^n(y))$. We define the metric ρ on $X \times Y$ by $\rho((x_1, y_1), (x_2, y_2)) = \rho_1(x_1, x_2) + \rho_2(y_1, y_2)$.

Lemma 5.1. *Let $T_1 : \mathbb{Z}^d \times X \to X$ and $T_2 : \mathbb{Z}^d \times Y \to Y$ be $\mathbb{Z}^d$-actions on metric spaces (X, ρ_1) and (Y, ρ_2) respectively. If T_1 or T_2 is k-type SDIC then $T_1 \times T_2$ is k-type SDIC. If $T_1 \times T_2$ is k-type SDIC then at least one of T_1 and T_2 is k-type SDIC.*

Proof. Suppose T_1 is k-type SDIC. Let $p = (x, y) \in X \times Y$ and let W be a neighborhood of (x, y). Then there exists neighborhood U of x in X and neighborhood V of y in Y such that $U \times V \subset W$. Since T_1 is k-type SDIC, there exists $\delta > 0$ such for some $x' \in U$ and $n >^k 0$, we have $\rho_1(T_1^n(x), T_1^n(x')) > \delta$. Let $y' \in V$ then $p' = (x', y') \in W$ and

$$\rho((T_1 \times T_2)^n(p), (T_1 \times T_2)^n(p')) = \rho_1((T_1)^n(x), T_1^n(x')) + \rho_2(T_2^n(y), T_2^n(y')) \geq \rho_1(T_1^n(x), T_1^n(x')) > \delta.$$

Hence $T_1 \times T_2$ has k-type SDIC.

Next, suppose $T_1 \times T_2$ is k-type SDIC but neither T_1 nor T_2 is k-type SDIC. Note that for any $\delta > 0$, there exist $x \in X$ and a neighborhood $N(x)$ of x in X such that $\rho_1(T_1^n(x), T_1^n(x')) < \frac{\delta}{2}$, for every $x' \in N(x)$ and for every $n >^k 0$. Similarly there exist $y \in Y$ and a neighborhood $N(y)$ of y in Y such that $\rho_2(T_2^n(y), T_2^n(y')) < \frac{\delta}{2}$, for every $y' \in N(y)$ and for every $n >^k 0$. Thus we have

$$\rho((T_1 \times T_2)^n(x, y), (T_1 \times T_2)^n(x', y')) = \rho_1(T_1^n(x), T_1^n(x')) + \rho_2(T_2^n(y), T_2^n(y')) < \delta,$$

for every $(x', y') \in N(x) \times N(y)$ implying $T_1 \times T_2$ is not k-type SDIC which is a contradiction.

Lemma 5.2. *Let $T_1 : \mathbb{Z}^d \times X \to X$ and $T_2 : \mathbb{Z}^d \times Y \to Y$ be $\mathbb{Z}^d$-actions on metric spaces (X, ρ_1) and (Y, ρ_2) respectively then $T_1 \times T_2$ is k-type mixing if and only if T_1 and T_2 are k-type mixing.* *Proof.* Suppose T_1 and T_2 are k-type mixing. Given open sets $G_1, G_2 \subset X \times Y$, there exist open sets $U_1, U_2 \subset X$ and $V_1, V_2 \subset Y$ such that $U_i \times V_i \subset G_i$ for $i = 1, 2$. By k-type mixing property of T_1 and T_2, there exist $n_1 >^k 0$, $n_2 >^k 0$ in $\mathbb{Z}^d$ such that

$$T_1^n(U_1) \cap U_2 \neq \phi, \text{ for } n >^k n_1$$
$$T_2^n(V_1) \cap V_2 \neq \phi, \text{ for } n >^k n_2$$

Choose $n_0 >^k 0$ such that $\| n_0 \| > max\{\| n_1 \|, \| n_2 \|\}$. Then

$$[(T_1 \times T_2)^n(U_1 \times V_1)] \cap (U_2 \times V_2) = [T_1^n(U_1) \cap U_2] \times [T_2^n(V_1) \cap V_2] \neq \phi$$

for all $n >^k n_0$. Conversely, suppose $T_1 \times T_2$ is k-type mixing. Let $U_1, U_2 \subset X$ and $V_1, V_2 \subset Y$. Then for the open sets $U_1 \times V_1$, $U_2 \times V_2$ by k-type mixing property of $T_1 \times T_2$, there exists $N >^k 0$ such that

$$[(T_1 \times T_2)^n(U_1 \times V_1)] \cap (U_2 \times V_2) \neq \phi, \text{ for all } n >^k N$$

This implies that $[T_1^n(U_1)\cap U_2]\times[T_2^n(V_1)\cap V_2] \neq \phi$, for all $n >^k N$ and hence $T_1^n(U_1)\cap U_2 \neq \phi$ and $T_2^n(V_1)\cap V_2 \neq \phi$, for all $n >^k N$, proving that T_1 and T_2 are k-type mixing.

By similar arguments, we have

Lemma 5.3. *Let $T_1 : \mathbb{Z}^d \times X \to X$ and $T_2 : \mathbb{Z}^d \times Y \to Y$ be two $\mathbb{Z}^d$-actions on metric spaces (X,ρ_1) and (Y,ρ_2) respectively. If $T_1 \times T_2$ is k-type transitive then T_1 and T_2 are k-type transitive.*

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