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C^1-Robust Topologically Mixing Solenoid-Like Attractors and Their Invisible Parts

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Abstract

The aim of this paper is to discuss statistical attractors of skew products over the solenoid which have an m-dimensional compact orientable manifold M as a fiber and their ε-invisible parts, i.e. a sizable portion of the attractor in which almost all orbits visit it with average frequency no greater than ε.

We show that for any $n \in \mathbb{N}$ large enough, there exists a ball D_n in the space of skew products over the solenoid with the fiber M such that each C^2-skew product map from D_n possesses a statistical attractor with an ε-invisible part, whose size of invisibility is comparable to that of the whole attractor. Also, it consists of structurally stable skew product maps.

In particular, small perturbations of these skew products in the space of all diffeomorphisms still have attractors with the same properties.
Our construction develops the example of (Ilyashenko & Negut, 2010) to skew products over the solenoid with an m-dimensional fiber, $m \geq 2$.

As a consequence, we provide a class of local diffeomorphisms acting on $S^1 \times M$ such that each map of this class admits a robustly topologically mixing maximal attractor.

Keywords: statistical attractor, maximal attractor, skew-product, invisible part, solenoid attractor.

1. Introduction and Preliminaries

The study of attractors is one of the major problems in the theory of dynamical systems. An attractor is a set of points in the phase space, invariant under the dynamics, towards which neighboring points in a given basin of attraction tend asymptotically. Roughly abusing of the language, we will use the word attractor referring to any closed invariant set satisfies two kinds of properties: it attracts many orbits and it is indecomposable. Therefore, there are various non-equivalent definitions of attractors of dynamical systems including global attractor, Milnor attractor, statistical attractor and etc. Some knowledge of attractors and their properties is available, see (Karabacak & Ashwin, 2011), (Kleptsyn, 2006), (Ilyashenko, 1991) and (Milnor, 1985).

In this article, we will treat the attractors of skew products over the solenoid and their invisible parts. Invisibility of attractors introduced by (Ilyashenko & Negut, 2010) is a new effect in the theory of dynamical systems. The systems with this property have large parts of attractors that can not be observed in numerical experiments of any reasonable duration.

Here, we will build a skew product over the solenoid which has a closed m-dimensional orientable manifold M as a fiber, $m \geq 2$. This skew product possesses an attractor with a large invisible part. Moreover, our example is robust, i.e. this property remains true for every small perturbation.

Our approach is motivated by the example by (Ilyashenko & Negut, 2010). The authors described an open set in the space of skew products over the solenoid with one dimensional fiber whose attractors had large unobservable parts. Then this result extended (Ghane & *et al.*, 2012) to an open set of skew products over the Bernoulli shift with an m-dimensional fiber.

In fact, we will provide an open class of skew products admitting statistical attractors. These attractors support a SRB measure. In particular, this property remains true for all nearby diffeomorphisms. Consequently, a class of local diffeomorphisms is also proposed so that every map of this class admits a robust topologically mixing attractor.

To be more precise, we need to introduce some notations and recall several background definitions and concepts.

The *maximal attractor* of F in a neighborhood U is an invariant set A_{max} of F such that

$$A_{max} = \cap_{n=0}^{\infty} F^n(U).$$

The *Milnor attractor* A_M of F is the minimal invariant closed set that contains the ω-limit sets of almost all points with respect to the Lebesgue measure.

The minimal closed set A_{stat} of F is called the *statistical attractor* if all orbits spend an average time of 1 in any neighborhood of it. The notion of the statistical attractor is one of the ways of describing what an observer will see if looking at a dynamical system for a long time.

An F-invariant measure μ_∞ is called Sinai-Ruelle-Bowen (SRB) if there exists a measurable set $E \subset X$, with $Leb(E) > 0$, such that for any test function $\phi \in C(X)$ and any $x \in E$ we have

$$\int_X \varphi d\mu_\infty = \lim_{k\to\infty} \frac{1}{k} \sum_{i=0}^{k-1} \varphi(F^i(x)).$$

The set E is called the basin of μ_∞.

An open set U is called *ε-invisible* if almost every orbit visits U with an average frequency ε or less:

$$\limsup_{n\to\infty} \frac{\{k;\ F^k(x) \in U,\ 0 \le k \le n\}}{n} \le \varepsilon, \quad for\ \ a.e\ x.$$

Throughout this paper we assume that M is an m-dimensional closed orientable manifold and its metric is geodesic distance and the measure is the Riemannian volume.

Let $f_i,\ i = 0, 1$, be diffeomorphisms of M. A *step skew-product* over the Bernoulli shift $\sigma : \Sigma^2 \to \Sigma^2$, is defined by

$$F : \Sigma^2 \times M \to \Sigma^2 \times M;\ \ (\omega, x) \to (\sigma\omega, f_{w_0}(x)), \tag{1}$$

where Σ^2 is the space of two-sided sequences of 2 symbols $\{0, 1\}$. Consider the following standard metric on Σ^2

$$d(\omega, \omega') = 2^{-n},$$

where $n = \min\{|k|; \omega_k \ne \omega'_k\}$ and $\omega, \omega' \in \Sigma^2$.
Also, we equip Σ^2 by $(\frac{1}{2}, \frac{1}{2})$-probability measure P. This means that

$$P(\{\omega : \omega_{i_1} = \alpha_1, ..., \omega_{i_k} = \alpha_k\}) = \frac{1}{2^k},$$

for any $i_1, ..., i_k \in \mathbb{Z}$ and any $\alpha_1, ..., \alpha_k \in \{0, 1\}$.

A *mild skew product* over the Bernoulli shift is a map

$$\mathcal{G} : \Sigma^2 \times M \to \Sigma^2 \times M;\ \ (\omega, x) \to (\sigma\omega, g_\omega(x)), \tag{2}$$

where the fiber maps g_ω are diffeomorphisms of the fiber into itself.

We would like to mention that in contrast to step skew products, the fiber maps of mild skew products depend on the whole sequence ω.

Skew products play an important role in the theory of dynamical systems. Many properties observed for these products appear to persist as properties of diffeomorphisms, for instance see (Gorodetsky & Ilyashenko, 1999) and (Gorodetsky & Ilyashenko, 2000).

In the following, we consider skew products over the Smale-Williams solenoid. Take $R \geq 2$ and let $B = B(R)$ denote the solid torus

$$B = S^1 \times D(R),\ S^1 = \mathbb{R}/\mathbb{Z},\ D(R) = \{z \in \mathbb{C} : |z| \leq R\}.$$

The *solenoid map* is defined as

$$h : B \to B,\ (y, z) \longmapsto (2y, e^{2\pi iy} + \lambda z),\ \lambda < 0.1. \tag{3}$$

Here, we consider the Cartesian product $X = B \times M$ with the natural projections $\pi : X \to M$ along B, $p : X \to B$ along M. The set B is the base, while M is the fiber. The measure on X is the Cartesian product of the measures of the base and of the fiber. The distance between two points of X is the sum of the distances between their projections onto the base and onto the fiber.

Consider maps of the form

$$\mathcal{F} : B \times M \to B \times M,\ \mathcal{F}(b, x) = (h(b), f_b(x)), \tag{4}$$

where h is a solenoid map as above. Denote by Λ the maximal attractor of h, which is called the *Smale-Williams solenoid*. Let us mentioned that the solenoid was introduced into dynamics by Smale as a hyperbolic attractor (Katok & Hasselblat, 1999).

We recall that a homeomorphism $\mathcal{F}$ of a metric space is called *L-bi-Lipschitz* if $Lip(\mathcal{F}^{\pm 1}) \leq L$, where Lip denotes Lipschitz constant. Here we shall consider only L-bi-Lipschitz maps $\mathcal{F}$, in order to guarantee that the phenomenon of ε-invisibility is not produced by any large extraordinary distortion (see Remark 1 of (Ilyashenko & Negut, 2010)).

Consider the Cartesian product $X = B \times M$. Let $D_L(X)$ (respectively, $C^1_{p,L}(X)$) denote the space of L-bi-Lipschitz smooth maps (respectively, smooth skew products). Also, $C^2(X)$ denote the space of all C^2-maps on X.

Suppose that D_n is a ball of radius $\frac{1}{n^2}$ of $\mathcal{F}$ in $C^1_{P,L}(X)$, the space of all C^1 L-bi-Lipschitz skew products on X, this means that if skew product $\mathcal{G} \in D_n$ then

$$d(\mathcal{F}, \mathcal{G}) = \sup_{b \in B} d_{C^1}(f_b^{\pm 1}, g_b^{\pm 1}) \leq \frac{1}{n^2} \tag{5}$$

We will now state our main result.

Theorem A *Consider* $n \geq 100m^2$. *Then there exists a ball* D_n, *of radius* $\frac{1}{n^2}$ *in the space* $C^1_{p,L}(X)$, $X = B \times M$, *having the following property.*
Any map $\mathcal{G} \in D_n \cap C^2(X)$, *has a statistical attractor* $\mathcal{A}_{stat} = \mathcal{A}_{stat}(\mathcal{G})$ *such that the followings hold*:

1. *The projection* $\pi(\mathcal{A}_{stat}) \subset M$ *has the property*

$$R \subset \pi(\mathcal{A}_{stat}) \subset R^*, \tag{6}$$

where $\pi : B \times M \to M$ *is the natural projection and* R, R^* *are the inverse images of the* m-dimensional cubes of $\mathbb{R}^m$ *under some local chart of* M.

2. *There exist a set* N *that is* ε*- invisible for* $\mathcal{G}$ *with* $\varepsilon = \frac{1}{2^n}$, *and the size of* N *is comparable to that of the whole attractor.*
Also, each $\mathcal{G} \in D_n \cap C^2(X)$ *is structurally stable in* $D^1(X)$, *where* $D^1(X)$ *is the space of all* C^1-diffeomorphisms *on* X. *Moreover, small perturbations of the maps from* $\mathcal{D}_n$ *in the space* $D_L(X)$ *of all diffeomorphisms have statistical attractors with the same properties.*

In this context the following questions are interesting.

Can we develop the example to provide a better rate of invisibility while keeping the same radius of the ball in the space of skew products?

Is it possible to obtain the rate of invisibility as a tower of exponents whose height grows with the dimension?

As a Consequence of the main result, we will also provide a class of endomorphisms defined on $S^1 \times M$ so that every endomorphism of this class admits a robust topologically mixing attractor.

To be more precise, let $\mathcal{E}(M)$ be the space of all skew product maps acting on $S^1 \times M$ of the following form

$$F : S^1 \times M \to S^1 \times M, \quad F(y, x) = (g(y), f_y(x)), \tag{7}$$

where $g(y) = ky$, $k \geq 2$, is an expanding circle map and fiber maps $x \mapsto f_y(x)$ are C^1- diffeomorphisms defined on closed manifold M.

Corollary B *There exists an open ball $\mathcal{D}_n \subset \mathcal{E}(M)$ such that any map $F \in \mathcal{D}_n \cap C^2(S^1 \times M)$ admits a maximal attractor which is the support of an invariant ergodic SRB measure. Also, it is robustly topologically mixing. Moreover, the projection of the maximal attractor on the fiber contains an m-cube.*

Here is an extremely brief indication of the proof of our main result.
We follow the approach suggested by (Ilyashenko & Negut, 2010) to provide an open ball D_n in the space of skew product maps over the solenoid such that it satisfies the requirements of Theorem A. However, we can not use this approach straightforwardly to settle our result.
In (Ilyashenko & Negut, 2010), the fiber maps defined on the circle S^1 and the fact that central direction is one dimensional is essential. In our setting, the central direction is m-dimensional and this makes some difficulties in the proof.
To specify the open ball D_n, we seek a single skew product map $\mathcal{F}$ which is the center of D_n. In order to introduce $\mathcal{F}$, we need to choose two diffeomorphisms f_i, $i = 0, 1$, in an appropriate way such that the set $\{f_0, f_1\}$ has covering property, i.e. there exists an open set U satisfying $U \subset f_0(U) \cup f_1(U)$. Hence, we should choose an m-cube R such that $U_{\frac{1}{n^2}}(R) = U$, see section 2.
Moreover, f_i, $i = 0, 1$, can be chosen so that the size of invisible part of the attractor is large enough.

This paper is organized as follows. First, an open set of skew products, North-South like skew products, is introduced in section 2. The sections after that will be concerned with the proof of the main result. In section 3, we assert that the maximal attractor and statistical attractor of skew products which have been chosen in an appropriate way are coincide. In section 4, the proof of statement (2) of Theorem A is presented. Moreover, the invisible part of the attractor is also specified. Also, we prove that small perturbations of the maps from D_n in the space $D_L(X)$ of all diffeomorphisms have statistical attractors with the same properties, in section 5. Finally, section 6 is devoted to prove Corollary B.

2. North-South Like Skew Products over the Solenoid

In this section, we will introduce an open set of skew products that will be studied in the paper.

For a closed m-dimensional manifold M, consider two disjoint open neighborhoods $U, W \subset M$ which are the domains of two local charts (W, φ) and (U, ψ). Take two C^2-gradient Morse-Smale vector fields on M, each of which possesses a unique hyperbolic repelling equilibrium q_i in W, a unique hyperbolic attracting equilibrium p_i in U, $i = 0, 1$, (see e.g. (Matsumoto, 2002), Theorem 3.35 for the existence of Morse functions with unique extrema) and finitely many saddle points r^i_j, $i = 0, 1$, $j = 1, ..., l$, which are contained in $M \setminus (U \cup W)$. Suppose that f_i, $i = 0, 1$, are their time-1 maps. Also, we require that they satisfy the following conditions.

(i) The mappings f_i, $i = 0, 1$, are coincide on U^c. So, we can take $q_0 = q_1$ and $r^0_j = r^1_j$, $j = 1, ..., l$. For simplicity, we take $r_j := r^i_j$, $j = 1, ..., l$. Moreover, we assume that they have no any saddle connection.

(ii) We may choose the coordinate functions φ and ψ such that they are isometries with the following properties:

$\psi : U \to C_3(0) \subset \mathbb{R}^m$, $\varphi : W \to C_2(0)$, where $C_r(0)$ is an m- dimensional cube $\Pi^m_{s=1}[-r, r]$ with $r > 0$. Also, let $V \subset W$ be a compact neighborhood containing the fixed points $q_0 = q_1$.

If we take $\widehat{f_i} := \psi \circ f_i \circ \psi^{-1}$, $i = 0, 1$, then $\widehat{f_i}$'s are affine maps which are defined by

$$\begin{aligned}
\widehat{f_0}(x_1, \ldots, x_m) &= (1 - \tfrac{1}{m^2 n})(x_m, x_1, \ldots, x_{m-1}) - (\tfrac{1}{mn} + \tfrac{2m}{n^2}, 0, \ldots, 0),\\
\widehat{f_1}(x_1, \ldots, x_m) &= (\tfrac{1}{n} x_m + 1 - \tfrac{2}{3n}, (1 - \tfrac{1}{m^2 n})x_1, \ldots, (1 - \tfrac{1}{m^2 n})x_{m-1}).
\end{aligned}$$

Also if we take $\overline{f}_i := \varphi \circ f_i \circ \varphi^{-1}$, $i = 0, 1$, then $\overline{f}_i(x_1, \ldots, x_m) = 2(x_1, \ldots, x_m)$.
Now consider the m-dimensional cubes $\widehat{R} = \Pi^m_{i=1}[-\alpha_i, \alpha_i]$ and $\widehat{R}^* = \Pi^m_{i=1}[-\gamma_i, \gamma_i]$, where

$$\alpha_i = 1 + \frac{m - i}{m^2 n} + \frac{2(m - i)}{n^2}, \; i = 1, \ldots, m - 1, \; \alpha_m = 1$$

and

$$\gamma_i = 2 + \frac{2(m-i)}{m^2 n} - \frac{m-i-2}{n^2} + \frac{2(m-i)}{m^4 n^2},\ i = 1, \ldots, m-1,\ \gamma_m = 2.$$

Let R and R^* be the inverse images of $\widehat{R}$ and $\widehat{R}^*$ under the local chart ψ, respectively. Clearly

$$\widehat{f_0}(\widehat{R}) \cup \widehat{f_1}(\widehat{R}) \supset \widehat{R},\ \widehat{f_i}(\widehat{R^*}) \subset \widehat{R}^*, i = 0, 1, \tag{8}$$

which implies that $R \subset f_0(R) \cup f_1(R)$, i.e. $\{f_0, f_1\}$ has covering property.

Note that these inclusions are robust, this means that they remain true for any maps $\widehat{g_0} := \psi \circ g_0 \circ \psi^{-1}$ and $\widehat{g_1} := \psi \circ g_1 \circ \psi^{-1}$, where (g_0, g_1) belongs to open ball $\mathcal{W}$ of (f_0, f_1) with radius $\frac{1}{n^2}$ in $Diff^1(M) \times Diff^1(M)$. Also let $V_j,\ j = 1, ..., l$, be disjoint compact neighborhoods of r_j contained in the domains of local charts of M.

The *iterated function system* $\mathfrak{F}(f_0, f_1)$ is the action of semigroup generated by $\{f_0, f_1\}$ on M. An orbit of $x \in M$ under the iterated function system $\mathfrak{F}(f_0, f_1)$ is a sequence

$$\{f_{i_1} \circ f_{i_2} \circ ... \circ f_{i_k}(x); i_j \in \{0, 1\},\ j = 1, \ldots, k,\ k \in \mathbb{N}\}.$$

The $\mathfrak{F}$-orbit of x denoted by $Orbit^+_{\mathfrak{F}}(x)$ is the set of points lying on some orbit of x under the iterated function system $\mathfrak{F}$.
We say that the iterated function system $\mathfrak{F}$ is *minimal* if the $\mathfrak{F}$-orbit of any point is dense in M. The iterated function system $\mathfrak{F}(f_0, f_1)$ has *covering property* if there exists an open set D such that

$$D \subset \bigcup_{i=0}^{1} f_i(D).$$

Now, we fix the diffeomorphisms $f_0, f_1 : M \to M$ as above. Then it is not hard to see that the iterated function system $\mathfrak{F}$ generated by f_0, f_1 admits a unique compact invariant set $\Delta = \Delta_{\mathfrak{F}}$ with nonempty interior so that the acting $\mathfrak{F}$ on Δ is minimal. In particular, this property is robust in C^1-topology, see (Homburg & Nassiri, 2013).

Let us consider the corresponding skew product map F with generators $f_i,\ i = 0, 1$, which are defined as follows:

$$F : \Sigma^2 \times M \to \Sigma^2 \times M;\ \ (\omega, x) \to (\sigma\omega, f_{\omega_0}(x)), \tag{9}$$

where $\sigma : \Sigma^2 \to \Sigma^2$ is the Bernoulli shift map. The skew product map F with generators $f_i,\ i = 0, 1$, which satisfies all properties mentioned above, is called *North-South like skew product map*.

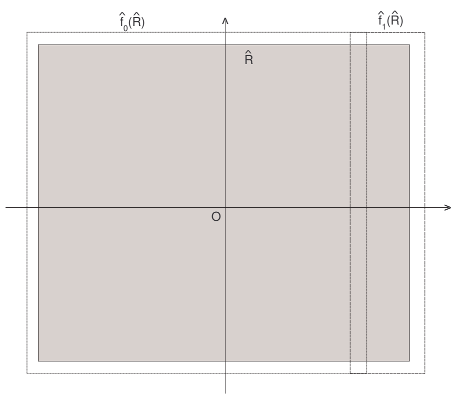

Figure 1. Description of the open domain $\widehat{R}$ contained in the attractor.

Lemma 2.1 *Consider the North-south like skew product map F of the form*

$$F : \Sigma^2 \times M \to \Sigma^2 \times M;\ \ (\omega, x) \to (\sigma\omega, f_{\omega_0}(x)),$$

as above. Then F is C^1 robustly topologically mixing on $\Sigma^2 \times \Delta$ under continuous perturbation of $\omega \mapsto f_\omega$ in the C^1-topology, where Δ is the unique compact invariant set of the corresponding iterated function system $\mathfrak{F}$ generated by $f_i : M \to M,\ i = 0, 1$, and the acting $\mathfrak{F}$ on Δ is minimal.

Proof. Let F be a North-south like skew product map with the fiber maps $f_i,\ i = 0, 1$, and with the corresponding iterated function system $\mathfrak{F}(f_0, f_1)$ satisfying the hypothesis. Then, $\mathfrak{F}$ admits a unique compact invariant set Δ such that the acting $\mathfrak{F}$ on Δ is minimal, see (Homburg & Nassiri, 2013).
Let us define

$$\mathcal{L}(\Delta) = \bigcup_{i=0}^{1} f_i(\Delta).$$

Take two open subsets $\Delta_{in} \subset \Delta \subset \Delta_{out}$ close enough to Δ so that $\widehat{f_i} := \psi \circ f \circ \psi^{-1},\ i = 0, 1$, are affine on them. Then

$$\Delta_{in} \subset \mathcal{L}(\Delta_{in}) \subset \Delta \subset \mathcal{L}(\Delta_{out}) \subset \Delta_{out}, \tag{10}$$

and the sequences $\mathcal{L}^i(\Delta_{in})$ and $\mathcal{L}^i(\Delta_{out})$ converge to Δ in the Hausdorff topology whenever $i \to \infty$.
Let us show that the skew product map $F(\omega, x) = (\sigma\omega, f_{\omega_0}(x))$ is topologically mixing on $\Sigma^2 \times \Delta$. For, take an open set $U \subset \Sigma^2 \times \Delta$, a high iterate $F^n(U)$ contain a strip $\Sigma^2 \times W$ in $\Sigma^2 \times \Delta$, where $W \subset M$ is an open set. Now $F^{n+1}(U)$ maps $\Sigma^2 \times W$ to 2 strips with total width larger than $c.vol(W)$, with

$$c = \sum_{i=0}^{1} (m(Df_i(p_i)))^m > 1,$$

where $vol(W)$ is the volume of W and $m(Df_i(p_i))$ is the co-norm of linear operator $Df_i(p_i)$ which is defined by $m(Df_i(p_i)) := \inf\{\|Df_i(p_i)(v)\| : \|v\| = 1\}$. Further iterates $F^{n+k}(U)$ contains 2^k strips of increasing width so that for some $k > 0$, $F^{n+k}(U)$ lies dense in $\Sigma^2 \times V$ for any neighborhood $V \subset \Delta$. This shows that F is topologically mixing on $\Sigma^2 \times \Delta$.
This reasoning also applies to small perturbations of F, where the fiber maps may depend on all ω instead of just ω_0, with any modifications. The inclusions (10) get replaced by

$$\Sigma^2 \times \Delta_{in} \subset F(\Sigma^2 \times \Delta_{in}),\ F(\Sigma^2 \times \Delta_{out}) \subset \Sigma^2 \times \Delta_{out}.$$

The map F acting on $\Sigma^2 \times \Delta_{out}$ acts by contractions in the fibers $\omega \times \Delta_{out}$. A high iterate $F^n(U)$ may not contain a product $\Sigma^2 \times\ W$ but contains a strip of some width ε lying between the graphs of 2 maps. Again $F^{n+1}(U)$ contains 2 strip of total width exceeding $c\varepsilon$ for some $c > 1$, and $F^{n+k}(U)$ contain 2^k strips of increasing total width. There exists a closed neighborhood $\widetilde{\Delta}$ near to Δ so that for some $k > 0$, $F^{n+k}(U)$ lies dense in $\Sigma^2 \times V$ for any $V \subset \widetilde{\Delta}$.

In the following, we will introduce an SRB measure on the Smale-Williams solenoid Λ. Consider the solenoid map h as above. Let $\Sigma^2_1 \subset \Sigma^2$ be the set of infinite sequences of $0'$s and $1'$s without a tail of $1'$s infinitely to the right (i.e sequences which have $0'$s arbitrary far to the right). Its metric and measure are inherited from the space Σ^2. Consider the fate map

$$\Phi : \Lambda \to \Sigma^2_1,\ \Phi(b) = (\ldots \omega_{-1}\omega_0\omega_1 \ldots),$$

where $\omega_k = 0$ if $y(h^k(b)) \in [0, \frac{1}{2})$ and $\omega_k = 1$ if $y(h^k(b)) \in [\frac{1}{2}, 1)$. The map Φ is a bijection with a continuous inverse. Moreover, it conjugates the map $h|_\Lambda$ with the Bernoulli shift σ on Σ^2_1:

$$\begin{array}{ccc} \Lambda & \xrightarrow{h} & \Lambda \\ \Phi \downarrow & & \downarrow \Phi \\ \Sigma^2_1 & \xrightarrow{\sigma} & \Sigma^2_1 \end{array}$$

In addition to the fate map Φ, we can define the *forward fate map* $\Phi^+(b) = (\omega_0\omega_1 \ldots)$, with $\omega_0\omega_1 \ldots$ described as above. The map $\Phi^+(b)$ is now defined for all b in the solid torus B, and it only depends on $y(b)$. More generally, if $h^{-k}(b)$ exists, then we can define $\Phi^+_{-k}(b) = (\omega_k \ldots \omega_0\omega_1)$.
It is not hard to see that the SRB measure on Λ is the pullback of the Bernoulli measure on Σ^2_1 under the fate map Φ, i.e. $\mu_\Lambda = \Phi^* P$. In fact, we set $f_b = f_{\Phi(b)_0}$. This means that f_b depends on the digit $\Phi(b)_0$ only, where $\Phi(b)_0 \in \{0, 1\}$. Note that this skew product would be discontinuous at $y(b) \in \{0, \frac{1}{2}\} \subset S^1$. In following, we apply the approach suggested in [8] to remove this discontinuity.
For, consider an isotopy

$$f_t : M \to M,\ t \in [0, 1)$$

between f_0, f_1 as follows. Since f_0 and f_1 are both orientation preserving on U, so we can take $\widehat{f_t} = (1-t^2)\widehat{f_0} + t^2\widehat{f_1}$ on $\psi(U)$ and $f_t = f_0 = f_1$, for each $t \in [0, 1]$ outside U . The choice of isotopy f_t implies that this family is C^1 in

y. In below, numbers in [0,1) are written in binary representation. For $y \in [0, 1)$, define

$$\widehat{f_y} := \begin{cases} \widehat{f_0} & \text{for y} \in [0,0.011), \\ \widehat{f_{8y-3}} & \text{for y} \in [0.011,0.1), \\ \widehat{f_1} & \text{for y} \in [0.1,0.111), \\ \widehat{f_{8-8y}} & \text{for y} \in [0.111,1). \end{cases} \tag{11}$$

We define the *almost step North-South like skew product* over the solenoid corresponding to the fiber maps f_0, f_1 by

$$\mathcal{F} : X \to X, \mathcal{F}(b, x) = (h(b), f_{y(b)}(x)),$$

where f_y's are introduced by (11). Note that if we consider a word $w = (\omega_0 \dots \omega_{k+1})$ that contains no cluster 11 and a sequence ω with the subword w starting at the zero position then

$$f_{h^{i-1}(b)} \circ \dots \circ f_b = f_{\omega_{i-1}} \circ \dots \circ f_{\omega_0}.$$

Indeed, the binary expansion of $y(h^i(b))$, for any $0 \le i \le k-1$, starts with the combination $\omega_i\omega_{i+1}\omega_{i+2}$ which is different from 011 or 111. Hence, by definition, $f_{h^i(b)} = f_{\omega_i}$.

Proposition 2.2 *Consider the almost step North-South like skew product $\mathcal{F}$ over the solenoid corresponding to the fiber maps f_0, f_1 as above. Let $\mathcal{A}_{max}(\mathcal{F}) = \bigcap_{k\ge0} \mathcal{F}^k(B \times R^*)$ be the maximal attractor of $\mathcal{F}$. Then*

$$\mathcal{A}_{max}(\mathcal{F}) = \Lambda \times \Delta,$$

where Λ is the maximal attractor (solenoid attractor) of the base map h and Δ is the compact invariant set of the corresponding iterated function system $\mathfrak{F}(f_0, f_1)$ such that the acting $\mathfrak{F}$ on Δ is minimal.

Proof. Consider the isotopy $\hat{f}_t = (1-t^2)\hat{f}_0 + t^2\hat{f}_1$, $t \in [0, 1]$, on $\psi(U)$, where $\hat{f}_i = \psi \circ f_i \circ \varphi^{-1}$, $i = 0, 1$. Then the construction shows that

$$\hat{f}_t(\hat{R}^*) \subset \hat{f}_0(\hat{R}^*) \cup \hat{f}_1(\hat{R}^*).$$

Therefore,

$$f_t(R^*) \subset f_0(R^*) \cup f_1(R^*).$$

We conclude that

$$\pi(\mathcal{A}_{max}(\mathcal{F})) = \pi(\bigcap_{k\ge0} \mathcal{F}^k(B \times R^*)) = \bigcap_{n\ge0} \mathcal{L}^n(R^*) = \Delta,$$

where $\mathcal{L}(R^*) = f_0(R^*) \cup f_1(R^*)$ and $\mathcal{L}^n(R^*) = \mathcal{L}^{n-1}(\mathcal{L}(R^*)$, (see (Homburg & Nassiri, 2013)).

Now, consider the ball D_n of radius $\frac{1}{n^2}$ centered at $\mathcal{F}$ in the space $C^1_{p,L}(X)$, $X = B \times M$, of skew products over the solenoid. This ball consists of skew products:

$$\mathcal{G} : B \times M \to B \times M, \;\; \mathcal{G}(b, x) = (h(b), g_b(x))$$

which satisfy

$$\max_B d_{C^1}(f_b^{\mp1}, g_b^{\mp1}) \le \frac{1}{n^2}.$$

Also, we set $C^2(X)$ the space of all C^2-maps on X.

Proposition 2.3 *Consider $n > 100m^2$. Then any $\mathcal{G} \in D_n \cap C^2(X)$ satisfies the following properties:*

(i) *For each $b \in B$, the fiber map g_b has one hyperbolic attracting fixed point $p(b)$, one hyperbolic repelling fixed point $q(b)$ and saddle fixed points $r_i(b)$, for $i = 1, \dots, l$,.*

(ii) *All the attractors of the maps g_b lie strictly inside R^*.*

(iii) *All the repellers of the maps g_b lie strictly inside W, the domain of the local chart (W, φ).*

(iv) *All the saddles of the maps g_b lie strictly inside V_i, for some $i \in \{1, ..., l\}$, where V_i is a compact subset which is contained in a domain of some local chart of M, as introduced before.*

(v) *All the maps g_b bring R^* into itself and they are contracting on R^*, uniformly in b. In particular, $g_b(R^*) \subset int(R^*)$. Moreover, the map g_b^{-1} is expanding on R^*, for all $b \in B$.*

(vi) *All the inverse maps g_b^{-1} bring W into itself and they are contracting on W uniformly in b. Moreover, the map g_b is expanding on W, for all $b \in B$.*

(vii) *The mappings g_b and g_b^{-1} depend on b continuously in the $Diff^1$-norm.*

Proof. First note that for any $t \in (0, 1)$, the map f_t has a unique attracting fixed point in R^* and it is contracting on R^*, it has a unique repelling fixed point in W and it is expanding on W. Also, it has a unique saddle in V_i, $i = 1, ..., l$, and has no other fixed points. Hence, $\mathcal{F}$ possesses all of the properties mentioned in the proposition.

Now let $\mathcal{G} \in D_n \cap C^2(X)$. We verify the property (v) for $\mathcal{G}$, the other properties follow immediately. We show that the rectangle R^* is mapped strictly inside R^* by g_b for any $b \in B$. We use notations $b(0), b(1)$ for any point b in B with $y(b)$ lies in $[0, 0.011)$ and $[0.1, 0.111)$, respectively.
For any $x \in \widehat{R^*}$, $j = 0, 1$ and $i = 2, \ldots, m$,

$$\begin{aligned}
\pi_i(\widehat{g}_{b(j)}(x)) < \pi_i(\widehat{f_j}(x)) + \tfrac{1}{n^2} \leq (1 - \tfrac{1}{m^2n})\gamma_{i-1} + \tfrac{1}{n^2} < \gamma_i,\\
\pi_1(\widehat{g}_{b(0)}(x)) < \pi_1(\widehat{f_0}(x)) + \tfrac{1}{n^2} \leq (1 - \tfrac{1}{m^2n})\gamma_m - \tfrac{1}{mn} - \tfrac{2m}{n^2} + \tfrac{1}{n^2} < \gamma_1,\\
\pi_1(\widehat{g}_{b(1)}(x)) < \pi_1(\widehat{f_1}(x)) + \tfrac{1}{n^2} < \tfrac{1}{n}\gamma_m + 1 - \tfrac{2}{3n} + \tfrac{1}{n^2} < \gamma_1,
\end{aligned}$$

where $\pi_i : \mathbb{R}^m \to \mathbb{R}$, $i = 1, ..., m$, are the natural projections on the ith-coordinate. The construction shows that these inequalities hold for any y_b replaced by $y_{b(j)}$. Note that $f_t = f_0 = f_1$ outside of U. Also, these estimates imply that

$$U_{\frac{1}{n^2}}(\widehat{f_b}(\widehat{R^*})) \subset int(\widehat{R^*}),$$

and

$$\widehat{g_b}(\widehat{R^*}) \subset int(\widehat{R^*}),$$

where $U_{\frac{1}{n^2}}(\widehat{f_b}(\widehat{R^*}))$ is a $\frac{1}{n^2}$-neighborhood of the set $\widehat{f_b}(\widehat{R^*})$. We conclude that

$$g_b(R^*) \subset R^*.$$

In particular, g_b is a contraction on R^* and it has a unique attracting fixed point $p(b) \in R^*$.

In the sequel, we say that a skew product is *North-South like skew product* if it possesses all of the properties mentioned above. Also, we require that

$$n \geq 100m^2. \tag{12}$$

3. Statistical Attractors of North-South Like Almost Step Skew Products

The following theorem is needed to prove the main result.

Theorem 3.1 *Let $\mathcal{G} : X \to X$ be a North-South like skew product over the solenoid. Then*

(a) *The statistical attractor of $\mathcal{G}$ lies inside $B \times R^*$, and is the graph of a continuous map $\Gamma_{\mathcal{G}} : \Lambda_{\mathcal{G}} \to R^*$, where $\Lambda_{\mathcal{G}}$ is an invariant set of $\mathcal{G}$ homeomorphic to the solenoid attractor Λ* of *$\mathcal{F}$. Under the projection*

$$p : \mathcal{A}_{stat} \to \Lambda_{\mathcal{G}}, \tag{13}$$

the restriction $\mathcal{G}|_{\mathcal{A}_{stat}}$ becomes conjugated to the solenoid map on $\Lambda_{\mathcal{G}}$:

$$\begin{array}{ccc}
 & \mathcal{G} & \\
\mathcal{A}_{stat} & \longrightarrow & \mathcal{A}_{stat}\\
p\downarrow & & p\downarrow\\
\Lambda_{\mathcal{G}} & \longrightarrow & \Lambda_{\mathcal{G}}\\
 & h &
\end{array}$$

(b) *There exists an SRB measure μ_∞ on $X = B \times M$. This measure is concentrated on $\mathcal{A}_{stat}$ and is precisely the pull back of the Bernoulli measure P on Σ_1^2 under the isomorphism $\Phi\ o\ p : \mathcal{A}_{stat} \to \Sigma_1^2$.*

This theorem is proved in the same way as Theorem 4 of (Ilyashenko & Negut, 2010), see also the proof of Theorem 2 of (Ghane & *et al.*, 2012). So we present only a sketch of proof.
For proof, we claim that

$$\mathcal{A}_{max} = \mathcal{A}_{stat}.$$

The claim and the following lemma imply statement (a) of the theorem.

Lemma 3.2 *The attractor $\mathcal{A}_{max}$ is the graph of a continuous function $\Gamma : B \to R^*$.*

Proof. See proposition 3 of (Ghane & *et al.*, 2012).

Let us now prove the claim. The proof relies on the following lemma.

Lemma 3.3 *For almost all $(b, x) \in B \times M$, there exist $k \in \mathbb{N}$ such that $\mathcal{G}^k(b, x) \in B \times R^*$.*

First, consider the following invariant sets:

$$\mathcal{S}_i = \bigcap_{k=-\infty}^{\infty} \mathcal{G}^k(\Lambda_{\mathcal{G}} \times V_i),\ i = 1, ..., l. \tag{14}$$

Recall that if S is an invariant set of $\mathcal{G}$ then we set

$$W^s S = \{q \in S^2 \times M;\ d(\mathcal{G}^k(q), S) \to 0\ as\ k \to \infty\}.$$

To prove the lemma, we need to show that $W^s(S_i)$, $i = 1, ..., l$, have Lebesgue measure zero. For, we apply the Bowen's theorem (Bowen, 1995).

Theorem 3.4 [**Bowen**] *Consider a C^2 diffeomorphism of a compact manifold M and a hyperbolic invariant set S of this diffeomorphism which is not a maximal attractor in its neighborhood. Then the attracting set $W^s S$ has Lebesgue measure zero.*

Now the hyperbolicity of the invariant sets

$$\mathcal{S}_i = \bigcap_{k=-\infty}^{\infty} \mathcal{G}^k(\Lambda_{\mathcal{G}} \times V_i),\ i = 1, ..., l,$$

follow as Proposition 4 of (Ghane & *et al.*, 2012), see also Lemma 2 and Proposition 6 of (Ilyashenko & Negut, 2010).

Let us show that how the proof of Lemma 3.3 follows from the hyperbolicity of the invariant sets

$$\mathcal{S}_i = \bigcap_{k=-\infty}^{\infty} \mathcal{G}^k(\Lambda_{\mathcal{G}} \times V_i),\ i = 1, ..., l,$$

and Bowen's theorem.

Proof. [Proof of Lemma 3.3] First, we mention that $\mathcal{S}_i$, $i = 1, ..., l$, are locally maximal hyperbolic set. Notice that all the fiber maps g_b push points away from W and V_i, $i = 1, ..., l$, and into R^* or V_j, for some $j \in \{1, ..., l\}$.
Therefore, the statement of lemma fails only for elements of the sets

$$\mathcal{R} = \bigcap_{k=0}^{\infty} \mathcal{G}^{-k}(\Lambda_{\mathcal{G}} \times V),\ W^s(\mathcal{S}_i),\ i = 1, ..., l,$$

where

$$\mathcal{S}_i = \bigcap_{k=-\infty}^{\infty} \mathcal{G}^{-k}(\Lambda_{\mathcal{G}} \times V_i),\ i = 1, ..., l.$$

In fact, since any point whose orbit stays forever in $\Lambda_{\mathcal{G}} \times V_i$, $i = 1, ..., l$, will be attracted to $\mathcal{S}_i$, $i = 1, ..., l$.
Now, we apply the Bowen' theorem to conclude that

$$mes(W^s(\mathcal{S}_i)) = 0,\ i = 1, ..., l.$$

Also, according to Lemma 1 of (Ilyashenko & Negut, 2010), $mes(\mathcal{R}) = 0$. This terminates the proof.

Let us to verify that $\mathcal{A}_{max} = \mathcal{A}_{stat}$. By Lemma 3.3, the ω-limit sets of almost all points in $\Lambda_{\mathcal{G}} \times M$ belong to $\mathcal{A}_{max}$. So $\mathcal{A}_{max}$ is the Milnor attractor of $\mathcal{G}$, and thus contains $\mathcal{A}_{stat}$. Now, we show that $\mathcal{A}_{max}$ is precisely equal to $\mathcal{A}_{stat}$. Suppose that μ_∞ is any good measure of $\mathcal{G}$. For each measurable set $A \subset \Lambda_{\mathcal{G}}$, we have

$$\mathcal{G}^{-1}(A \times M) = h^{-1}(A) \times M$$

and therefore
$$\mathcal{G}_*\mu(A \times M) = \mu(h^{-1}(A) \times M) = \mu(A \times M).$$
By iterating it, we obtain
$$\mathcal{G}_*^k\mu(A \times M) = \mu(A \times M) = \mu_{\Lambda_{\mathcal{G}}}(A) = \phi^* P(\Sigma_1^2),$$
for all k. By the definition of a good measure, we conclude that
$$\mu_\infty(A \times M) = \mu_{\Lambda_{\mathcal{G}}}(A). \tag{15}$$
But any good measure supported on $\mathcal{A}_{stat}$, and therefore on $\mathcal{A}_{max}$. These facts imply that μ_∞ must be push-forward of $\mu_{\Lambda_{\mathcal{G}}} = \Phi^* p$ under the isomorphism $(p|\mathcal{A}_{max})^{-1}$. In particular, the support of μ_∞ is the whole of $\mathcal{A}_{max}$.
So, the only good measure is μ_∞. Its support $\mathcal{A}_{max}$ coincide with the minimal attractor $\mathcal{A}_{min}$. Therefore, by the inclusions between attractors, we conclude that
$$\mathcal{A}_{min} = \mathcal{A}_{stat} = \mathcal{A}_{max}, \tag{16}$$
as desired. This proves statement (a) of Theorem 3.1. The proof of statement (b) is similar to Theorem 3 of (Ilyashenko & Negut, 2010).

4. Large ε-Invisible Parts of Attractors for Skew Products over the Solenoid

Suppose that $n \geq 100m^2$, $\mathcal{F}$ is the almost step skew product as introduced in section 2, and $\mathcal{G} \in D_n$. Hence,
$$d(\mathcal{F}, \mathcal{G}) = \sup_{b \in B} d_{C^1}(f_b^{\pm 1}, g_b^{\pm 1}) \leq \frac{1}{n^2}. \tag{17}$$
In particular, $\mathcal{G}$ is a North-South like skew product which satisfies the properties of Proposition 2.3.
By definition of $f_i,\ i = 0, 1,$
$$\widehat{f_0}(\widehat{R}) \cup \widehat{f_1}(\widehat{R}) \supset \widehat{R},\ \widehat{f_i}(\widehat{R^*}) \subset \widehat{R^*}, i = 0, 1,$$
where $\widehat{f_i} = \psi \circ f_i \circ \psi^{-1}$ (see (Ilyashenko & Negut, 2010)). So, it is easy to see that
$$R \subset \pi(\mathcal{F}^i(B \times R^*)) \subset R^*,$$
for each $i \in \mathbb{N}$. This implies that
$$R \subset \pi(\mathcal{A}_{max}) \subset R^*.$$
Since $\mathcal{A}_{max} = \mathcal{A}_{stat}$, then
$$R \subset \pi(\mathcal{A}_{stat}) \subset R^*.$$
Note that
$$\widehat{f_0}(\widehat{R}) \cup \widehat{f_1}(\widehat{R}) \supset U_{\frac{1}{n^2}}\widehat{R}.$$
Moreover, the above inclusions are robust, this means that they remain true for any maps $\widehat{g_0} = \psi \circ g_0 \circ \psi^{-1}$ and $\widehat{g_1} = \psi \circ g_1 \circ \psi^{-1}$, where (g_0, g_1) belongs to an open ball $\mathcal{W}$ of (f_0, f_1) with radius $\frac{1}{n^2}$ in $Diff^1(M) \times Diff^1(M)$. Therefore,
$$R \subset \pi(\mathcal{A}_{stat}(\mathcal{G})) \subset R^*.$$
Now, statement (1) of the main theorem follows by this fact and Theorem 3.1.
Let us prove statement (2).
Indeed we must show that the set $N = \pi^{-1}(\psi^{-1}(A))$ is $\varepsilon - invisible$ for $\varepsilon = 2^{-n}$, where
$$A = \{x \in \widehat{R}^* | \pi_i(x) < 1 - \frac{1}{m}, i = 1, \dots, m\},$$
$\pi_i : \mathbb{R}^m \to \mathbb{R},\ (x_1, ..., x_m) \mapsto x_i,\ i = 1, ..., m,$ are the natural projections, and $\pi : B \times M \to M$ is the projection on the fiber.

In other words, we must show that the orbits of almost all points $(b, x) \in B \times M$ visit N with frequency at most ε. By lemma 3.3, we restrict attention to $(b, x) \in B \times R^*$. Let $\mathcal{U}$ be the set of finite words of length $2n$ which do not contain the two-digit sequence 10. These words have the form $\underbrace{0\dots0}_{s}\underbrace{1\dots1}_{t}$, $(0 \leq s, t \leq 2n,\ s + t = 2n)$. The cardinality of $\mathcal{U}$ is $2n + 1$.

Proposition 4.1 *Let $k \geq 2n$ and $(b, x) \in B \times R^*$ such that $\mathcal{G}^k(b, x) \in N$. If $\omega = \Phi^+(b)$, then $(\omega_{k-2n} \dots \omega_{k-1}) \in \mathcal{U}$.*

Proof. To get a contradiction, suppose that $j \leq 2n$ is the minimum integer satisfies $\omega_{k-j}\omega_{k-j+1} = 10$. Then $g_{h^{k-j}b}$ is $\frac{1}{n^2}$ close to f_1.
Clearly, for each $(x_1, x_2, \dots, x_m) \in \widehat{R}^*$,

$$\pi_1(\widehat{f_1}(x_1, x_2, \dots, x_m)) > 1 - \frac{4}{3n}.$$

So

$$\pi_1(\widehat{g}_{h^{k-j}b}(x_1, x_2, \dots, x_m)) > 1 - \frac{4}{3n} - \frac{1}{n^2} > 1 - \frac{2}{n}.$$

Moreover, we have the following statements:
(i) For all $(x_1, x_2, \dots, x_m) \in R^*$ with $1 - \frac{1}{m} < x_i < 1 - \frac{2}{n},\ i = 1, \dots, m-1$,

$$\pi_{i+1}(\widehat{f_j}(x_1, x_2, \dots, x_m)) > x_i(1 - \frac{1}{m^2n}),\ j = 0, 1.$$

Note that the above inequality persists under linear homotopy. Hence, it holds for any fiber map f_b of almost step skew product $\mathcal{F}$. Hence for each $b \in B$

$$\pi_{i+1}(\widehat{g_b}(x_1, x_2, \dots, x_m)) > x_i(1 - \frac{1}{m^2n}) - \frac{1}{n^2}.$$

(ii) For all $(x_1, \dots, x_m) \in R^*$ with $1 - \frac{1}{m} < x_i < 1 - \frac{2}{n},\ i = 1, \dots, m-1$,

$$\pi_1(\widehat{f_0}(x_1, x_2, \dots, x_m)) > x_m(1 - \frac{1}{m^2n}) - \frac{1}{mn} - \frac{2m}{n^2}.$$

So for each $b \in B$

$$\pi_1(\widehat{g_b}(x_1, x_2, \dots, x_m)) > x_m(1 - \frac{1}{m^2n}) - \frac{1}{mn} - \frac{2m+1}{n^2}.$$

By induction, it is easy to see that

$$\pi_j(\widehat{f_0^i}(\widehat{f_1}(x))) < 1 - \frac{2}{n},\ \ for\ i = j(mod\ m),\ i = 0, 1, 2, \dots, 2n,\ x \in \widehat{R}^*.$$

So by applying statements (i) and (ii), we conclude that the following holds:

$$\pi_s(\psi \circ \pi(\mathcal{G}^k(b, x)) = \pi_s(\widehat{g}_{h^{k-1}b} \circ \dots \widehat{g}_{h^{k-j+1}b} \circ \psi \circ \pi \circ \mathcal{G}^{k-j+1}(b, x)) >$$
$$(1 - \tfrac{1}{m^2n})^j(1 - \tfrac{2}{n}) - \tfrac{1}{n^2}\textstyle\sum_{t=0}^{j-1}(1 - \tfrac{1}{m^2n})^t - (\tfrac{1}{mn} + \tfrac{2m}{n^2})(\sum_{t=0}^{[\frac{j}{m}]-1}(1 - \tfrac{1}{m^2n})^{mt}) >$$
$$(1 - \tfrac{1}{m^2n})^{2n}(1 - \tfrac{2}{n}) - \tfrac{1}{n^2}\textstyle\sum_{t=0}^{2n-1}(1 - \tfrac{1}{m^2n})^t - (\tfrac{1}{mn} + \tfrac{2m}{n^2})(\sum_{t=0}^{[\frac{2n}{m}]-1}(1 - \tfrac{1}{m^2n})^{mt}),$$

for $s = j\ (mod\ m)$, $0 \leq j \leq 2n$, see figure 2. The limit of the right side of the last inequality is $\frac{2}{\sqrt[m^2]{e^2}} - 1$ which is greater than $1 - \frac{1}{m}$ for $n \geq 100m^2$. This contradicts the assumptions of the proposition.

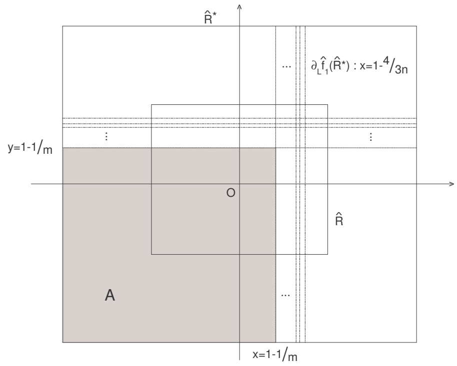

Figure 2. Invisible part of the attractor.

The ergodicity of Bernoulli shift implies that the subwords in $\mathcal{U}$ are met in almost all forward sequences $\omega = (\omega_0\omega_1\omega_2\ldots)$ with frequency 2^{-2n}. But almost all sequences ω, correspond under Φ^+ to almost all $b \in B$. Thus we conclude that, for almost all $b \in B$, subwords in $\mathcal{U}$ are met in $\Phi^+(b)$ with frequency at most $(2n+1).2^{-2n} < 2^{-n} = \varepsilon$. This fact and the previous proposition imply that almost all orbits visit N with frequency at most $\varepsilon = 2^{-n}$. Hence N is ε- invisible, as desired.

Now we are going to prove the last statement of the main theorem. Let us, consider the North-South like skew product $\mathcal{G} \in D_n(X) \cap C^2(X)$, $X = B \times M$, which is $\frac{1}{n^2}$-close to $\mathcal{F}$. We show that $\mathcal{G}$ is structurally stable in $D^1(X)$.

According to the criterion of structural stability, we need to check two things:

(1) The non-wandering set of $\mathcal{G}$ is hyperbolic and periodic points are dense in it (Axiom A).

(2) The stable and unstable manifolds of the non-wandering points are transversal.
The choices of $\mathcal{F}$ and $\mathcal{G}$ cause that the non-wandering set of $\mathcal{G}$ is the union of the invariant sets

$$\mathcal{A} = \bigcap_{k=0}^{\infty} \mathcal{G}^k(B \times R^*),\ \mathcal{R} = \bigcap_{k=0}^{\infty} \mathcal{G}^{-k}(\Lambda_{\mathcal{G}} \times W),\ \ \mathcal{S}_i = \bigcap_{k=-\infty}^{\infty} \mathcal{G}^k(\Lambda_{\mathcal{G}} \times V_i),\ i = 1, ..., l.$$

Now, we can apply an argument similar to Proposition 4.6 of (Ghane & *et al.*, 2012) to conclude that $\mathcal{A}, \mathcal{R}$ and $\mathcal{S}_i, i = 1, \ldots, l$ are hyperbolic (see also Lemma 2 of (Ilyashenko & Negut, 2010)).
Also, the dynamics on $\mathcal{A}, \mathcal{R}$ and $\mathcal{S}_i, i = 1, \ldots, l$ are conjugate to the Bernoulli shift and it is known that it has a dense set of periodic points. These facts imply that statement (1) is justified.

The proof of statement (2) is similar to proof of statement (c) of Theorem 4 of (Ilyashenko & Negut, 2010).

5. Perturbations

Here, to complete the proof of the main theorem, we will show that the assertions of Theorem A hold for all nearby diffeomorphisms $\mathcal{H} \in D_L(X)$, $X = B \times M$. We will use the approach suggested in (Ilyashenko & Negut, 2010) with any modification.

Consider the solenoid map h which has the maximal attractor Λ (the solenoid attractor). It is a hyperbolic invariant set with contraction coefficient $\lambda < 0.1$ and expansion coefficient $\mu^{-1} = 2$.
Now we recall the concept of modified dominated splitting condition. We say that the skew product $\mathcal{G}$ over the solenoid of the form (4) with contraction coefficient λ and expansion coefficient μ^{-1} satisfies the *modified dominated splitting condition* if

$$\max(\max(\lambda, \mu) + \|\frac{\partial g_b^{\mp 1}}{\partial b}\|_{C^0(X)}, \|\frac{\partial g_b^{\mp 1}}{\partial x}\|_{C^0(X)}) =: L < \min(\lambda^{-1}, \mu^{-1}),$$

see (Ilyashenko & Negut, 2010). Now consider the almost step skew product $\mathcal{F}$ over the solenoid with the solenoid map $h : \Lambda \to \Lambda$ and the fiber M, as introduced in section 2. It is a North-South like skew product such that the bundle maps $f_b, b \in B$, possess all properties mentioned in Proposition 2.3. By construction, $\mathcal{F}$ satisfies the modified dominated splitting condition.

Now, we fix $\mathcal{G} \in D_n$. We recall that $\mathcal{G}$ is a North-South like skew product over the solenoid. Moreover, $\mathcal{G}$ satisfying the modified dominated splitting condition.

Suppose that $\mathcal{H}$ is any C^2-diffeomorphism which is C^1-close to $\mathcal{G}$. Let us note that small perturbations of skew products are not necessarily skew product anymore. However, one can show that they are conjugate to skew products, and moreover the conjugation map satisfies a Hölder continuity property. The following theorem is cited from (Ilyashenko & Negut, 2010).

Theorem 5.1 *Consider a skew product $\mathcal{G}$* on $X := B \times M$ as in (4) *over the solenoid map h in the base, satisfying the modified dominated splitting condition, where B is the solid torus and M is a closed m-dimensional manifold. Then for small enough* $\rho > 0$, *any ρ-perturbation $\mathcal{H}$ of $\mathcal{G}$ has the following properties:*

a) *There exists a $\mathcal{H}$-invariant set $\mathcal{Y} \subset X$ and a continuous map $p : \mathcal{Y} \to B$ such that the diagram*

$$\begin{array}{ccc} & \mathcal{H} & \\ \mathcal{Y} & \longrightarrow & \mathcal{Y} \\ p\downarrow & & p\downarrow \\ \Lambda_{\mathcal{G}} & \longrightarrow & \Lambda_{\mathcal{G}} \\ & h & \end{array}$$

commutes, where $\Lambda_{\mathcal{G}}$ is the solenoid attractor corresponding to $\mathcal{G}$. Moreover, the map

$$K : \mathcal{Y} \to \Lambda_{\mathcal{G}} \times M,\ K(b, x) = (p(b, x), x)$$

is a homeomorphism.

b) The fibers $p^{-1}(b)$ are Lipschitz close to vertical fibers and Hölder continuous in b. Moreover, the map K^{-1} is also Hölder continuous.

As we have mentioned before, $\mathcal{H}$ is a priori not a skew product anymore. However, letting $H = K \circ \mathcal{H}|_{\mathcal{Y}} \circ K^{-1}$, statement ($a$) of the above theorem implies that

$$H : \Lambda_{\mathcal{G}} \times M \to \Lambda_{\mathcal{G}} \times M$$

is indeed a skew product:

$$H(b, x) = (h(b), h_b(x)).$$

One can then study the dynamical properties of $\mathcal{H}|_{\mathcal{Y}}$ by studying the dynamical properties of its conjugate skew product H.
The fiber maps h_b of the skew product H are C^1-close to those of the skew product $\mathcal{G}$, in the following sense:

$$d_{C^1}(h_b^{\mp 1}, g_b^{\mp 1}) \leq O(\rho).$$

These facts and the inclusion

$$\mathcal{G}(B \times R^*) \subset B \times R^*$$

imply that

$$\mathcal{H}(B \times R^*) \subset B \times R^*,\ H(B \times R^*) \subset B \times R^*,$$

where the cubes R and R^* are introduced in section 2.

Consider the maximal attractors of $\mathcal{H}|B \times R^*$ and $H|B \times R^*$:

$$\mathcal{A}_{max}(\mathcal{H}) = \bigcap_{k=0}^{\infty} \mathcal{H}^k(B \times R^*),\ \mathcal{A}_{max}(H) = \bigcap_{k=0}^{\infty} H^k(B \times R^*),$$

respectively. These attractors are connected, since $B \times R^*$ is connected. Let us mention that if the fiber M is 1-dimensional, $M := S^1$, then the connectivity of $\pi(\mathcal{A}_{max}(\mathcal{H}))$ implies that it must be an arc without any holes. However, in general case, $\pi(\mathcal{A}_{max}(\mathcal{H}))$ may have some holes. In this setting, we need to apply Theorem 5.1 and statement (1) of Theorem A to conclude that

$$R \subset \pi(\mathcal{A}_{max}(H)) \subset R^*,$$

and therefore

$$R \subset \pi(\mathcal{A}_{max}(\mathcal{H})) \subset R^*,$$

provided that $\rho > 0$ is small enough.
The rest arguments goes roughly as follows. The hyperbolicity of $\mathcal{A}_{max}(\mathcal{H})$ will be provided by the structural stability of the hyperbolic attractors. Now, since $\mathcal{H}$ is a C^2-diffeomorphism, a theorem in (Gorodetsky, 1999) yields the following equation

$$\mathcal{A}_{max}(\mathcal{H}) = \mathcal{A}_{stat}(\mathcal{H}).$$

Hence, statement (1) of Theorem A is proved for all nearby diffeomorphisms.

To prove statement (2), let $\mu_{\infty}^{\mathcal{G}}$ denote the SRB measure for $\mathcal{G}$, which is described in Theorem 3.1. By statement (b) of Theorem 3.1 and Proposition 1 of (Gorodetsky, 1999), we conclude that

$$\mu_{\infty}^{\mathcal{G}}(N) \leq \varepsilon.$$

The Ruelle theorem on the differentiability of the SRB measures (Ruelle, 1997) implies that any small perturbation $\mathcal{H}$ of $\mathcal{G}$ has an SRB measure $\mu_{\infty}^{\mathcal{H}}$, and that this measure depends differentiably on $\mathcal{H}$. In particular, it follows that for $\mathcal{H}$ close enough to $\mathcal{G}$ we will still have

$$\mu_{\infty}^{\mathcal{H}}(N) \leq \varepsilon.$$

By applying Proposition 1 of (Gorodetsky, 1999) again, it follows that N is ε-invisible for $\mathcal{H}$.

6. Natural Extensions

This section is devoted to prove Corollary B. We recall that $\mathcal{E}(M)$ is the space of all skew product maps acting on $S^1 \times M$ of the form

$$F : S^1 \times M \to S^1 \times M, \quad F(y, x) = (g(y), f_y(x)),$$

where $g(y) = ky$, $k \geq 2$, is an expanding circle map and fiber maps $x \mapsto f_y(x)$ are C^1- diffeomorphism defined on a closed manifold M. We equip $\mathcal{E}(M)$ by the metric

$$d(F, G) = \sup_{y \in S^1} d_{C^1}(f_y^{\pm 1}, g_y^{\pm 1}).$$

For each $F \in \mathcal{E}(M)$, we consider its natural extension as the following form

$$\mathcal{F} : B \times M \to B \times M, \quad (b, x) \to (h(b), f_y(x)),$$

where $b = (y, z)$ belongs to the solid torus $B := S^1 \times D$ and $x \in M$. Then $\mathcal{F}$ is a skew product map over the solenoid. Also we observe that the fiber maps f_y do not depend on z. This permits us to consider $\mathcal{F}$ as a skew product map over g with fiber $D \times M$. Let q be the projection map along D,

$$q : B \times M \to S^1 \times M, \quad q(y, z, x) = (y, x).$$

Then

$$q \circ \mathcal{F} = F \circ q \tag{18}$$

Note that in (Homburg, 2012), the author establishes some facts on extension of skew product endomorphisms to skew product maps over the solenoid.

Let us take the skew product map $F \in \mathcal{E}(M)$ acting on $S^1 \times M$ and its natural extension $\mathcal{F}$ such that the fiber maps f_y defined by (11). Then $\mathcal{F}$ is an almost step North-South like skew product map which is introduced in section 2. Therefore we can apply Theorem 3.1 for $\mathcal{F}$ to conclude that it has a statistical attractor $\mathcal{A}_{stat}(\mathcal{F})$ which is equal to $\mathcal{A}_{max}(\mathcal{F}) = \bigcap_{k=0}^{\infty} \mathcal{F}^k(B \times R^*)$. Also it is the graph of a continuous function $\Gamma_{\mathcal{F}}$. Moreover, there is an SRB measure μ_∞ on $B \times M$ which is concentrated on $\mathcal{A}_{max}(\mathcal{F})$.
The equation (16) and these facts imply that $F(S^1 \times R^*) \subset (S^1 \times R^*)$, hence we can consider

$$A_{max}(F) = \bigcap_{k=0}^{\infty} F^k(S^1 \times R^*).$$

Now, we show that the maximal attractor $A_{max}(F)$ is robustly topologically mixing. Consider $\sum_2^+ = \{0, 1\}^{\mathbb{N}}$ endowed with the product topology and let $\sigma : \sum_2^+ \to \sum_2^+$ be the left shift.
The base map g (or some iterate of it) admits an invariant Cantor set on which the dynamics is topologically conjugate to $\sigma : \sum_2^+ \to \sum_2^+$. Therefore the skew product F is also conjugate to a step skew product over $\sum_2^+$ with fiber maps f_i, $i = 0, 1$. By Proposition 2.2, the maximal attractor $\mathcal{A}_{max}(\mathcal{F}) = \Lambda \times \Delta$. Therefore,

$$A_{max}(F) = q(\mathcal{A}_{max}(\mathcal{F})) = q(\Lambda \times \Delta) = S^1 \times \Delta.$$

This observations and Lemma 2.1 imply that $A_{max}(F)$ is robustly topologically mixing.
Indeed, take an open set U in $\sum_2^+ \times \Delta$. The construction in Lemma 2.1 gives that $\bigcup_{n \in \mathbb{N}} F^n(U)$ is open and dense in $\sum_2^+ \times \Delta$. Now take open sets $U, V \subset S^1 \times \Delta$. As g is expanding, some iterate of U under F intersects $\sum_2^+ \times \Delta$. Again as g is expanding, a higher iterate will intersect V, establishing topological mixing of $F : S^1 \times \Delta \to S^1 \times \Delta$.

Also statement (1) of Theorem A and equation (18) imply that

$$S^1 \times R \subset A_{max}(F).$$

In particular, if we set $\mu := q_*(\mu_\infty)$ then μ is an SRB measure for F with $Supp(\mu) = A_{max}(F)$.
Let us consider an open ball $\mathcal{D}_n \subset \mathcal{E}(M)$ with the center F and radius $\frac{1}{n^2}$. Since every $G \in \mathcal{D}_n \cap C^2(S^1 \times M)$ has a natural extension $\mathcal{G}$ close to $\mathcal{F}$, then G satisfies all properties mentioned above. This terminates the proof of Corollary B.

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