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$L^{\Phi} - L^{\infty}$ Inequalities and Applications

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Abstract

In this paper we prove some $L^{\Phi} - L^{\Phi}$ and $L^{\Phi} - L^{\infty}$ inequalities for quasi-minima of scalar integral functionals defined in Orlicz-Sobolev space $W^1 L^{\Phi}(\Omega)$, where Φ is a N-function and $\Phi \in \triangle_2$. Moreover, if $\Phi \in \triangle'$ or if $\Phi \in \triangle_2 \cap \nabla_2$, we prove that quasi-minima are Hölder continuous functions.

Keywords: variational inequalities, regularity, Quasi Minima, Hölder Continuityd

1. Introduction

In this paper we show a regularity theorem for quasi-minima of scalar integral functionals of the Calculus of Variations with general growth conditions.

Let us consider functionals as the following form

$$\mathcal{F}[u,\Omega] = \int_{\Omega} f(x, u(x), \nabla u(x))\, dx \tag{1.1}$$

where $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function satisfying the inequalities

$$c_1 \Phi(|z|) - b(x)\Phi(|s|) - a(x) \le f(x, s, z) \le c_2 \Phi(|z|) + b(x)\Phi(|s|) + a(x) \tag{1.2}$$

for each $z \in \mathbb{R}^N$, $s \in \mathbb{R}$ and for $\mathcal{L}^N$-a. e. $x \in \Omega$, where c_1 and c_2 are two positive real constants, with $c_1 < c_2$, Ω is an open subset of $\mathbb{R}^N$, $N \ge 2$, $b(x)$, $a(x) \in L^{\beta}(\Omega)$ with $\beta = \frac{N}{1-N\epsilon}$ and $0 < \epsilon < \frac{1}{N}$. The functional (1.1) is defined on the Orlicz-Sobolev space $W_0^1 L^{\Phi}(\Omega) + g$ where $g \in W^1 L^{\Phi}(\Omega)$, Φ is a N-function which satisfies some additional hypotheses that we well show later.

The first result of this paper is the following maximal $L^{\Phi} - L^{\Phi}$ inequality.

Theorem 1. *Let Φ be a N-function and $\Phi \in \triangle_2$. If $u \in W^1 L^{\Phi}(\Omega)$ is a quasi-minimum of the functional (1.1) with the growths (1.2) then u is locally bounded on Ω. Furthermore, for each $x_0 \in \Omega$ and $0 < R \le \min(R_0, d(x_0, \partial\Omega), 1)$ there exists an universal constant $c_M = c_M(\alpha, N, m, H, \chi)$ such that for any $h_0 \in \mathbb{R}$*

$$ess - \sup_{Q_{\frac{R}{2}}(x)} \left(\frac{v - h_0}{R}\right) \le 2\Phi^{-1}\left(c_M \left(\frac{V(h_0, R)}{R^N}\right)^{\alpha} \left(\frac{1}{R^N} \int_{A(h_0,R)} \Phi\left(\frac{v - h_0}{R}\right) dy\right)\right) \tag{1.3}$$

where R_0, H, χ are positive real constants introduced in Theorem 4 [Caccioppoli's Inequality] , $v = u - \chi R^{\frac{N\epsilon}{m}}$, $h_0 = k_0 - \chi R^{\frac{N\epsilon}{m}}$, $V(h_0, R) = \mathcal{L}^N(A(h_0, R))$ and $\alpha = \frac{-1+\sqrt{5}}{2}$.

Since Φ is not a homogeneity function, the Inequality (1.3) resolves so many tipical homogeneity problems of the general growth conditions and it is the first necessary step to extend the results introduced in (Giaquinta et al., 1982; Granucci, 2014; Lieberman, 1991; Mascolo et al., 1996; Moscariello et al., 1991).

As first consequence of the Theorem 1 we get the following regularity theorem.

Theorem 2. *Let $\Phi \in \triangle_2 \cap \nabla_2$ if $u \in W^1L^{\Phi}(\Omega)$ is a quasi-minimum of the functional (1.1) then u is locally hölder continuous.*

In (1996) E. Mascolo and G. Papi have determined an Harnack inequality for the minimizer of the functional (1.1) under the following conditions: $f(z) = \Phi(|z|)$ where Φ is a N-function and $\Phi \in \triangle_2 \cap \nabla_2$. We observe that $\Phi \in \triangle_2 \cap \nabla_2$ implies

$$c_3 t^p - c_4 < \Phi(t) < c_5 t^m + c_6 \qquad \text{for } t > 0 \tag{1.4}$$

with real positive constants c_3, c_4, c_5,c_6 and $1 < p \leq m$. Classical regularity theorem for functionals with standard growth conditions ($p = m$) has been proved in (Giaquinta, et al., 1982) (we refer also to (Ambrosio, Lecture Notes on Partial Differential Equations) and (Giusti, 1994)). In (Moscariello, et al., 1991), G. Moscariello and L. Nania proved the local boundedness of the minimizer of functional (1.1) with $f(z) = \Phi(|z|)$, $\Phi \in \triangle_2$ and the growth conditions (1.4) with $1 < p \leq m < \frac{Np}{N-p}$, moreover in (1991) G. Moscariello and L. Nania proved the hölder continuity of the minimizer of functional (1.1) with $f(z) = \Phi(|z|)$, $\Phi \in \triangle_2 \cap \nabla_2$. In 1991, G. M. Lieberman proved an Harnack inequality for the minimizer of the functional (1.1) with $\Phi \in C^2$ suth that

$$c_7 \leq \frac{t\ddot{\Phi}(t)}{\dot{\Phi}(t)} \leq c_8 \qquad \text{for } t > 0$$

with $0 < c_7 < c_8$. Moreover in (2000) V. S. Klimov studies this problem when Φ satisfies ∇_2 but not a $\triangle_2$ condition.

Therefore our technique allows to unify the approaches to the regularity of quasi-minima with general growth, introduced in (Lieberman, 1991; Mascolo et al., 1996; Moscariello et al., 1991), with those introduced in (Giaquinta et al., 1982). Moreover, if we assume that the following hypotheses are given:

H 1) Φ globally satisfies the $\triangle'$- condition in $[0, +\infty)$,

H 2) there exists a constant $c_{H_2} > 0$

$$\dot{\Phi}(t)\,\dot{\Phi}\left(\frac{1}{t}\right) \leq c_{H_2} \tag{1.5}$$

for every $t \in (0, 1)$,

H 3) there exists a constant $c_{H_3} > 0$

$$\Phi^{-1}(t) \leq c_{H_3} t^{\frac{1}{m}} \tag{1.6}$$

for every $t \in (0, 1)$,

then we get the following regularity theorem.

Theorem 3. *If $u \in W^1L^{\Phi}(\Omega)$ is a quasi-minimum of the functional (1.1) and Φ fulfils H 1, H 2 and H 3 then u is locally hölder continuous.*

The class of functions to which we can apply Theorem 2 and Theorem 3 is slightly wider that the one discussed in (Granucci, 2014; Lieberman, 1991; Mascolo et al., 1996; Moscariello et al., 1991). This makes us think that the introduced techniques are the first step to prove the regularity in the general case $\Phi \in \triangle_2$. Moreover, the author thinks that hypothesis H2 and H3 are removable. The author, using Theorem 1 and some new tricks, hopes to exstend Theorem 2 and Theorem 3 under the general growth condition $\Phi \in \triangle'$ and, probably, also in the case $\Phi \in \triangle_2$.

2. Definitions

We now introduce some definitions.

Definition 1. *A continuous and convex function $\Phi : [0, +\infty) \to [0, +\infty)$ is called N-function if it satisfies*

$$\begin{aligned} &\Phi(0) = 0 \text{ and } \Phi(t) > 0 \text{ if } t > 0; \\ &\lim_{t \to o^+} \tfrac{\Phi(t)}{t} = 0; \\ &\lim_{t \to +\infty} \tfrac{\Phi(t)}{t} = +\infty. \end{aligned} \tag{2.1}$$

Let Φ be a N-function then there exists a real valued function p defined on $[0, +\infty)$ and having the following properties: $p(0) = 0$, $p(t) > 0$ if $t > 0$, p is increasing and right continuous on $(0, +\infty)$ such that

$$\Phi(t) = \int_0^t p(s)\, ds \qquad \text{for every } t \in (0, +\infty)$$

and

$$\dot{\Phi}_+(t) = p(t) \qquad \text{a.e. on } (0, +\infty).$$

Definition 2. *Let p be a real valued function defined on $[0, +\infty)$ and having the following properties: $p(0) = 0$, $p(t) > 0$ if $t > 0$, p is increasing and right continuous on $(0, +\infty)$. We define*

$$\widetilde{p}(s) = \sup_{p(t) \le s} (t)$$

and

$$\widetilde{\Phi}(t) = \int_0^t \widetilde{p}(s)\, ds.$$

The N-functions Φ and $\widetilde{\Phi}$ are complementary N-functoins.

Particularly from Definition 2 we get the following Young inequality

$$ab \le \widetilde{\Phi}(a) + \Phi(b). \tag{2.2}$$

Let us introduce an important class of N-functions.

Definition 3. *A N-function Φ is of class $\triangle_2$ globally in $(0, +\infty)$ if exists $k > 1$ such that*

$$\Phi(2t) \le k\Phi(t) \qquad \forall t \in (0, +\infty). \tag{2.3}$$

Definition 4. *A N-function Φ is of class $\triangle_2^m$ globally in $(0, +\infty)$, with $m > 1$, if for every $\lambda > 1$*

$$\Phi(\lambda t) \le \lambda^m \Phi(t) \qquad \forall t \in (0, +\infty). \tag{2.4}$$

Definition 5. *A N-function Φ is of class ∇_2 globally in $(0, +\infty)$ if exists $l > 1$ such that*

$$\Phi(t) \le \frac{\Phi(lt)}{2l} \qquad \forall t \in (0, +\infty). \tag{2.5}$$

Definition 6. *A N-function Φ is of class ∇_2^r globally in $(0, +\infty)$, with $r > 1$, if for every $\lambda > 1$*

$$\lambda^r \Phi(t) \le \Phi(\lambda t) \qquad \forall t \in (0, +\infty). \tag{2.6}$$

The N-functions $\Phi \in \triangle_2^m$ are characterized by the following result.

Lemma 1. *Let Φ be a N-function and let $\dot{\Phi}_+$ be its right derivative. For $m > 1$ the following properties are equivalent:*

(i) *$\Phi(\lambda t) \le \lambda^m \Phi(t)$, for every $t \ge 0$, for every $\lambda > 1$;*

(ii) *$t\dot{\Phi}_+(t) \le m\Phi(t)$, for every $t \ge 0$;*

(iii) *the function $\frac{\Phi(t)}{t^m}$ is nonincreasing on $(0, +\infty)$.*

The N-functions $\Phi \in \nabla_2^r$ are characterized by the following result.

Lemma 2. *Let Φ be a N-function and let $\dot{\Phi}_-$ be its left derivative. For $r > 1$ the following properties are equivalent:*

(i)' $\Phi(\lambda t) \geq \lambda^r \Phi(t)$, *for every* $t \geq 0$, *for every* $\lambda > 1$;

(ii)' $t\dot{\Phi}_-(t) \geq r\Phi(t)$, *for every* $t \geq 0$;

(iii)' *the function* $\frac{\Phi(t)}{t^m}$ *is nondecreasing on* $(0, +\infty)$.

We observe that

$$\Delta_2 = \bigcup_{m>1} \triangle_2^m$$

and

$$\nabla_2 = \bigcup_{r>1} \nabla_2^r.$$

Moreover we get

$$t\dot{\Phi}(t) \leq t\dot{\Phi}_+(t) \leq m\Phi(t), \text{ for every } t \geq 0;$$

where $\dot{\Phi}$ is the weak derivative of Φ.

The following condition is very important to us.

Definition 7. *We say that the N-function* Φ *satisfies the* $\triangle'$*- condition if there exist positive constants - c and* t_0 *- such that*

$$\Phi(ts) \leq c_4 \Phi(t) \Phi(s) \tag{2.7}$$

for every $t, s \geq t_0$.

Particularly, the regularity Theorem 2 is based on the following class of N-functions:

Definition 8. *We say that the N-function* Φ *globally satisfies the* $\triangle'$*- condition in* $[0, +\infty)$ *if (2.7) holds for every* $t, s \geq 0$.

Lemma 3. *If the N-function* Φ *satisfies the* $\triangle'$*- condition then it also satisfies the* $\triangle_2$*- condition.*

Example 1. *The N-functions*

$$\begin{array}{ll} \Phi_1(t) = t^p & \text{with } p > 1 \\ \Phi_2(t) = t^p(|\ln(t)| + 1) & \text{with } p > 1 \\ \Phi_3(t) = (1+t)\ln(1+t) - t & \end{array}$$

satisfy the $\triangle'$*-condition. Moreover* Φ_1 *and* Φ_2 *satisfy the* $\triangle'$*-condition globally in* $[0, +\infty)$ *and belong to the class* ∇_2 *globally in* $[0, +\infty)$. *The function* Φ_3 *does not satisfy* $\triangle'$*-condition for all* $t, s \geq 0$ *and* $\Phi_3 \notin \nabla_2$. *Assuming* Φ *equivalent to* Φ_3, *we show that a regularity theorem is valid.*

For details see (Adams, 1975), (Krasnoswl'kiĭ et al., 1961) and (Rao et al., 1991).

Now we can introduce Orlicz spaces and Orlicz Sobolev Spaces, L^Φ and W^1L^Φ. Let $\Omega \subseteq \mathbb{R}^N$ be a bounded and open set, the Orlicz class $K^\Phi(\Omega)$ is the set of all measurable functions $u : \Omega \to \mathbb{R}$ satisfying $\int_\Omega \Phi(|u|)\, d\mathcal{L}^N < +\infty$.

The Orlicz space $L^\Phi(\Omega)$ is defined to be the linear hull of $K^\Phi(\Omega)$, thus it consists of all measurable functions u such that $\lambda u \in K^\Phi(\Omega)$ for some $\lambda > 0$. Moreover, the equality $K^\Phi(\Omega) \equiv L^\Phi(\Omega)$ holds if and only if $\Phi \in \triangle_2$.

Definition 9. *If* $\Omega \subset \mathbb{R}^N$ *is a bounded open set and* $\Phi \in \triangle_2$ *then*

$$W^1L^\Phi(\Omega) = \left\{ u \in L^\Phi(\Omega) : \partial_i u \in L^\Phi(\Omega) \text{ for } i = 1, ..., N \right\}$$

where $\partial_i u$ *are the weak derivatives of u for* $i = 1, ..., N$.

Lemma 4. *Let* $\Phi \in \triangle_2$, *then* $L^\Phi(\Omega)$ *and* $W^1L^\Phi(\Omega)$ *are Banach spaces with the following norms*

$$\|u\|_{\Phi,\Omega} = \inf\left(k > 0 : \int_\Omega \Phi\left(\frac{|u|}{k}\right) d\mathcal{L}^N \leq 1\right)$$

and

$$\|u\|_{1,\Phi,\Omega} = \|u\|_{\Phi,\Omega} + \sum_{i=1}^N \|\partial_i u\|_{\Phi,\Omega}.$$

For details see (Adams, 1975), (Krasnoswl'kiĭ et al., 1961) and (Rao et al., 1991).

3. Caccioppoli Inequalities

3.1 Lemmas

In order to proof the Caccioppoli's inquality (3.18) we need the following Lemmas.

Lemma 5. *Let $g(t)$, $h(t)$ be a non-negative and increasing functions on $[0,+\infty)$ then*

$$g(t)\,h(s) \le g(t)\,h(t) + g(s)\,h(s)$$

for every s, $t \in [0,+\infty)$.

Lemma 6. *Let f be a nonnegative bounded function defined in $[\tau_0,\tau_1]$, $\tau_0 \ge 0$. Suppose that for all t, s with $\tau_0 \le t < s \le \tau_1$ we have*

$$f(t) \le \theta f(s) + \Phi\left(\frac{A}{s-t}\right) + B \tag{3.1}$$

where A, B, θ are nonnegative constants, $0 \le \theta < 1$, Φ is a N-function and $\Phi \in \triangle_2^m$ with $m > 1$. Then for all ϱ, R, $\tau_0 \le \varrho < R \le \tau_1$ we have

$$f(\varrho) \le c\left[\Phi\left(\frac{A}{R-\varrho}\right) + B\right] \tag{3.2}$$

where c is a constant depending only on θ and m.

Proof. Consider the sequence $\{t_i\}_{i\in\mathbb{N}}$ defined by $t_0 = \varrho$ and $t_{i+1} = t_i + (1-\lambda)\,\lambda^i\,(R-\varrho)$ where $0 < \lambda < 1$. By (3.1) we get

$$f(t_0) \le \theta f(t_1) + \Phi\left(\frac{A}{(1-\lambda)(R-\varrho)}\right) + B, \tag{3.3}$$

since $\Phi \in \triangle_2^m$ with $m > 1$ it follows

$$f(t_0) \le \theta f(t_1) + \frac{1}{(1-\lambda)^m}\Phi\left(\frac{A}{R-\varrho}\right) + B \tag{3.4}$$

and

$$f(t_i) \le \theta f(t_{i+1}) + \frac{1}{(1-\lambda)^m\,\lambda^{mi}}\Phi\left(\frac{A}{R-\varrho}\right) + B. \tag{3.5}$$

By (3.5) we have

$$f(t_0) \le \theta^k f(t_k) + \left[\frac{1}{(1-\lambda)^m}\Phi\left(\frac{A}{R-\varrho}\right) + B\right]\sum_{i=0}^{k-1}\left(\theta\lambda^{-m}\right)^i. \tag{3.6}$$

If we now choose λ in such a way that $\theta\lambda^{-m} < 1$ and go to the limit for $k \to +\infty$ we get (3.2) with $c = c_{\theta,m} = (1-\lambda)^{-m}\left(1-\theta\lambda^{-m}\right)^{-1}$. □

Lemma generalizes the Lemma 6.1 of (Giusti, 1994).

Lemma 7. *Let $F(k,\cdot)$ be a nonnegative bounded function defined in $[\tau_0,\tau_1]$, $\tau_0 \ge 0$. Suppose that for all t, s with $\tau_0 \le t < s \le \tau_1$ we have*

$$F(k,t) \le \theta F(k,s) + A\int_{A(k,s)}\Phi\left(\frac{u-k}{s-t}\right)dx + B \tag{3.7}$$

where A, B, θ are nonnegative constants, $0 \le \theta < 1$, Φ is a N-function and $\Phi \in \triangle_2^m$ with $m > 1$. Then for all ϱ, R, $\tau_0 \le \varrho < R \le \tau_1$ we have

$$F(k,\varrho) \le c\left[\int_{A(k,R)}\Phi\left(\frac{u-k}{R-\varrho}\right)dx + B\right] \tag{3.8}$$

where c is a constant depending only on θ and m.

Proof. Consider the sequence $\{t_i\}_{i\in\mathbb{N}}$ defined by $t_0 = \varrho$ and $t_{i+1} = t_i + (1-\lambda)\lambda^i (R-\varrho)$ where $0 < \lambda < 1$. By (3.7) we get

$$\mathcal{F}(k, t_0) \leq \theta \mathcal{F}(k, t_1) + A \int_{A(k,t_1)} \Phi\left(\frac{u-k}{t_1 - t_0}\right) dx + B, \tag{3.9}$$

since $\Phi \in \triangle_2^m$ with $m > 1$ it follows

$$\mathcal{F}(k, t_0) \leq \theta \mathcal{F}(k, t_1) + \frac{A}{(1-\lambda)^m} \int_{A(k,t_1)} \Phi\left(\frac{u-k}{R-\varrho}\right) dx + B \tag{3.10}$$

and by iteration we obtain

$$\mathcal{F}(k, t_0) \leq \theta^j \mathcal{F}\left(k, t_j\right) + \frac{A}{(1-\lambda)^m} I_j + \frac{B}{(1-\lambda)^m} II_j \tag{3.11}$$

where

$$I_j = \sum_{i=1}^{j} \left[\left(\frac{\theta}{\lambda^m}\right)^{i-1} \int_{A(k,t_i)} \Phi\left(\frac{u-k}{R-\varrho}\right) dx\right] \tag{3.12}$$

and

$$II_j = \sum_{i=1}^{j} \left(\frac{\theta}{\lambda^m}\right)^{i-1}. \tag{3.13}$$

Since $t_i < R$ for all $i \geq 1$ we get $A(k, t_i) \subset A(k, R)$, $\int_{A(k,t_i)} \Phi\left(\frac{u-k}{R-\varrho}\right) dx \leq \int_{A(k,R)} \Phi\left(\frac{u-k}{R-\varrho}\right) dx$ and

$$I_j \leq II_j \int_{A(k,R)} \Phi\left(\frac{u-k}{R-\varrho}\right) dx. \tag{3.14}$$

for all $i \geq 1$. By (3.11), (3.12), (3.13) and (3.14) it follows

$$\mathcal{F}(k, t_0) \leq \theta^j \mathcal{F}\left(k, t_j\right) + \frac{1}{(1-\lambda)^m} \left[A \int_{A(k,R)} \Phi\left(\frac{u-k}{R-\varrho}\right) dx + B\right] \sum_{i=0}^{j-1} \left(\frac{\theta}{\lambda^m}\right)^i \tag{3.15}$$

If we now choose λ in such a way that $\theta\lambda^{-m} < 1$ and go to the limit for $k \to +\infty$ we get (4.8) with $c = c_{\theta,m} = (1-\lambda)^{-m} (1-\theta\lambda^{-m})^{-1}$. □

Lemma 6 and Lemma 7 generalize Lemma 6.1 of (Giusti, 1994)

3.2 Caccioppoli's Inequality

Now we can proof the Caccioppoli's inequality.

Definition 10. *A function $u \in W^1_{loc}L^\Phi(\Omega)$ is a quasi-minimum of the functional (1.1), with costant $Q \geq 1$, if for every function $\varphi \in W^1_{loc}L^\Phi(\Omega)$, with the support $K \subset \Omega$, then*

$$F(u, K) \leq QF(u + \varphi, K).$$

Definition 11. *Let $u \in W^1_{loc}L^\Phi(\Omega)$ it is a sub-quasi-minimum of the functional (1.1), with constant $Q \geq 1$, if for all not-positive function $\varphi \in W^1_{loc}L^\Phi(\Omega)$, with support $K \subset \Omega$, we have*

$$F(u, K) \leq QF(u + \varphi, K). \tag{3.16}$$

Definition 12. *Let $u \in W^1_{loc}L^\Phi(\Omega)$ it is a super-quasi-minimum of the functional (1.1), with constant $Q \geq 1$, if for all not-negative function $\varphi \in W^1_{loc}L^\Phi(\Omega)$, with support $K \subset \Omega$, we have*

$$F(u, K) \leq QF(u + \varphi, K). \tag{3.17}$$

Remark 1. *Quasi-minima are at the same time sub - and a super-quasi-minima.*

If $u \in W^1_{loc}L^{\Phi}(\Omega)$, k is a real number and $Q_R(x_0)$ is a cube strictly contained in Ω we set

$$A(k,R) = \{x \in Q_R : u(x) > k\} = \{u > k\} \cap Q_R,$$
$$B(k,R) = \{x \in Q_R : u(x) < k\} = \{u < k\} \cap Q_R.$$

Remark 2. *We have $|A(R,k)| = |Q_R| - |B(R,k)|$ for almost every $k \in \mathbb{R}$, so that when necessary we can assume without loss of generality that all the vcalues k under consideration will satisfy this relation.*

For dettails refer (Ambrosio; E De Giorgi, 1957; Giaquinta et al., 1982) and (Giusti, 1994).

Theorem 4. *(Caccioppoli's inequality) Let $u \in W^1_{loc}L^{\Phi}(\Omega)$ be a sub-quasi-minimum for the functional (1.1) and let the growths (1.2) hold. If $\Phi \in \triangle_2$; then there exists a real number $R_0 > 0$ such that for every $x_0 \in \Omega$, every $R, \varrho \in \mathbb{R}$ with $0 < \varrho < R < \min\left\{R_0, \frac{d(x_0,\partial\Omega)}{2\sqrt{2^N}}\right\}$ and every $k \geq k_0 \geq 0$ we have*

$$\begin{aligned}\int_{A(k,\varrho)} \Phi(|\nabla u|)\, dx &\leq c_{Cac,1} \int_{A(k,R)} \Phi\left(\frac{u-k}{R-\varrho}\right) dx + \\ &+ c_{Cac,2}\left[R^{-\epsilon N}\Phi(k) + \|a\|_{\beta}\right] |A(k,R)|^{1-\frac{1}{N}+\epsilon}\end{aligned} \tag{3.18}$$

where $c_{Cac,1} = c_{12}(N,m,Q)$ and $c_{Cac,2} = c_{13}(N,m,Q)$ are two positive real constant.

Proof. The proof follows using the Lemma 6, the techniques introduced in (Giusti et al., 1994; Mascolo et al., 1996) and the Lemma 8. □

Remark 3. *Let $u \in W^1_{loc}L^{\Phi}(\Omega)$ be a sub-quasi-minimum for the functional (1.1) and let the growths (1.2) hold. If $\Phi \in \triangle_2$; then $-u$ will be a sub-quasi-minimum for the functional*

$$\overline{\mathcal{F}}[u,\Omega] = \int_{\Omega} \overline{f}(x, u(x), \nabla u(x))\, dx \tag{3.19}$$

with $\overline{f}(x,s,z) = f(x,-s,-z)$. Since $\overline{f}$ satisfies conditions (1.2) then (4.18) holds for $-u$ and for every $k \leq -k_0 \leq 0$ we have

$$\begin{aligned}\int_{B(k,\varrho)} \Phi(|\nabla u|)\, dx &\leq c_{Cac,1} \int_{B(k,R)} \Phi\left(\frac{k-u}{R-\varrho}\right) dx + \\ &+ c_{Cac,2}\tilde{G}(R,k)\end{aligned} \tag{3.20}$$

where

$$\tilde{G}(R,k) = \left[R^{-\epsilon N}\Phi(k) + \|a\|_{\beta}\right] |B(k,R)|^{1-\frac{1}{N}+\epsilon}.$$

and $c_{Cac,1} = c_{12}(N,m,Q)$ and $c_{Cac,2} = c_{13}(N,m,Q)$ are two positive real costants.

Theorem 5. *If $u \in W^1_{loc}L^{\Phi}(\Omega)$ is a quasi-minimum for the functional (1.1) and let the growths (1.2) hold. If $\Phi \in \triangle_2$; then there exists a real number $R_0 > 0$ such that for every $x_0 \in \Omega$, for every $R, \varrho \in \mathbb{R}$ with $0 < \varrho < R < \min\left\{R_0, \frac{d(x_0,\partial\Omega)}{2\sqrt{2^N}}\right\}$, for every $k \in \mathbb{R}$ the function u satisfies the Caccioppoli Inequality (4.18) and (4.44).*

We can now introduce the adequate De Giorgi classes relating to the functional (1.1).

Definition 13. *Let Φ be a N-function and $\Phi \in \triangle_2$. Let $u \in W^1_{loc}L^{\Phi}(\Omega)$; we say that $u \in DG^+_{\Phi}(\Omega, H_1, H_2, \chi, \epsilon, R_0, k_0)$ if for every couple of concentric cubes $Q_\varrho \subset Q_R \subset Q_{R_0} \Subset \Omega$, with $\varrho < R < R_0$, and for every $k \geq k_0 \geq 0$ we have*

$$\begin{aligned}\int_{A(k,\varrho)} \Phi(|\nabla u|)\, dx &\leq H_1 \int_{A(k,R)} \Phi\left(\frac{u-k}{R-\varrho}\right) dx + \\ &+ H_2\left[R^{-\epsilon N}\Phi(k) + \chi\right] |A(k,R)|^{1-\frac{1}{N}+\epsilon}.\end{aligned} \tag{3.21}$$

Definition 14. *Let Φ be a N-function and $\Phi \in \triangle_2$. Let $u \in W^1_{loc}L^{\Phi}(\Omega)$; we say that $u \in DG^-_{\Phi}(\Omega, H_1, H_2, \chi, \epsilon, R_0, k_0)$ if for every couple of concentric cubes $Q_\varrho \subset Q_R \subset Q_{R_0} \Subset \Omega$, with $\varrho < R < R_0$, and for every $k \leq -k_0 \leq 0$ we have*

$$\begin{aligned}\int_{B(k,\varrho)} \Phi(|\nabla u|)\, dx &\leq H_1 \int_{B(k,R)} \Phi\left(\frac{k-u}{R-\varrho}\right) dx + \\ &+ H_2\left[R^{-\epsilon N}\Phi(k) + \chi\right] |B(k,R)|^{1-\frac{1}{N}+\epsilon}.\end{aligned} \tag{3.22}$$

Definition 15. *Let Φ be a N-function and $\Phi \in \triangle_2$. Let $u \in W^1_{loc}L^{\Phi}(\Omega)$,we say that $u \in DG_{\Phi}(\Omega, H_1, H_2, \chi, R_0)$ if $u \in DG^{\pm}_{\Phi}(\Omega, H_1, H_2, \chi, R_0)$, that is*

$$DG_{\Phi}(\Omega, H_1, H_2, \chi, \epsilon, R_0) = DG^{+}_{\Phi}(\Omega, H_1, H_2, \chi, \epsilon, R_0) \cap DG^{-}_{\Phi}(\Omega, H_1, H_2, \chi, \epsilon, R_0).$$

Remark 4. *The relationships (4.45) and (4.46) can be written this way:*

$$\begin{aligned}\int_{A(k,\varrho)} \Phi(|\nabla u|)\, dx \quad &\leq H_1 \int_{A(k,R)} \Phi\left(\tfrac{u-k}{R-\varrho}\right) dx + \\ &+ H_2 R^{-\epsilon N}\left[\Phi\left(k + \chi_1 R^{\frac{N\epsilon}{m}}\right)\right] |A(k,R)|^{1-\frac{1}{N}+\epsilon}\end{aligned} \tag{3.23}$$

and

$$\begin{aligned}\int_{B(k,\varrho)} \Phi(|\nabla u|)\, dx \quad &\leq H_1 \int_{B(k,R)} \Phi\left(\tfrac{k-u}{R-\varrho}\right) dx + \\ &+ H_2 R^{-\epsilon N}\left[\Phi\left(k + \chi_1 R^{\frac{N\epsilon}{m}}\right)\right] |B(k,R)|^{1-\frac{1}{N}+\epsilon}.\end{aligned} \tag{3.24}$$

where $\chi_1 = \Phi^{-1}(\chi)$. Therefore, replacing $v = u - \chi_1 R^{\frac{N\epsilon}{m}}$ and $h = k - \chi_1 R^{\frac{N\epsilon}{m}}$, we get

$$\int_{A(h,\varrho)} \Phi(|\nabla v|)\, dx \leq H_1 \int_{A(h,R)} \Phi\left(\frac{v-h}{R-\varrho}\right) dx + H_2 R^{-\epsilon N}\Phi(h)\, |A(h,R)|^{1-\frac{1}{N}+\epsilon} \tag{3.25}$$

and

$$\int_{B(h,\varrho)} \Phi(|\nabla v|)\, dx \leq H_1 \int_{B(h,R)} \Phi\left(\frac{h-v}{R-\varrho}\right) dx + H_2 R^{-\epsilon N}\Phi(h)\, |B(h,R)|^{1-\frac{1}{N}+\epsilon}. \tag{3.26}$$

In the sequel it will be useful to associate to u the function

$$w_R(y) = \frac{u(Ry)}{R} \tag{3.27}$$

and we get the following Caccioppoli's inequality.

Corollary 1. *If $u \in W^1_{loc}L^{\Phi}(\Omega)$ is a quasi-minimum for the functional (1.1) and let the growths (1.2) hold. If $\Phi \in \triangle_2$ and $1 \geq \sigma > \tau \geq \frac{1}{2}$ then*

$$\int_{A(h/R,\tau)} \Phi(|\nabla w_R|)\, dy \leq H_1 \int_{A(h/R,\sigma)} \Phi\left(\frac{w_R - h/R}{\sigma - \tau}\right) dy + \tilde{H}_2 \Phi(h)\, |A(h,\sigma)|^{1-\frac{1}{N}+\epsilon} \tag{3.28}$$

and

$$\int_{B(h/R,\tau)} \Phi(|\nabla w_R|)\, dy \leq H_1 \int_{B(h/R,\sigma)} \Phi\left(\frac{h/R - w_R}{\sigma - \tau}\right) dy + \tilde{H}_2 \Phi(h)\, |B(h,\sigma)|^{1-\frac{1}{N}+\epsilon} \tag{3.29}$$

where $\tilde{H}_2 = 2^{\epsilon N} H_2$.

Proof. Let $w_R(y) = \frac{u(Ry)}{R}$ then

$$\int_{A(h/R,\tau)} \Phi(|\nabla w_R|)\, dy = \frac{1}{R^N} \int_{A(h,\tau R)} \Phi(|\nabla u|)\, dx \tag{3.30}$$

and

$$\int_{A(h/R,\sigma)} \Phi\left(\frac{w_R - h/R}{\sigma - \tau}\right) dy = \frac{1}{R^N} \int_{A(h,\sigma)} \Phi\left(\frac{u-h}{\sigma R - \tau R}\right) dx \tag{3.31}$$

by Caccioppoli inequalities (3.23) and (3.24) we obtain (3.28) and (3.29). □

4. $L^{\Phi} - L^{\infty}$ Inequalities I

Let us remember the following lemma:

Lemma 8. *Let both $\lambda > 0$ and $\{x_i\}_{i\in\mathbb{N}}$ a set of positive real numbers, such that*

$$x_{i+1} \le CB^i x_i^{1+\lambda} \tag{4.1}$$

with $C > 0$ and $B > 1$. Then, if $x_0 \le C^{-\frac{1}{\lambda}} B^{-\frac{1}{\lambda^2}}$, we have

$$x_i \le B^{-\frac{i}{\lambda}} x_0 \tag{4.2}$$

and consequently, in particular, we have

$$\lim_{i\to+\infty} x_i = 0. \tag{4.3}$$

Proof. Refer to Lemma 7.1 of (Giusti, 1994). □

Theorem 6. *If $u \in DG_{\Phi}^{+}(\Omega, H_1, H_2, \chi, \epsilon, R_0)$ then u is locally bounded from above in Ω. Furthermore, for every $x_0 \in \Omega$ and $0 < R \le \min\left(R_0, \frac{d(x_0,\partial\Omega)}{2\sqrt{2^N}}, 1\right)$ there exists an universal constant $c_{14} = c_{14}(N, m, H_1, H_2, \chi)$ such that*

$$ess - \sup_{Q_{\frac{R}{2}}} \left(\frac{v - h_0}{R}\right) \le 2\Phi^{-1}\left(\frac{c_{14}}{|Q_R|}\int_{Q_R} \Phi\left(\frac{(v-h_0)_+}{R}\right) dx\right) \tag{4.4}$$

where $v = u - \chi R^{N\epsilon}$, $h = k - \chi R^{N\epsilon}$ and $h_0 = k_0 - \chi R^{N\epsilon}$.

Proof. Let $\frac{1}{2} \le \tau < \sigma \le 1$ and $\zeta = \eta \max\left\{\frac{v(Ry)-h}{R}, 0\right\} = \eta\left(\frac{v(Ry)-h}{R}\right)_+$ where $\eta \in C_c^{\infty}\left(Q_{\frac{\sigma+\tau}{2}}\right)$ with $0 \le \eta \le 1$ on $Q_{\frac{\sigma+\tau}{2}}$, $\eta = 1$ on Q_τ and $|\nabla\eta| \le \frac{2}{(\sigma-\tau)}$ on $Q_{\frac{\sigma+\tau}{2}}$. Setting

$$I = \left(\int_{Q_{\frac{\tau+\sigma}{2}}} (\Phi(\zeta))^{\frac{N}{N-1}}\, dy\right)^{\frac{N-1}{N}} \tag{4.5}$$

using Holder and Sobolev inequalities it follows

$$\int_{A(h,\tau)} \Phi\left(\left(\frac{v(Ry)-h}{R}\right)_+\right) dy \le |A(h,\tau)|^{\frac{1}{N}}\, I \tag{4.6}$$

and

$$I \le C_{SN}\left(\int_{Q_{\frac{\tau+\sigma}{2}}} \dot{\Phi}(\zeta)\,|\nabla\zeta|\, dy\right) \tag{4.7}$$

then we have

$$\begin{aligned}\int_{A(h,\tau)} \Phi\left(\left(\tfrac{v(Ry)-h}{R}\right)_+\right) dy \quad &\le c\,|A(h,\tau)|^{\frac{1}{N}} \cdot \\ &\cdot \int_{Q_{\frac{\tau+\sigma}{2}}} \dot{\Phi}(\zeta)\,|\nabla\zeta|\, dy\end{aligned} \tag{4.8}$$

where $c = C_{SN}$. Since

$$|\nabla\zeta| \le \eta\,|\nabla w_R| + \left(\frac{v(Ry)-h}{R}\right)_+ |\nabla\eta| \tag{4.9}$$

and

$$\dot{\Phi}(a)\, b \le \dot{\Phi}(b)\, b + \dot{\Phi}(a)\, a \le m\,(\Phi(a) + \Phi(b)) \tag{4.10}$$

we get

$$\begin{aligned} &\int\limits_{Q_{\frac{\tau+\sigma}{2}}} \dot{\Phi}(\zeta)\,|\nabla\varpi|\;dy \leq \\ &\leq \int\limits_{Q_{\frac{\tau+\sigma}{2}}} \dot{\Phi}(\zeta)\left(\eta\,|\nabla w_R| + \left(\tfrac{v(Ry)-h}{R}\right)_+|\nabla\eta|\right)dy. \end{aligned} \tag{4.11}$$

Setting

$$II = \int\limits_{Q_{\frac{\tau+\sigma}{2}}} \dot{\Phi}(\zeta)\left(\eta\,|\nabla w_R| + \left(\frac{v(Ry)-h}{R}\right)_+|\nabla\eta|\right)dy \tag{4.12}$$

it follows

$$\begin{aligned} II \quad &\leq \int\limits_{Q_{\frac{\tau+\sigma}{2}}} \eta\dot{\Phi}(\zeta)\,|\nabla w_R|\;dy + \\ &+ \int\limits_{Q_{\frac{\tau+\sigma}{2}}} \dot{\Phi}(\zeta)\left(\tfrac{v(Ry)-h}{R}\right)_+|\nabla\eta|\;dy \\ &\leq \int\limits_{Q_{\frac{\tau+\sigma}{2}}} \dot{\Phi}\left(\left(\tfrac{v(Ry)-h}{R}\right)_+\right)|\nabla w_R|\;dy + \\ &+\tfrac{4}{\sigma-\tau}\int\limits_{Q_{\frac{\tau+\sigma}{2}}} \dot{\Phi}\left(\left(\tfrac{v(Ry)-h}{R}\right)_+\right)\left(\tfrac{v(Ry)-h}{R}\right)_+\;dy \\ &\leq m\int\limits_{Q_{\frac{\tau+\sigma}{2}}} \Phi\left(\left(\tfrac{v(Ry)-h}{R}\right)_+\right)dy + m\int\limits_{Q_{\frac{\tau+\sigma}{2}}} \Phi(|\nabla w_R|)\;dy + \\ &+\tfrac{4m}{\sigma-\tau}\int\limits_{Q_{\frac{\tau+\sigma}{2}}} \Phi\left(\left(\tfrac{v(Ry)-h}{R}\right)_+\right)dy \\ &\leq \left(m+\tfrac{4m}{\sigma-\tau}\right)\int\limits_{A(h,\sigma)} \Phi\left(\left(\tfrac{v(Ry)-h}{R}\right)_+\right)dy + m\int\limits_{Q_{\frac{\tau+\sigma}{2}}} \Phi(|\nabla w_R|)\;dy \end{aligned} \tag{4.13}$$

Using (3.28) we obtain

$$\begin{aligned} \int\limits_{A(h,\tau)} \Phi\left(\left(\tfrac{v(Ry)-h}{R}\right)_+\right)dy \quad &\leq C_{SN}\,|A(h,\tau)|^{\frac{1}{N}}\left(\left(m+\tfrac{4m}{\sigma-\tau}\right)\cdot\right. \\ &\cdot\int\limits_{A(h,\sigma)} \Phi\left(\left(\tfrac{v(Ry)-h}{R}\right)_+\right)dy + \\ &+mH_1\int\limits_{A(h,\sigma)} \Phi\left(\tfrac{w_R-h/R}{\sigma-\tau}\right)dx + \\ &\left.+m\tilde{H}_2\Phi(h)\,|A(h,\sigma)|^{1-\frac{1}{N}+\epsilon}\right) \\ &\leq mC_{SN}\,|A(h,\tau)|^{\frac{1}{N}}\left(\left(1+\tfrac{4}{\sigma-\tau}+\tfrac{H_1}{(\sigma-\tau)^m}\right)\cdot\right. \\ &\cdot\int\limits_{A(h,\sigma)} \Phi\left(\left(\tfrac{v(Ry)-h}{R}\right)_+\right)dy + \\ &\left.+\tilde{H}_2\Phi(h)\,|A(h,\sigma)|^{1-\frac{1}{N}+\epsilon}\right). \end{aligned} \tag{4.14}$$

Since

$$|A(k,\tau)| \leq \frac{1}{\Phi\left(\frac{k-h}{R}\right)}\int\limits_{A(k,\tau)} \Phi\left(\frac{v(Ry)-k}{R}\right)dy \tag{4.15}$$

we have

$$\begin{aligned} &\int\limits_{A(h,\tau)} \Phi\left(\left(\tfrac{v(Ry)-h}{R}\right)_+\right)dy \leq \left[\left(1+\tfrac{4}{\sigma-\tau}+\tfrac{H_1}{(\sigma-\tau)^m}\right)|A(k,\tau)|^{\frac{1}{N}-\epsilon}+\tfrac{\tilde{H}_2\Phi(h)}{\Phi\left(\frac{k-h}{R}\right)}\right]\cdot \\ &\cdot mC_{SN}\cdot\left(\tfrac{1}{\Phi\left(\frac{k-h}{R}\right)}\int\limits_{A(k,\tau)} \Phi\left(\tfrac{v-k}{R}\right)dy\right)^{\epsilon}\int\limits_{A(h,\sigma)} \Phi\left(\left(\tfrac{v(Ry)-h}{R}\right)_+\right)dy \end{aligned} \tag{4.16}$$

Since

$$\int\limits_{A(h,\sigma)} \Phi\left(\left(\frac{v(Ry)-h}{R}\right)_+\right)dy \leq \int\limits_{A(k,\sigma)} \Phi\left(\frac{v(Ry)-k}{R}\right)dy \tag{4.17}$$

by (4.16) we get

$$\begin{aligned}\int_{A(h,\tau)} \Phi\left(\left(\tfrac{v(Ry)-h}{R}\right)_+\right) dy \le \left(\tfrac{1}{\Phi\left(\frac{k-h}{R}\right)}\right)^{\epsilon}\left(\int_{A(k,\sigma)} \Phi\left(\tfrac{v-k}{R}\right) dy\right)^{1+\epsilon}\cdot \\ \cdot mC_{SN}\cdot\left[\left(1+\tfrac{4}{\sigma-\tau}+\tfrac{H_1}{(\sigma-\tau)^m}\right)|A(k,\tau)|^{\frac{1}{N}-\epsilon}+\tfrac{\tilde{H}_2\Phi(h)}{\Phi\left(\frac{k-h}{R}\right)}\right].\end{aligned} \tag{4.18}$$

Let $\tilde{h}=\frac{h}{R}$ and $\tilde{k}=\frac{k}{R}$ then $\int_{A(h,\tau)} \Phi\left(\left(\frac{v(Ry)-h}{R}\right)_+\right) dy = \int_{A(\tilde{h},\tau)} \Phi\left(\left(w_R-\tilde{h}\right)_+\right) dy$, by (4.18) it follows

$$\begin{aligned}\int_{A(\tilde{h},\tau)} \Phi\left(\left(w_R-\tilde{h}\right)_+\right) dy \le \left(\tfrac{1}{\Phi(\tilde{k}-\tilde{h})}\right)^{\epsilon}\left(\int_{A(\tilde{k},\sigma)} \Phi\left(\left(w_R-\tilde{k}\right)_+\right) dy\right)^{1+\epsilon}\cdot \\ \cdot mC_{SN}\cdot\left[\left(1+\tfrac{4}{\sigma-\tau}+\tfrac{H_1}{(\sigma-\tau)^m}\right)R^{1-\epsilon N}+\tfrac{\tilde{H}_2\Phi(\tilde{h})}{\Phi(\tilde{k}-\tilde{h})}\right].\end{aligned} \tag{4.19}$$

Let us define

$$\begin{aligned}&\tilde{k}_0=\tfrac{d}{R}\\ &\tilde{k}_{i+1}=\tilde{k}_i+\Phi^{-1}\left(\tfrac{\Phi\left(\frac{d}{R}\right)}{2^{im}}\right)\quad \text{for } i\ge 1\end{aligned}$$

and

$$r_i=\frac{1}{2}\left(1+2^{-i}\right)\ \text{for } i\in\mathbb{N}.$$

by inequality (4.19) we have

$$\begin{aligned}U_i \quad &\le mC_{SN}\left[\left(1+\tfrac{4}{r_i-r_{i+1}}+\tfrac{H_1}{(r_i-r_{i+1})^m}\right)R^{1-\epsilon N}+\tfrac{\tilde{H}_2\Phi(\tilde{k}_i)}{\Phi(\tilde{k}_{i+1}-\tilde{k}_i)}\right]\cdot \\ &\cdot\left(\tfrac{1}{\Phi(\tilde{k}_{i+1}-\tilde{k}_i)}\right)^{\epsilon}U_{i+1}^{1+\epsilon}\end{aligned} \tag{4.20}$$

and

$$U_i \le mC_{SN}\left[cR^{1-\epsilon N}+\frac{\tilde{H}_2\Phi\left(\tilde{k}_i\right)}{\Phi\left(\tilde{k}_{i+1}-\tilde{k}_i\right)}\right]\left(\frac{1}{\Phi\left(\frac{d}{R}\right)}\right)^{\epsilon}2^{m\epsilon i}U_{i+1}^{1+\epsilon} \tag{4.21}$$

where $U_i=\int_{A(\tilde{k}_i,r_i)} \Phi\left(\left(w_R-\tilde{k}_i\right)_+\right) dy$. Since $\Phi^{-1}\left(\frac{1}{2^m}t\right)\le\frac{1}{2}\Phi^{-1}(t)$ for $t>0$ then $\tilde{k}_i=\frac{d}{R}+\sum_{j=1}^{i}\Phi^{-1}\left(\frac{\Phi\left(\frac{d}{R}\right)}{2^{jm}}\right)\le\frac{d}{R}+\sum_{j=1}^{i}\frac{\frac{d}{R}}{2^j}\le 2\frac{d}{R}$, it follows

$$U_i\le mC_{SN}\left[\xi+2^m\tilde{H}_2\right]\left(\frac{1}{\Phi\left(\frac{d}{R}\right)}\right)^{\epsilon}2^{m(\epsilon+1)i}U_{i+1}^{1+\epsilon} \tag{4.22}$$

Applying Lemma 8 get that $\lim_{i\to+\infty}U_i=0$ if

$$U_0\le\left(mC_{SN}\left[\xi+2^m\tilde{H}_2\right]\right)^{-\frac{1}{\epsilon}}\Phi\left(\frac{d}{R}\right)2^{-\frac{m(\epsilon+1)}{\epsilon^2}},$$

that is

$$2^{\frac{m(\epsilon+1)}{\epsilon^2}}\left(mC_{SN}\left[\xi+2^m\tilde{H}_2\right]\right)^{\frac{1}{\epsilon}}\int_{A\left(\frac{d}{R},1\right)}\Phi\left(\left(w_R-\frac{d}{R}\right)_+\right)dy\le\Phi\left(\frac{d}{R}\right). \tag{4.23}$$

It is easy to check that (4.23) is satisfies if we choose

$$\Phi^{-1}\left(2^{\frac{m(\epsilon+1)}{\epsilon^2}}\left(mC_{SN}\left[\xi+2^m\tilde{H}_2\right]\right)^{\frac{1}{\epsilon}}\int_{A(0,1)}\Phi\left((w_R)_+\right)dy\right)\le\frac{d}{R}. \tag{4.24}$$

Since $\Phi^{-1}\left(\frac{1}{2^m}t\right)\le\frac{1}{2}\Phi^{-1}(t)$ for $t>0$ then $\tilde{k}_i=\frac{d}{R}+\sum_{j=1}^{i}\Phi^{-1}\left(\frac{\Phi\left(\frac{d}{R}\right)}{2^{jm}}\right)\le\frac{d}{R}+\sum_{j=1}^{i}\frac{\frac{d}{R}}{2^j}\le 2\frac{d}{R}$, it follows $A\left(\tilde{k}_i,r_i\right)\supset A\left(2\frac{d}{R},\frac{1}{2}\right)$, hence, since $\lim_{i\to+\infty}U_i=0$, we have $\left|A\left(2\frac{d}{R},\frac{1}{2}\right)\right|=0$, which gives

$$\sup_{Q_{\frac{1}{2}}}\{w_R\}\le 2\frac{d}{R}=2\Phi^{-1}\left(2^{\frac{m(\epsilon+1)}{\epsilon^2}}\left(mC_{SN}\left[\xi+2^m\tilde{H}_2\right]\right)^{\frac{1}{\epsilon}}\int_{A(0,1)}\Phi\left((w_R)_+\right)dy\right) \tag{4.25}$$

and therefore (4.4). □

Theorem 7. *If $u \in DG_\Phi(\Omega, H_1, H_2, \chi, \epsilon, R_0)$ then u is locally bounded on Ω. Furthermore, for each $x_0 \in \Omega$ and $0 < R \le \min\left(R_0, \frac{d(x_0,\partial\Omega)}{2\sqrt{2^N}}, 1\right)$ there exists an universal constant $c_{15} = c_{15}(N, m, H_1, H_2, \chi)$ such that*

$$ess-\sup_{Q_{\frac{R}{2}}} (|u(x)|) \le 2R\Phi^{-1}\left(\frac{c_{15}}{|Q_R|}\int_{Q_R} \Phi\left(\frac{|u|}{R}\right) dx\right) + k_0 + \chi R^{\frac{N\epsilon}{m}} \tag{4.26}$$

Proof. If $u \in DG_\Phi^-(\Omega, H_1, H_2, \chi, \epsilon, R_0, \kappa_0)$ then $-u \in DG_\Phi^+(\Omega, H_1, H_2, \chi, \epsilon, R_0)$ and the proof follows by Theorem 6. □

Theorem 8. *Let $u \in W^1_{loc}L^\Phi(\Omega)$ be a quasi-minimum of the functional (1.1) with the growths (1.2). If $\Phi \in \triangle_2$; then there exists a real number $R_0 > 0$ such that for every $x_0 \in \Omega$, for every $R, \varrho \in \mathbb{R}$ with $0 < \varrho < R < \min\left\{R_0, \frac{d(x_0,\partial\Omega)}{2\sqrt{2^N}}\right\}$, for every $k \in \mathbb{R}$ the following inequalities hold*

$$\begin{aligned} ess-\sup_{Q_{\frac{R}{2}}} (u-k) &\le 2R\Phi^{-1}\left(\tfrac{c_{14}}{|Q_R|}\int_{Q_R} \Phi\left(\tfrac{(u-k)_+}{R}\right) dx\right) + \chi R^{\frac{N\epsilon}{m}} \\ ess-\inf_{Q_{\frac{R}{2}}} (k-u) &\le 2R\Phi^{-1}\left(\tfrac{c_{14}}{|Q_R|}\int_{Q_R} \Phi\left(\tfrac{(k-u)_-}{R}\right) dx\right) + \chi R^{\frac{N\epsilon}{m}} \end{aligned} \tag{4.27}$$

and

$$\begin{aligned} ess-\sup_{Q_{\frac{R}{2}}} (u-k) &\le 2\Phi^{-1}\left(\tfrac{c_{14}}{|Q_R|}\int_{Q_R} \Phi\left((u-k)_+\right) dx\right) + \chi R^{\frac{N\epsilon}{m}} \\ ess-\inf_{Q_{\frac{R}{2}}} (k-u) &\le 2\Phi^{-1}\left(\tfrac{c_{14}}{|Q_R|}\int_{Q_R} \Phi\left((k-u)_-\right) dx\right) + \chi R^{\frac{N\epsilon}{m}} \end{aligned} \tag{4.28}$$

Proof. Inequalities (4.27) follow by Theorem 6. Inequalities (4.28) follow using the demostration methods presented in (Mascolo et al., 1996). □

Remark 5. *Inequalties (4.27) and (4.28) are equivalent if $\Phi(t) = t^p$ with $p > 1$.*

Moreover the following lemma is valid.

Lemma 9. *If $u \in DG_\Phi^+(\Omega, H_1, H_2, \chi, \epsilon, R_0)$ then u is locally bounded above on Ω. Let $v = u - \chi R^{\frac{N\epsilon}{m}}$ the we get the foolowing $L^\Phi - L^\Phi$ estimation: for each $x_0 \in \Omega$ and $0 < R \le \min\left(R_0, \frac{d(x_0,\partial\Omega)}{2\sqrt{2^N}}, 1\right)$ there exists an universal constant $c_{16} = c_{16}(N, m, H_1, H_2, \chi)$ such that for any $k > k_0$*

$$ess-\sup_{Q_\varrho}\left(\frac{v-k}{R}\right) \le 2\Phi^{-1}\left(\frac{c_{16}}{(R-\varrho)^N}\int_{Q_R} \Phi\left(\frac{(v-k)_+}{R}\right) dx\right) \tag{4.29}$$

for each $Q_\varrho \subset Q_R$ e $0 < \varrho < R$. Furthermore, for each $x_0 \in \Omega$ and $R \le \min\left(R_0, \frac{d(x_0,\partial\Omega)}{2\sqrt{2^N}}, 1\right)$ there exists an universal constant $c_{16} = c_{16}(N, m, H_1, H_2, \chi)$ such that for any $k > k_0$

$$ess-\sup_{Q_\varrho} (u-k) \le 2R\Phi^{-1}\left(\frac{c_{16}}{(R-\varrho)^N}\int_{Q_R} \Phi\left(\frac{(u-k)_+}{R}\right) dx\right) + k_0 + \chi R^{\frac{N\epsilon}{m}} \tag{4.30}$$

for each $Q_\varrho \subset\subset Q_R$ e $0 < \varrho < R$.

Proof. Observe that for every $0 < t < 1$ there exists $x_0 \in Q_{tR}$ such that

$$ess - \sup_{Q_{tR}} \left\{ \frac{v-k}{R} \right\} \leq ess - \sup_{Q_{\frac{(1-t)R}{2}}(x_0)} \left\{ \frac{v-k}{R} \right\} \tag{4.31}$$

therefore (4.27) gives

$$ess - \sup_{Q_{tR}} \left\{ \frac{v-k}{R} \right\} \leq 2\Phi^{-1} \left(\frac{c_{16}}{(1-t)^N R^N} \int_{Q_R} \Phi \left(\frac{(v-k)_+}{R} \right) dx \right) \tag{4.32}$$

by (4.32) it follows (4.29) and (4.30). □

Using Theorem 4 and Theorem 6 we get the following theorem.

Theorem 9. *Let $u \in W^1_{loc} L^\Phi(\Omega)$ be a quasi-minimum of the functional (1.1) with the growths (1.2). If $\Phi \in \triangle_2$; then there exists a real number $R_0 > 0$ such that for every $x_0 \in \Omega$, for every $R, \varrho \in \mathbb{R}$ with $0 < \varrho < R < \min \left\{ R_0, \frac{d(x_0, \partial\Omega)}{2\sqrt{2^N}} \right\}$, for every $k \in \mathbb{R}$ the following Caccioppoli inequalities hold*

$$\begin{aligned} \int_{A(k,\varrho)} \Phi(|\nabla u|)\, dx \quad &\leq \widetilde{c}_{Cac,1} \int_{A(k,R)} \Phi\left(\tfrac{u-k}{R-\varrho}\right) dx + \\ &+ \widetilde{c}_{Cac,2} \, |A(k,R)|^{1-\frac{1}{N}+\epsilon} \end{aligned} \tag{4.33}$$

and

$$\begin{aligned} \int_{B(k,\varrho)} \Phi(|\nabla u|)\, dx \quad &\leq \widetilde{c}_{Cap,1} \int_{B(k,R)} \Phi\left(\tfrac{k-u}{R-\varrho}\right) dx + \\ &+ \widetilde{c}_{Cac,2} \, |B(k,R)|^{1-\frac{1}{N}+\epsilon} \end{aligned} \tag{4.34}$$

where $\widetilde{c}_{Cac,1} = c_1(N, m, \gamma, Q, M)$ and $\widetilde{c}_{Cac,2} = c_2(N, m, \gamma, Q, M)$ are two real positive constants with $M = 2\sup_{Q_{2R}}(u)$.

Proof. The proof follows from the Theorem 4 and Theorem 6 as in (Giusti, 1994). □

Remark 6. *The preceding relationships (4.33) and (4.34) they are worth with the same constants for $u - A$ if $|A| + \sup_{Q_{2R}}(u) \leq M$.*

5. $L^\Phi - L^\infty$ Inequalities II

5.1 The maximal $L^\Phi - L^\Phi$ inequality

Lemma 10. *Let $\Phi \in \triangle_2$. If $u \in DG_\Phi(\Omega, H_1, H_2, \chi, \epsilon, R_0)$ then there exists an universal constant $c_{17} = c_{17}(N, m, H_1, H_2, \chi)$ such that for each $k, h > k_0$ and $k > h$ we have*

$$\int_{A(k,\varrho)} \Phi\left(\frac{v-k}{R}\right) dx \leq c_{17} \frac{V(h,\varrho)}{(\tau-\varrho)^N} \int_{A(h,\tau)} \Phi\left(\frac{v-h}{R}\right) dx \tag{5.1}$$

where $v = u - \chi R^{\frac{N\epsilon}{m}}$.

Proof. Assuming that $1 < s < N$ and $k_0 < h < k$; let ϱ and τ be two real numbers such that $\frac{R}{2} \leq \varrho < \tau \leq R$. Let $\eta \in C_c^\infty\left(Q_{\frac{\varrho+\tau}{2}}\right)$ with $0 \leq \eta \leq 1$ on $Q_{\frac{\varrho+\tau}{2}}$, $\eta = 1$ on Q_ϱ and $|\nabla\eta| \leq \frac{2}{(\tau-\varrho)}$ on $Q_{\frac{\varrho+\tau}{2}}$. Let us then define $\zeta = \eta\frac{(v-k)_+}{R}$, from Theorem 8 we know that $(\Phi(\zeta))^{\frac{N-1}{N-\alpha}} \in W^{1,1}\left(Q_{\frac{\varrho+\tau}{2}}\right)$ and using Hölder inequality we obtain

$$\int_{A(k,\varrho)} \Phi\left(\frac{v-k}{R}\right) dx \leq |A(k,\varrho)|^{\frac{s}{N}} \left[\int_{A(k,\varrho)} \left(\Phi\left(\frac{v-k}{R}\right)\right)^{\frac{N}{N-s}} dx \right]^{\frac{N-s}{N}}. \tag{5.2}$$

Since

$$\int_{A(k,\varrho)} \left(\Phi\left(\frac{v-k}{R}\right)\right)^{\frac{N}{N-s}} dx \leq \int_{Q_{\frac{\varrho+\tau}{2}}} \left[(\Phi(\zeta))^{\frac{N-1}{N-s}}\right]^{\frac{N}{N-1}} dx \tag{5.3}$$

then from Sobolev Inequality and Chain Rule Theorem it follows

$$\begin{aligned}&\left[\int_{Q_{\frac{\varrho+\tau}{2}}}\left[(\Phi(\zeta))^{\frac{N-1}{N-s}}\right]^{\frac{N}{N-1}}dx\right]^{\frac{N-s}{N}}\leq\\&\leq\left[C_{SN}\frac{N-1}{N-s}\int_{Q_{\frac{\varrho+\tau}{2}}}(\Phi(\zeta))^{\frac{s-1}{N-s}}\dot{\Phi}(\zeta)\left(|\nabla\eta|\frac{(v-k)_+}{R}+\frac{\eta}{R}|\nabla(v-k)_+|\right)dx\right]^{\frac{N-s}{N-1}}.\end{aligned}\tag{5.4}$$

Since

$$|A(k,\tau)|\leq\frac{R}{\Phi(k-h)}\int_{A(\tau,h)}\Phi\left(\frac{v-h}{R}\right)dx\tag{5.5}$$

then, using Lemma 1 (i), Lemma 5, Lemma 9 and relations (3.18), (5.4) and (5.5), we have

$$\begin{aligned}&\left[\int_{Q_{\frac{\varrho+\tau}{2}}}\left[(\Phi(\zeta))^{\frac{N-1}{N-s}}\right]^{\frac{N}{N-1}}dx\right]^{\frac{N-s}{N}}\leq\left[C_{SN}\frac{N-1}{N-s}\left(\frac{c_{16}}{(\tau-\varrho)^N}\int_{Q_\tau}\Phi\left(\frac{(v-k)_+}{R}\right)dx\right)^{\frac{s-1}{N-s}}\cdot\right.\\&\left.\cdot\left(\int_{Q_{\frac{\varrho+\tau}{2}}}\dot{\Phi}(\zeta)\left(|\nabla\eta|\frac{(v-k)_+}{R}+\frac{\eta}{R}|\nabla(v-k)_+|\right)dx\right)\right]^{\frac{N-s}{N-1}}\leq\\&\leq\left[2C_{SN}\frac{N-1}{N-s}\left(\frac{c_{16}}{(\tau-\varrho)^N}\int_{Q_\tau}\Phi\left(\frac{(v-k)_+}{R}\right)dx\right)^{\frac{s-1}{N-s}}\cdot\right.\\&\left.\cdot\left(\left(\frac{m}{\tau-\varrho}+\frac{m}{R}\right)\int_{Q_\tau}\Phi\left(\frac{(v-k)_+}{R}\right)dx+\frac{m}{R}\int_{Q_{\frac{\varrho+\tau}{2}}}\Phi(|\nabla(v-k)_+|)\,dx\right)\right]^{\frac{N-s}{N-1}}\leq\\&\leq\left(2C_{SN}\frac{N-1}{N-s}\right)^{\frac{N-s}{N-1}}\left(\frac{c_{16}}{(\tau-\varrho)^N}\right)^{\frac{s-1}{N-1}}\left(\int_{A(\tau,h)}\Phi\left(\frac{v-h}{R}\right)dx\right)^{\frac{s-1}{N-1}}\cdot\\&\cdot\left(\left(\frac{m}{\tau-\varrho}+\frac{m}{R}\right)\int_{A(\tau,h)}\Phi\left(\frac{v-k}{R}\right)dx+\frac{m}{R}\int_{A\left(\frac{\varrho+\tau}{2},k\right)}\Phi(|\nabla v|)\,dx\right)^{\frac{N-s}{N-1}}.\end{aligned}$$

It follows

$$\begin{aligned}&\left[\int_{Q_{\frac{\varrho+\tau}{2}}}\left[(\Phi(\zeta))^{\frac{N-1}{N-s}}\right]^{\frac{N}{N-1}}dx\right]^{\frac{N-s}{N}}\leq\\&\leq\left(\frac{m}{\tau-\varrho}+\frac{m}{R}+\frac{mH}{R(\tau-\varrho)^m}+\frac{m\widetilde{H}_2R^{-\epsilon N}\left[\Phi(k)+\chi\right]|B(k,R)|^{-\frac{1}{N}+\epsilon}}{R\Phi(k-h)}\right)^{\frac{N-s}{N-1}}\cdot\\&\cdot\left(2C_{SN}\frac{N-1}{N-s}\right)^{\frac{N-s}{N-1}}\left(\frac{c_{16}}{(\tau-\varrho)^N}\right)^{\frac{s-1}{N-1}}\left(\int_{A(\tau,h)}\Phi\left(\frac{v-h}{R}\right)dx\right).\end{aligned}\tag{5.6}$$

Using relation (5.2) and (5.6) we obtain

$$\begin{aligned}&\int_{A(k,\varrho)}\Phi\left(\frac{(v-k)_+}{R}\right)dx\leq\\&\leq G(s)\,|A(k,\varrho)|^{\frac{s}{N}}\left(\int_{A(\tau,h)}\Phi\left(\frac{v-h}{R}\right)dx\right)\quad\text{for every } s\in(1,N),\end{aligned}\tag{5.7}$$

where

$$\begin{aligned}G(s)=&\left(2C_{SN}\frac{N-1}{N-s}\right)^{\frac{N-s}{N-1}}\left(\frac{c_{16}}{(\tau-\varrho)^N}\right)^{\frac{s-1}{N-1}}\cdot\\&\cdot\left(\frac{m}{\tau-\varrho}+\frac{m}{R}+\frac{mH}{R(\tau-\varrho)^m}+\frac{m\widetilde{H}_2R^{-\epsilon N}\left[\Phi(k)+\chi\right]|B(k,R)|^{-\frac{1}{N}+\epsilon}}{R\Phi(k-h)}\right)^{\frac{N-s}{N-1}}\end{aligned}\tag{5.8}$$

is a continuous function in $s\in(1,N)$. From $s\to N^-$ we obtain (5.1). □

5.2 The Maximal $L^{\Phi} - L^{\infty}$ Inequality

Theorem 10. *Let Φ be a N-function and $\Phi \in \triangle_2$. If $u \in DG_{\Phi}(\Omega, H_1, H_2, \chi, \epsilon, R_0))$ then u is locally bounded on Ω. Furthermore, for each $x_0 \in \Omega$ and $0 < R \leq \min(R_0, d(x_0, \partial\Omega), 1)$ there exists an universal constant $c_M = c_M(\alpha, N, m, H, \chi)$ such that for any $h_0 \in \mathbb{R}$*

$$ess - \sup_{Q_{\frac{R}{2}}(x)} \left(\frac{v - h_0}{R}\right) \leq 2\Phi^{-1}\left(c_M\left(\frac{V(h_0, R)}{R^N}\right)^{\alpha}\left(\frac{1}{R^N}\int_{A(h_0,R)} \Phi\left(\frac{v - h_0}{R}\right) dy\right)\right) \tag{5.9}$$

where $v = u - \chi R^{\frac{N\epsilon}{m}}$, $h_0 = k_0 - \chi R^{\frac{N\epsilon}{m}}$, $V(h_0, R) = \mathcal{L}^N(A(h_0, R))$ and $\alpha = \frac{-1+\sqrt{5}}{2}$.

Proof. Let us consider the following sequences:

$$\begin{aligned} k_0 &= d \\ k_{i+1} &= k_i + \Phi^{-1}\left(\tfrac{\Phi(d)}{2^{im}}\right) \quad \text{for } i \geq 1 \end{aligned}$$

and

$$r_i = \frac{R}{2}\left(1 + 2^{-i}\right) \text{ for } i \in \mathbb{N}.$$

Let us define

$$\vartheta(k, r) = (V(k, r))^{\alpha}\int_{A(k,r)} \Phi\left(\frac{v - k}{R}\right) dx; \tag{5.10}$$

and $\vartheta_i = \vartheta(k_i, r_i)$. From relations (5.1) and (5.10) we obtain

$$\vartheta_{i+1} \leq c_{18} 2^{(n+m\alpha)i} R^{-N} (\Phi(d))^{-\alpha} (\vartheta_i)^{1+\alpha}$$

where $\alpha = \frac{-1+\sqrt{5}}{2}$ is a solution of $\alpha = \frac{1}{1+\alpha}$; remembering that

$$\begin{aligned} \vartheta_0 &= |A(d, R)|^{\alpha} \int_{A(d,R)} \Phi\left(\tfrac{v-d}{R}\right) dx \\ &\leq |A(0, R)|^{\alpha} \int_{A(0,R)} \Phi\left(\tfrac{v}{R}\right) dx \end{aligned}$$

and considering

$$|A(0, R)|^{\alpha} \int_{A(0,R)} \Phi(v)\, dx \leq \frac{1}{(c_{14})^{\alpha}\, 2^{\frac{1+\alpha}{\alpha^2}}} R^{\frac{N}{\alpha}} \Phi(d),$$

that is

$$d \geq \Phi^{-1}\left(C(2, \alpha) R^{-\frac{N}{\alpha}} |A(0, R)|^{\alpha} \int_{A(0,R)} \Phi\left(\frac{v}{R}\right) dx\right)$$

where $C(2, \alpha) = (c_{18})^{\alpha}\, 2^{\frac{1+\alpha}{\alpha^2}}$, then from Lemma 8 it comes

$$\lim_{i \to +\infty} \vartheta_i = 0. \tag{5.11}$$

Given that $\Phi^{-1}(a^m t) \leq a\Phi^{-1}(t)$ for each $t > 0$ and $a < 1$ (Refer [Mascolo et al., 1996]) then we have

$$k_i = d + \sum_{j=0}^{i} \Phi^{-1}\left(\frac{\Phi(d)}{2^{mi}}\right) \leq 2d$$

so, for each $i > 0$, we obtain $A\left(2d, \frac{R}{2}\right) \subset A(k_i, r_i)$ and

$$\left|A\left(2d, \frac{R}{2}\right)\right|^{\alpha} \int_{A\left(2d, \frac{R}{2}\right)} \Phi\left(\frac{v - 2d}{R}\right) dx \leq \vartheta_i.$$

From relation (5.11) it follows

$$ess - \sup_{Q_{\frac{R}{2}}} \left\{ \frac{v}{R} \right\} \leq 2d$$

given

$$d = \Phi^{-1} \left(C(2,\alpha) R^{-\frac{N}{\alpha}} |A(0,R)|^{\alpha} \int_{A(0,R)} \Phi\left(\frac{v}{R}\right) dx \right)$$

we have

$$ess - \sup_{Q_{\frac{R}{2}}} \left\{ \frac{v}{R} \right\} \leq 2\Phi^{-1} \left(C(2,\alpha) R^{-\frac{N}{\alpha}} |A(0,R)|^{\alpha} \int_{A(0,R)} \Phi\left(\frac{v}{R}\right) dx \right).$$

Since $\alpha = \frac{-1+\sqrt{5}}{2}$ then $R^{\frac{N}{\alpha}} = R^{N\frac{\sqrt{5}+1}{2}} = R^{N+N\alpha}$, consequently we obtain

$$ess - \sup_{Q_{\frac{R}{2}}} \left\{ \frac{v}{R} \right\} \leq 2\Phi^{-1} \left(C(2,\alpha) R^{-N} \left(\frac{|A(0,R)|}{R^N} \right)^{\alpha} \int_{A(0,R)} \Phi\left(\frac{v}{R}\right) dx \right)$$

and (6.9). □

Now we can prove Theorem 1.

Proof. (Proof of Theorem 1) It follows using Theorem 9 and Teorem 10. □

6. Decay Lemma

We divide this paragraph into two parts.

Moreover we suppose $0 < R \leq \min(R_0, d(x_0, \partial\Omega), 1)$ and

$$osc(u,r) = M(r) - m(r)$$

where $M(r) = M(u,r) = \sup_{Q_r}(u)$ and $m(r) = m(u,r) = \inf_{Q_r}(u)$.

6.1 Case One

If $\Phi \in \triangle_2 \cap \nabla_2$ then the following theorem hold.

Lemma 11. *We assume that u is a bounded function and (4.33) is valid for every $k \in \mathbb{R}$. We set $2k_0 = M(2R) + m(2R)$ and we assume that $|A(k_0,R)| \leq \gamma |Q_R|$ for some $\gamma \in (0,1)$. If for some integer v we have*

$$osc(u,2R) \geq 2^{v+1} \chi \Phi^{-1}\left(R^{\epsilon N}\right)$$

then, if $k_v = M(2R) - 2^{-v-1} osc(u,2R)$, we obtain

$$|A(k_v,R)| \leq \left(\frac{c_{17}}{\sqrt{v}} + c_{18} \frac{1}{\sqrt[2\bar{m}]{v}} \right) R^N$$

where c_{17} and c_{18} are universal constants.

Proof. We consider $v \in \mathbb{N}$, then we define

$$\begin{aligned} k_i &= M(2R) - 2^{-i-1} osc(u,2R) \quad \text{for } i = 1,...,v \\ h_i &= M(2R) - 2^{-i} osc(u,2R) \quad \text{for } i = 1,...,v \end{aligned}$$

and

$$v_i(x) = \begin{cases} \frac{k_i - h_i}{R} & \text{if } u > k_i \\ \frac{u(x)-h_i}{R} & \text{if } h_i < u \leq k_i \\ 0 & \text{if } u \leq h_i \end{cases} \quad \text{for } i = 1,...,v.$$

In particular, we have that $\Phi(v_i(x)) \in W^{1,1}(Q_R)$ and $v_i = 0$ on $Q_R \backslash A(R, \kappa_0)$ where $k_0 = h_1 = \frac{M(2R)+m(2R)}{2}$ for all $i = 1, ..., \nu$. Using the Hölder Inequality we have

$$\begin{aligned} |A(k_i, R)|\,\Phi\left(\tfrac{k_i-k_{i-1}}{R}\right) &\leq \int\limits_{A(k_{i-1},R)} \Phi(v_i)\, dx \\ &\leq |A(k_{i-1}, R)|^{\frac{1}{N}} \left[\int\limits_{A(k_{i-1},R)} [\Phi(v_i)]^{\frac{N}{N-1}}\, dx\right]^{\frac{N-1}{N}}. \end{aligned}$$

Since $\Phi(v_i) = 0$ on $Q_R \backslash A(R, \kappa_0)$ where $k_0 = h_1 = \frac{M(2R)+m(2R)}{2}$ for all $i = 1, ..., \nu$, using the Sobolev Inequality it follows

$$|A(k_i, R)|\,\Phi\left(\tfrac{k_i-k_{i-1}}{R}\right) \leq c_{SN} \int\limits_{\Delta_i} \dot{\Phi}(u - k_{i-1})\,|\nabla u|\, dx.$$

For every $\varepsilon > 0$,

$$\int\limits_{\Delta_i} \dot{\Phi}(u - k_{i-1})\,|\nabla u|\, dx = m \int\limits_{\Delta_i} \frac{\dot{\Phi}(u - k_{i-1})}{\varepsilon m}\varepsilon\,|\nabla u|\, dx$$

then, since Φ is a N-function from the Young inequality $ab \leq \widetilde{\Phi}(a) + \Phi(b)$, we get

$$|A(k_i, R)|\,\Phi(k_i - k_{i-1}) \quad \leq mc_{SN} \int\limits_{\Delta_i} \widetilde{\Phi}\left(\tfrac{\dot{\Phi}(u-k_{i-1})}{\varepsilon m}\right) dx + mc_{SN} \int\limits_{\Delta_i} \Phi(\varepsilon\,|\nabla u|)\, dx.$$

Since $\Phi \in \triangle_2 \cap \nabla_2$ then $\widetilde{\Phi} \in \triangle_2 \cap \nabla_2$ and

$$\begin{aligned} \widetilde{\Phi}\left(\tfrac{\dot{\Phi}(u-k_{i-1})}{\varepsilon m}\right) &\leq \tfrac{1}{\varepsilon^{\tilde{m}}}\widetilde{\Phi}\left(\tfrac{(u-k_{i-1})\dot{\Phi}(u-k_{i-1})}{m(u-k_{i-1})}\right) \\ &\leq \widetilde{\Phi}\left(\tfrac{\Phi(u-k_{i-1})}{(u-k_{i-1})}\right) \end{aligned}$$

from the inequality

$$\widetilde{\Phi}\left(\frac{\Phi(t)}{t}\right) < \Phi(t)$$

(see inequality (6), page 230 of [Adams, 1975]) we have

$$\begin{aligned} |A(k_i, R)|\,\Phi\left(\frac{k_i - k_{i-1}}{R}\right) \quad &\leq \quad \frac{mc_{SN}}{\varepsilon} R \int\limits_{\Delta_i} \Phi\left(\frac{u - k_{i-1}}{R}\right) dx + \\ &+ mc_{SN}c_4^2\varepsilon \int\limits_{\Delta_i} \Phi(|\nabla u|)\, dx. \end{aligned}$$

Using the Caccioppoli's inequality (4.33) we obtain

$$\begin{aligned} |A(k_i, R)|\,\Phi\left(\tfrac{k_i-k_{i-1}}{R}\right) &\leq \tfrac{c_{SN}m}{\varepsilon^{\tilde{m}}}\,|\Delta_i|\,\Phi\left(\tfrac{osc(u,2R)}{2^i R}\right) + \\ &+ mc_{SN}c_4^2\varepsilon\,(H + \chi)\,\Phi\left(\tfrac{osc(u,2R)}{2^i R}\right) R^N \end{aligned}$$

and

$$|A(k_\nu, R)| \leq 2^m \left(\frac{c_{SN}m}{\varepsilon^{\tilde{m}}}\,|\Delta_i| + mc_{SN}c_4^2\varepsilon\,(H + \chi)\,R^N\right). \tag{6.1}$$

Using (6.1) we have

$$\begin{aligned} \nu\,|A(k_\nu, R)| &\leq 2^m \left(\tfrac{c_{SN}m}{\varepsilon^{\tilde{m}}} \sum_{i=1}^{\nu} |\Delta_i| + mc_{SN}c_4^2\varepsilon\,(H + \chi)\,R^N \nu\right) \\ &\leq 2^m \tfrac{c_{SN}m}{\varepsilon^{\tilde{m}}}\,|A(k_0, R)| + \nu 2^m mc_{SN}c_4^2\varepsilon\,(H + \chi)\,R^N \end{aligned}$$

and

$$|A(k_\nu, R)| \leq 2^m \frac{c_{SN}m}{\nu\varepsilon^{\tilde{m}}}\,|A(k_0, R)| + 2^m mc_{SN}c_4^2\varepsilon\,(H + \chi)\,R^N.$$

Fixed $\varepsilon = \nu^{-\frac{1}{2\tilde{m}}}$ it follows

$$|A(k_\nu, R)| \leq \left(\frac{2^m c_{SN}m}{\sqrt{\nu}} + mc_{SN}c_4^2\,(H + \chi)\,2^m \frac{1}{\sqrt[2\tilde{m}]{\nu}}\right) R^N.$$

□

6.2 Case Two

If Φ satisfies the hypotheses H-1, H-2 and H-3 then the following theorem hold.

Lemma 12. *We assume that u is a bounded function and (4.33) is valid for every $k \in \mathbb{R}$. We set $2k_0 = M(2R) + m(2R)$ and we assume that $|A(k_0, R)| \leq \gamma |Q_R|$ for some $\gamma \in (0,1)$. If for some integer ν we have*

$$osc(u, 2R) \geq 2^{\nu+1} \chi \Phi^{-1}\left(R^{\epsilon N}\right)$$

then, if $k_\nu = M(2R) - 2^{-\nu-1} osc(u, 2R)$, we obtain

$$|A(k_\nu, R)| \leq \left(\frac{c_{17}}{\sqrt{\nu}} + c_{18}\dot{\Phi}\left(\frac{1}{\sqrt{\nu}}\right)\right) R^N$$

where c_{17} and c_{18} are universal constants.

Proof. We consider $\nu \in \mathbb{N}$, then we define

$$\begin{aligned} k_i &= M(2R) - 2^{-i-1} osc(u, 2R) \quad \text{for } i = 1, ..., \nu \\ h_i &= M(2R) - 2^{-i} osc(u, 2R) \quad \text{for } i = 1, ..., \nu \end{aligned}$$

and

$$v_i(x) = \begin{cases} k_i - h_i & \text{if } u > k_i \\ u(x) - h_i & \text{if } h_i < u \leq k_i \\ 0 & \text{if } u \leq h_i \end{cases} \quad \text{for } i = 1, ..., \nu.$$

In particular, we have that $\Phi(v_i(x)) \in W^{1,1}(Q_R)$ and $v_i = 0$ on $Q_R \backslash A(R, \kappa_0)$ where $k_0 = h_1 = \frac{M(2R)+m(2R)}{2}$ for all $i = 1, ..., \nu$. Using the Hölder Inequality we have

$$\begin{aligned} |A(k_i, R)| \Phi(k_i - k_{i-1}) &\leq \int_{A(k_{i-1},R)} \Phi(v_i)\, dx \\ &\leq |A(k_{i-1}, R)|^{\frac{1}{N}} \left[\int_{A(k_{i-1},R)} [\Phi(v_i)]^{\frac{N}{N-1}}\, dx\right]^{\frac{N-1}{N}}. \end{aligned}$$

Since $\Phi(v_i) = 0$ on $Q_R \backslash A(R, \kappa_0)$ where $k_0 = h_1 = \frac{M(2R)+m(2R)}{2}$ for all $i = 1, ..., \nu$, using the Sobolev Inequality it follows

$$|A(k_i, R)| \Phi(k_i - k_{i-1}) \leq c_{SN} \int_{\Delta_i} \dot{\Phi}(u - k_{i-1}) R |\nabla u|\, dx.$$

For every $\varepsilon > 0$,

$$\int_{\Delta_i} \dot{\Phi}(u - k_{i-1}) R |\nabla u|\, dx = \frac{m}{\varepsilon} \int_{\Delta_i} \frac{\dot{\Phi}(u - k_{i-1})}{m} \varepsilon R |\nabla u|\, dx$$

then, since Φ is a N-function from the Young inequality $ab \leq \widetilde{\Phi}(a) + \Phi(b)$, we get

$$|A(k_i, R)| \Phi(k_i - k_{i-1}) \leq \frac{mc_{SN}}{\varepsilon} \int_{\Delta_i} \widetilde{\Phi}\left(\frac{\dot{\Phi}(u-k_{i-1})}{m}\right) dx + \frac{mc_{SN}}{\varepsilon} \int_{\Delta_i} \Phi(\varepsilon R |\nabla u|)\, dx.$$

Since

$$\begin{aligned} \widetilde{\Phi}\left(\frac{\dot{\Phi}(u-k_{i-1})}{m}\right) &\leq \widetilde{\Phi}\left(\frac{(u-k_{i-1})\dot{\Phi}(u-k_{i-1})}{m(u-k_{i-1})}\right) \\ &\leq \widetilde{\Phi}\left(\frac{\Phi(u-k_{i-1})}{(u-k_{i-1})}\right) \end{aligned}$$

from the inequality

$$\widetilde{\Phi}\left(\frac{\Phi(t)}{t}\right) < \Phi(t)$$

(see inequality (6), page 230 of [Adams, 1975]) we have

$$\begin{aligned} |A(k_i, R)| \Phi(k_i - k_{i-1}) \leq{}& \frac{mc_{SN}}{\varepsilon} \int_{\Delta_i} \Phi(u - k_{i-1})\, dx + \\ &+ \frac{mc_{SN}c_4^2}{\varepsilon} \Phi(\varepsilon) \Phi(R) \int_{\Delta_i} \Phi(|\nabla u|)\, dx. \end{aligned}$$

Using the Caccioppoli's inequality (4.33) we obtain

$$\begin{aligned} |A(k_i,R)|\,\Phi(k_i-k_{i-1}) &\le \tfrac{c_{SN}m}{\varepsilon}|\Delta_i|\,\Phi\left(\tfrac{osc(u,2R)}{2^i}\right)+\\ &+\tfrac{mc_{SN}c_4^2}{\varepsilon}\Phi(\varepsilon)\,\Phi(R)\,\Phi\left(\tfrac{1}{R}\right)(H+\chi)\,\Phi\left(\tfrac{osc(u,2R)}{2^i}\right)R^N \end{aligned}$$

and

$$|A(k_\nu,R)|\le 2^m\left(\frac{c_{SN}m}{\varepsilon}|\Delta_i|+m\frac{c_{SN}c_4^2}{\varepsilon}\Phi(\varepsilon)\,\Phi(R)\,\Phi\left(\frac{1}{R}\right)(H+\chi)\,R^N\right). \tag{6.2}$$

Using (6.2) we have

$$\begin{aligned} \nu\,|A(k_\nu,R)| &\le 2^m\left(\tfrac{c_{SN}m}{\varepsilon}\sum_{i=1}^{\nu}|\Delta_i|+m\tfrac{c_{SN}c_4^2}{\varepsilon}\Phi(\varepsilon)\,\Phi(R)\,\Phi\left(\tfrac{1}{R}\right)(H+\chi)\,R^N\nu\right)\\ &\le 2^m\tfrac{c_{SN}m}{\varepsilon}|A(k_0,R)|+\nu\tfrac{2^m mc_{SN}c_4^2}{\varepsilon}\Phi(\varepsilon)\,\Phi(R)\,\Phi\left(\tfrac{1}{R}\right)(H+\chi)\,R^N \end{aligned}$$

and

$$|A(k_\nu,R)|\le 2^m\frac{c_{SN}m}{\nu\varepsilon}|A(k_0,R)|+\frac{2^m mc_{SN}c_4^2}{\varepsilon}\Phi(\varepsilon)\,\dot{\Phi}(R)\,\dot{\Phi}\left(\frac{1}{R}\right)(H+\chi)\,R^N.$$

Since, for the hypotesis H2, $\dot{\Phi}(R)\,\dot{\Phi}\left(\frac{1}{R}\right)\le c_{H_2}$ we get

$$|A(k_\nu,R)|\le 2^m\frac{c_{SN}m}{\nu\varepsilon}|A(k_0,R)|+mc_{SN}c_4^2\dot{\Phi}(\varepsilon)\,c_{H_2}(H+\chi)\,2^mR^N.$$

Fixed $\varepsilon=\nu^{-\frac{1}{2}}$ it follows

$$|A(k_\nu,R)|\le\left(\frac{2^m c_{SN}m}{\sqrt{\nu}}+mc_{SN}c_4^2c_{H_2}(H+\chi)\,2^m\dot{\Phi}\left(\frac{1}{\sqrt{\nu}}\right)\right)R^N.$$

□

7. Proof of the Regularity Theorem

Let us start remembering the following lemma:

Lemma 13. *Let $\varphi(t)$ be a positive and increasing function. We assume that there exists a number $\tau\in(0,1)$ such that*

$$\varphi(\tau R)\le\tau^\delta\varphi(R)+BR^\beta,\ \textit{for}\ 0<R<R_0$$

with $0<\beta<\delta$ and $B\ge 0$. Then for every $\varrho<R<R_0$ we get

$$\varphi(\varrho)\le C\left[\left(\frac{\varrho}{R}\right)^\beta\varphi(R)+B\varrho^\beta\right]$$

where $C>0$ is a constant depending only on τ, δ and β.

Proof. See Lemma 7.3 of (Giusti, 1994). □

7.1 Case One

If $\Phi\in\triangle_2\cap\nabla_2$ then our proof is founded upon the following result:

Theorem 11. *If u is a bounded function such that (4.33) and (4.34) hold for every $k\in\mathbb{R}$; then u is locally hölder continuous on Ω.*

Proof. Let us consider $0<R\le\min(R_0,d(x_0,\partial\Omega),1)$ and $2k_0=M(2R)+m(2R)$, then, without losing generality, we can assume that $|A(k_0,R)|\le\frac{1}{2}|Q_R|$ since otherwise we would have $|B(k_0,R)|=|Q_R|-|A(k_0,R)|\le\frac{1}{2}|Q_R|$ and

we can consider $-u$ instead of u. Let us consider $k_\nu = M(2R) - 2^{-\nu-1} osc(u, 2R)$, where ν is an integer that we fix later in appropriate way, then we have $k_\nu > k_0$ and from (1.4) it follows

$$\begin{aligned}\sup_{B_{\frac{R}{2}}}(u-k_\nu) &\leq 2R\Phi^{-1}\left(\frac{c_8}{R^N}\left(\frac{|A(k_\nu,R)|}{R^N}\right)^\alpha \int_{A(k_\nu,R)} \Phi\left(\frac{(u-k_\nu)_+}{R}\right)dx\right)+\\ &\quad +2\Phi^{-1}\left(R^{N\epsilon}\right).\end{aligned}$$

Using Lemma 11, if

$$osc(u,2R) \geq 2^{\nu+1}\chi\Phi^{-1}\left(R^{N\epsilon}\right)$$

for some integer ν, then we get

$$|A(k_\nu,R)| \leq \left(\frac{c_{17}}{\sqrt{\nu}} + c_{18}\frac{1}{\sqrt[2\bar{m}]{\nu}}\right)R^N$$

and

$$\begin{aligned}\sup_{B_{\frac{R}{2}}}(u-k_\nu) &\leq 2R\Phi^{-1}\left(\frac{c_8}{R^N}\left(\left(\frac{c_{17}}{\sqrt{\nu}} + c_{18}\frac{1}{\sqrt[2\bar{m}]{\nu}}\right)R^N\right)^\alpha \int_{A(k_\nu,R)} \Phi\left(\frac{(u-k_\nu)_+}{R}\right)dx\right)+\\ &\quad +c_{19}\Phi^{-1}\left(R^{N\epsilon}\right),\end{aligned}$$

where $c_{19} = 2\Phi^{-1}(1)$. Let us take $c_{20} = \max\{c_{17}, c_{18}\}$ and $\zeta(\nu) = \max\left\{\frac{1}{\sqrt{\nu}}, \frac{1}{\sqrt[2\bar{m}]{\nu}}\right\}$ then $\frac{c_{17}}{\sqrt{\nu}} + c_{18}\frac{1}{\sqrt[2\bar{m}]{\nu}} \leq 2c_{20}\zeta(\nu)$. Now we consider $2c_{20}\zeta(\nu) \leq \left(\frac{1}{2^{2m}C(N,\alpha)}\right)^{\frac{1}{\alpha}}$ and we get

$$\zeta(\nu) \leq \frac{1}{2c_{20}}\left(\frac{1}{2^{2m}C(N,\alpha)}\right)^{\frac{1}{\alpha}} \tag{7.1}$$

From the relation (7.1) it follows $\frac{1}{\sqrt{\nu}} \leq \frac{1}{2c_{20}}\left(\frac{1}{2^{2m}C(N,\alpha)}\right)^{\frac{1}{\alpha}}$ or $\dot{\Phi}\left(\frac{1}{\sqrt{\nu}}\right) \leq \frac{1}{2c_{20}}\left(\frac{1}{2^{2m}C(N,\alpha)}\right)^{\frac{1}{\alpha}}$ then

$$\left(2c_{20}\left(2^{2m}C(N,\alpha)\right)^{\frac{1}{\alpha}}\right)^2 \leq \nu$$

or, since $\dot{\Phi}$ is incresing and invertible,

$$\left(\dot{\Phi}^{-1}\left(\frac{1}{2c_{20}}\left(\frac{1}{2^{2m}C(N,\alpha)}\right)^{\frac{1}{\alpha}}\right)\right)^{-2} \leq \nu.$$

Then we choose

$$\nu \geq \max\{A+1, B+1\},$$

where $A = \left[\left(\dot{\Phi}^{-1}\left(\frac{1}{2c_{20}}\left(\frac{1}{2^{2m}C(N,\alpha)}\right)^{\frac{1}{\alpha}}\right)\right)^{-2}\right]$ and $B = \left[\left(2c_{20}\left(2^{2m}C(N,\alpha)\right)^{\frac{1}{\alpha}}\right)^2\right]$ are the integer part of $\left(\dot{\Phi}^{-1}\left(\frac{1}{2c_{20}}\left(\frac{1}{2^{2m}C(N,\alpha)}\right)^{\frac{1}{\alpha}}\right)\right)^{-2}$ and $\left(2c_{20}\left(2^{2m}C(N,\alpha)\right)^{\frac{1}{\alpha}}\right)^2$. If $osc(u,2R) \geq 2^{\nu+1}\chi R$, since $\Phi^{-1}(a^m t) \leq a\Phi^{-1}(t)$ for every $t > 0$ and every $a < 1$ (Refer [Mascolo et al., 1996]), we have

$$\begin{aligned}\sup_{B_{\frac{R}{2}}}(u-k_\nu) &\leq 2R\Phi^{-1}\left(\frac{1}{2^{2m}R^N}\int_{A(k_\nu,R)} \Phi\left(\frac{(u-k_\nu)_+}{R}\right)dx\right) + c_{19}\Phi^{-1}\left(R^{N\epsilon}\right)\\ &\leq \frac{1}{2}R\Phi^{-1}\left(\frac{1}{R^N}\int_{A(k_\nu,R)} \Phi\left(\frac{(u-k_\nu)_+}{R}\right)dx\right) + c_{19}\Phi^{-1}\left(R^{N\epsilon}\right)\\ &\leq \frac{1}{2}\sup_{B_R}(u-k_\nu) + c_{19}\Phi^{-1}\left(R^{N\epsilon}\right)\\ &\leq \frac{1}{2}\sup_{B_{2R}}(u-k_\nu) + c_{19}\Phi^{-1}\left(R^{N\epsilon}\right)\end{aligned}$$

from which, using the hypothesis H3, it follows

$$osc\left(u,\frac{R}{2}\right)\leq\left(1-\frac{1}{2^{\nu+2}}\right)osc\,(u,2R)\,.$$

In conclusion, either the function $osc\,(u,R)$ verifies the previous relationship, or else

$$osc\,(u,2R)\leq 2^{\nu+1}\chi\Phi^{-1}\left(R^{N\epsilon}\right).$$

Anyway we have

$$osc\left(u,\frac{R}{2}\right)\leq\left(1-\frac{1}{2^{\nu+2}}\right)osc\,(u,2R)+c_{20}2^{\nu}\chi R^{\frac{N\epsilon}{m}}.$$

Let us consider $\tau=\frac{1}{4}$ and $\delta=\log_{\tau}\left(1-\frac{1}{2^{\nu+2}}\right)$ then we have

$$osc\left(u,\frac{R}{2}\right)\leq\left(1-\frac{1}{2^{\nu+2}}\right)osc\,(u,2R)+c_{20}2^{\nu}\chi R^{\gamma}$$

with $\gamma<\min\left\{\delta,\frac{N\epsilon}{m}\right\}$. Now we can apply Lemma 13 and we have

$$osc\,(u,\varrho)\leq c_{21}\left[\left(\frac{\varrho}{R}\right)^{\gamma}osc\,(u,R)+c_{22}\chi\varrho^{\gamma}\right]$$

for every $0<\varrho<R<\min\left(1,dist(x_0,\partial\Omega)\right)$. □

From the previous theorem we conclude with the proof of the Theorem 2.

Proof. (Proof of Theorem 2) It comes from Theorem 11 since quasi-minima confirm relation (4.33) and (4.34). □

7.2 Case Two

If Φ satisfies the hypotheses H-1, H-2 and H-3 then our proof is founded upon the following result:

Theorem 12. *If u is a bounded function such that (4.33) and (4.34) hold for every $k\in\mathbb{R}$; then u is locally hölder continuous on Ω.*

Proof. Let us consider $0<R\leq\min\left(R_0,d\left(x_0,\partial\Omega\right),1\right)$ and $2k_0=M\,(2R)+m\,(2R)$, then, without losing generality, we can assume that $|A\,(k_0,R)|\leq\frac{1}{2}|Q_R|$ since otherwise we would have $|B\,(k_0,R)|=|Q_R|-|A\,(k_0,R)|\leq\frac{1}{2}|Q_R|$ and we can consider $-u$ instead of u. Let us consider $k_\nu=M\,(2R)-2^{-\nu-1}osc\,(u,2R)$, where ν is an integer that we fix later in appropriate way, then we have $k_\nu>k_0$ and from (1.4) it follows

$$\begin{aligned}\sup_{B_{\frac{R}{2}}}(u-k_\nu)\quad&\leq\quad 2\Phi^{-1}\left(\frac{c_8}{R^N}\left(\frac{|A\,(k_\nu,R)|}{R^N}\right)^{\alpha}\int\limits_{A(k_\nu,R)}\Phi\left((u-k_\nu)_+\right)\,dx\right)+\\&\quad +2\Phi^{-1}\left(R^{N\epsilon}\right).\end{aligned}$$

Using Lemma 12 if

$$osc\,(u,2R)\geq 2^{\nu+1}\chi\Phi^{-1}\left(R^{N\epsilon}\right)$$

for some integer ν, then we get

$$|A\,(k_\nu,R)|\leq\left(\frac{c_{17}}{\sqrt{\nu}}+c_{18}\dot{\Phi}\left(\frac{1}{\sqrt{\nu}}\right)\right)R^N$$

and

$$\begin{aligned}\sup_{B_{\frac{R}{2}}}(u-k_\nu)\quad&\leq\quad 2\Phi^{-1}\left(\frac{c_8}{R^N}\left(\frac{c_{17}}{\sqrt{\nu}}+c_{18}\dot{\Phi}\left(\frac{1}{\sqrt{\nu}}\right)\right)^{\alpha}\int\limits_{A(k_\nu,R)}\Phi\left((u-k_\nu)_+\right)\,dx\right)+\\&\quad +c_{19}\Phi^{-1}\left(R^{N\epsilon}\right)\end{aligned}$$

where $c_{19} = 2\Phi^{-1}(1)$. Let us take $c_{20} = \max\{c_{17}, c_{18}\}$ and $\zeta(\nu) = \max\left\{\frac{1}{\sqrt{\nu}}, \dot{\Phi}\left(\frac{1}{\sqrt{\nu}}\right)\right\}$ then $\frac{c_{17}}{\sqrt{\nu}} + c_{18}\dot{\Phi}\left(\frac{1}{\sqrt{\nu}}\right) \leq 2c_{20}\zeta(\nu)$. Now we consider $2c_{20}\zeta(\nu) \leq \left(\frac{1}{2^{2m}C(N,\alpha)}\right)^{\frac{1}{\alpha}}$ and we get

$$\zeta(\nu) \leq \frac{1}{2c_{20}}\left(\frac{1}{2^{2m}C(N,\alpha)}\right)^{\frac{1}{\alpha}} \tag{7.2}$$

From the relation (7.2) it follows $\frac{1}{\sqrt{\nu}} \leq \frac{1}{2c_{20}}\left(\frac{1}{2^{2m}C(N,\alpha)}\right)^{\frac{1}{\alpha}}$ or $\dot{\Phi}\left(\frac{1}{\sqrt{\nu}}\right) \leq \frac{1}{2c_{20}}\left(\frac{1}{2^{2m}C(N,\alpha)}\right)^{\frac{1}{\alpha}}$ then

$$\left(2c_{20}\left(2^{2m}C(N,\alpha)\right)^{\frac{1}{\alpha}}\right)^2 \leq \nu$$

or, since $\dot{\Phi}$ is incresing and invertible,

$$\left(\dot{\Phi}^{-1}\left(\frac{1}{2c_{20}}\left(\frac{1}{2^{2m}C(N,\alpha)}\right)^{\frac{1}{\alpha}}\right)\right)^{-2} \leq \nu.$$

Then we choose

$$\nu \geq \max\{C+1, D+1\},$$

where $C = \left[\left(\dot{\Phi}^{-1}\left(\frac{1}{2c_{20}}\left(\frac{1}{2^{2m}C(N,\alpha)}\right)^{\frac{1}{\alpha}}\right)\right)^{-2}\right]$ and $D = \left[\left(2c_{20}\left(2^{2m}C(N,\alpha)\right)^{\frac{1}{\alpha}}\right)^2\right]$ are the integer part of $\left(\dot{\Phi}^{-1}\left(\frac{1}{2c_{20}}\left(\frac{1}{2^{2m}C(N,\alpha)}\right)^{\frac{1}{\alpha}}\right)\right)^{-2}$ and $\left(2c_{20}\left(2^{2m}C(N,\alpha)\right)^{\frac{1}{\alpha}}\right)^2$. If $osc(u, 2R) \geq 2^{\nu+1}\chi R$, since $\Phi^{-1}(a^m t) \leq a\Phi^{-1}(t)$ for every $t > 0$ and every $a < 1$ (Refer [Mascolo et al., 1996]), we have

$$\begin{aligned}
\sup_{B_{\frac{R}{2}}}(u - k_\nu) &\leq 2\Phi^{-1}\left(\frac{1}{2^{2m}R^N}\int_{A(k_\nu,R)}\Phi\left((u-k_\nu)_+\right)dx\right) + c_{19}\Phi^{-1}\left(R^{N\epsilon}\right)\\
&\leq \tfrac{1}{2}\Phi^{-1}\left(\frac{1}{R^N}\int_{A(k_\nu,R)}\Phi\left((u-k_\nu)_+\right)dx\right) + c_{19}\Phi^{-1}\left(R^{N\epsilon}\right)\\
&\leq \tfrac{1}{2}\sup_{B_R}(u-k_\nu) + c_{19}\Phi^{-1}\left(R^{N\epsilon}\right)\\
&\leq \tfrac{1}{2}\sup_{B_{2R}}(u-k_\nu) + c_{19}\Phi^{-1}\left(R^{N\epsilon}\right)
\end{aligned}$$

from which, using the hypothesis H3, it follows

$$osc\left(u, \frac{R}{2}\right) \leq \left(1 - \frac{1}{2^{\nu+2}}\right) osc(u, 2R).$$

In conclusion, either the function $osc(u, R)$ verifies the previous relationship, or else

$$osc(u, 2R) \leq 2^{\nu+1}\chi\Phi^{-1}\left(R^{N\epsilon}\right).$$

Anyway we have

$$osc\left(u, \frac{R}{2}\right) \leq \left(1 - \frac{1}{2^{\nu+2}}\right) osc(u, 2R) + c_{20}2^\nu\chi R^{\frac{N\epsilon}{m}}.$$

Let us consider $\tau = \frac{1}{4}$ and $\delta = \log_\tau\left(1 - \frac{1}{2^{\nu+2}}\right)$ then we have

$$osc\left(u, \frac{R}{2}\right) \leq \left(1 - \frac{1}{2^{\nu+2}}\right) osc(u, 2R) + c_{20}2^\nu\chi R^\gamma$$

with $\gamma < \min\left\{\delta, \frac{N\epsilon}{m}\right\}$. Now we can apply Lemma 13 and we have

$$osc(u, \varrho) \leq c_{21}\left[\left(\frac{\varrho}{R}\right)^\gamma osc(u, R) + c_{22}\chi\varrho^\gamma\right]$$

for every $0 < \varrho < R < \min(1, dist(x_0, \partial\Omega))$. □

From the previous theorem we conclude with the proof of the Theorem 3.

Proof of the Main Theorem. (Proof of Theorem 3) It comes from Theorem 15 since quasi-minima confirm relation (4.33) and (4.34). □

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