Journal of Mathematics Research; Vol. 7, No. 3; 2015
ISSN 1916-9795 E-ISSN 1916-9809
Published by Canadian Center of Science and Education

Multiple-soliton Solutions for Nonlinear Partial Differential Equations

Yaning Tang[1], & Weijian Zai[1]

[1]Department of Applied Mathematics, Northwestern Polytechnical University, Xi'an, China

Correspondence: Yaning Tang, Department of Applied Mathematics, Northwestern Polytechnical University, Xi'an, China. E-mail: tyaning@nwpu.edu.cn

Received: May 20, 2015 Accepted: June 10, 2015 Online Published: July 11, 2015

doi:10.5539/ijsp.v4n3p75 URL: http://dx.doi.org/10.5539/ijsp.v4n3p75

Abstract

Based on the scale transformation and the multiple exp-function method, the (3+1)-dimensional Boiti-Leon-Manna-Pempinelli (BLMP) equation and a generalized Shallow Water Equation have been solved. The exponential wave solutions which include one-wave, two-wave and three-wave solutions have been obtained. In addition, by comparing the solutions obtained in this paper with those solved in the references, we find that our results are more general.

Keywords: Multiple exp-function method, BLMP equation, generalized Shallow Water Equation, multiple-soliton solutions

1. Introduction

Nonlinear partial differential equations (NLPDEs) have a vital role in almost all the branches of physics, for example optical waveguide fibers, fluid mechanics and plasma physics. What's more, NLPDEs are very significant in applied mathematics and chemistry. If we need to know the essence of phenomena that have showed by NLPDEs, exact solutions of these equations have to be studied. The study of solutions of NLPDEs plays a major part in soliton theory and nonlinear science.

The solutions can explain various phenomena in many scientific fields. In recent years, many approaches for solving exact traveling and solitary wave solutions to NLPDEs have been investigated. Among these methods, the Painlevé analysis (Hong, 2010), the Darboux transformation (DT) (Gu & Zhou, 1994), the inverse scattering transformation (Vakhnenko, Parkes, & Morrison, 2003), the sine–cosine method (Yan, 1996), the Bäcklund transformation (BT) (Nimmo, 1983), the extended tanh method (Fan, 2000), the variable separation approach (Hu, Tang, Lou, & Liu, 2004), the homogeneous balance method (HBM) (Fan, 1998), the Wronskian technique (Ma, 2004) are effective and general. However, these methods are mostly involved traveling wave solutions for NLPDEs. As we all know, there are multiple wave solutions to NLPDEs, for example, multiple periodic wave solutions for the Hirota bilinear equations (Ma, Zhou, & Gao, 2009) and so on. The multiple exp-function method (Ma, Huang, & Zhang, 2010) has been proposed and it offers a straightforward and efficient way to solve multiple wave solutions of NLPDEs (Ma et al., 2010, Su, Tang, & Zhao, 2013, Ma & Zhu, 2012). In this paper, based on the scale transformation and this method, the multiple soliton solutions to the (3+1)-dimensional BLMP equation and a generalized Shallow Water Equation are investigated. It is worth pointing out that a calculator program really helps in the process of solving the intricate resulting algebraic equations.

The rest of this work is planned as follow: at the beginning, a brief explanation of the multiple exp-function method is made. In Section 3, we solve the (3+1)-dimensional BLMP equation by the multiple exp-function method. In Section 4, we apply the multiple exp-function method on a generalized Shallow Water Equation. In the final section, we conclude the paper.

2. Methodology

The authors summarized the procedure of this method (Ma et al., 2010). In a word, the processes can be summarized as: Defining NLPDEs, Transforming NLPDEs and then we solve the algebraic equations. The emphasis of the method is to find rational solutions in new variables defining individual waves. We apply the method to solve specific one-wave, two-wave and three-wave solutions for high-dimensional nonlinear equations. The main steps are shown as follow, Ma et al., (2010) give more details on steps 1-3.

Step 1. Defining solvable NLPDEs.

To formulate our solution steps in the method, consider a nonlinear evolution equation as

$$f(x, y, z, t, u_x, u_y, u_z, u_t, \cdots) = 0. \tag{1}$$

We introduce new variables $\xi_i = \xi_i(x, y, z, t)$, $1 \le i \le n$, by solvable NLPDEs, for example, the linear partial differential equations

$$\xi_{i,x} = k_i \xi_i, \;\; \xi_{i,y} = l_i \xi_i, \;\; \xi_{i,z} = m_i \xi_i, \;\; \xi_{i,t} = -\omega_i \xi_i, \;\; 1 \le i \le n. \tag{2}$$

where k_i, are the angular wave numbers and ω_i, are the wave frequencies. Solving Eq. (2) leads to the following results

$$\xi_i = c_i e^{\eta_i}, \;\; \eta_i = k_i x + l_i y + m_i z - \omega_i t, \;\; 1 \le i \le n, \tag{3}$$

where c_i are arbitrary constants. Each of ξ_i, $1 \le i \le n$ describes a single wave.

Step 2. Transforming NLPDEs

We consider rational solutions in $\xi_i = \xi_i(x, y, z, t)$, $1 \le i \le n$:

$$u(x, y, z, t) = \frac{p(\xi_1, \xi_2, \cdots, \xi_n)}{q(\xi_1, \xi_2, \cdots, \xi_n)}, \tag{4}$$

$$\begin{cases} p(\xi_1, \xi_2, \cdots, \xi_n) = \displaystyle\sum_{r,\cdots,s=1}^{n} \sum_{i,\cdots,j=0}^{M} p_{r\cdots s,i\cdots j} \xi_r^i \cdots \xi_s^j, \\ q(\xi_1, \xi_2, \cdots, \xi_n) = \displaystyle\sum_{r,\cdots,s=1}^{n} \sum_{i,\cdots,j=0}^{N} q_{r\cdots s,i\cdots j} \xi_r^i \cdots \xi_s^j, \end{cases} \tag{5}$$

where $p_{r\cdots s,i\cdots j}$, $q_{r\cdots s,i\cdots j}$ are constants to be determined from the Eq.(1). We substitute Eq. (3) into Eq. (4), and the obtain the solution

$$u(x, y, z, t) = \frac{p(c_1 e^{k_1 x + l_1 y + m_1 z - \omega_1 t}, c_2 e^{k_2 x + l_2 y + m_2 z - \omega_2 t}, \cdots, c_n e^{k_n x + l_n y + m_n z - \omega_n t})}{q(c_1 e^{k_1 x + l_1 y + m_1 z - \omega_1 t}, c_2 e^{k_2 x + l_2 y + m_2 z - \omega_2 t}, \cdots, c_n e^{k_n x + l_n y + m_n z - \omega_n t})}. \tag{6}$$

Substituting Eq. (6) into Eq. (1), we obtain the equation

$$g(x, y, z, t, \xi_1, \cdots, \xi_n) = 0. \tag{7}$$

These transformations make it possible to solve solutions for differential equations directly by computer algebra systems.

Step 3. Solving algebraic systems

Finally, we assume the numerator of $g(x, y, z, t, \xi_1, \cdots, \xi_n)$ to zero. By solving these equations with the aid of Maple, we determine two polynomials $p_{r\cdots s,i\cdots j}$, $q_{r\cdots s,i\cdots j}$ and ξ_i, $1 \le i \le n$, and then, the multi-wave solutions $u(x, y, z, t)$ is computed and given by Eq. (6). In the following sections we solve the (3+1)-dimensional Boiti-Leon-Manna-Pempinelli equation and a (3+1)-dimensional generalized Shallow Water equation by the multiple exp-function method.

3. Application to the (3+1)-dimensional BLMP Equation

The (2+1)-dimensional BLMP equation

$$u_{yt} + u_{xxxy} - 3u_{xy} u_x - 3u_{xx} u_y = 0, \tag{8}$$

was derived by Gilson, Nimmo, and Willox (1993). Ma and Fang (2009) derived the variable separable solutions of Eq. (8) and investigate a lot of new localized excitations for example asmulti dromion-solitoffs and fractal-solitons.

The (3+1)-dimensional BLMP equation

$$u_{yt}+u_{zt}+u_{xxxy}+u_{xxxz}-3u_x(u_{xy}+u_{xz})-3u_{xx}(u_y+u_z)=0\,, \tag{9}$$

was proposed by Darvishi, Najafi, Kavitha, and Venkatesh, (2012) and some exact solutions were obtained. Wronskian determinant solutions of Eq. (9) were obtained by Ma and Bai (2013).

In this paper, in order to use the multiple exp-function approach to solve Eq. (9), we firstly make a scale transformation $x\to -x$, then the Eq. (9) is reduced equivalently to

$$u_{yt}+u_{zt}-u_{xxxy}-u_{xxxz}-3u_x(u_{xy}+u_{xz})-3u_{xx}(u_y+u_z)=0\,. \tag{10}$$

We use the multiple exp-function method to solve the Eq. (10), and then substitute $-x$ for x in the obtained results. The new solutions have checked in Eq. (9) and all solutions satisfy in the equation. Next, we will solve one-wave, two-wave and three-wave solutions for Eq. (9).

3.1 One-wave Solutions

Let’s assume the linear conditions to get one-wave solution as

$$\xi_{1,x}=k_1\xi_1,\;\; \xi_{1,y}=l_1\xi_1,\;\; \xi_{1,z}=m_1\xi_1,\;\; \xi_{1,t}=-\omega_1\xi_1, \tag{11}$$

where k_1, l_1, m_1 and ω_1 are constants. Based on the Eq. (3), we can obtain the exponential function solutions

$$\xi_1=c_1e^{k_1x+l_1y+m_1z-\omega_1t}\,. \tag{12}$$

We try the polynomials of degree one,

$$\begin{cases} p(\xi_1)=a_0+a_1\xi_1, \\ q(\xi_1)=b_0+b_1\xi_1. \end{cases} \tag{13}$$

where a_0, a_1, b_0 and b_1 are constants. Substituting Eq. (12) and Eq. (13) into Eq. (4), we obtain the equation

$$u(x,y,z,t)=\frac{a_0+a_1c_1e^{k_1x+l_1y+m_1z-\omega_1t}}{b_0+b_1c_1e^{k_1x+l_1y+m_1z-\omega_1t}}\,. \tag{14}$$

Solving the above polynomials with Eq. (11), we obtain the following relations for the algebraic equations.

$$a_1=\frac{b_1(2b_0k_1+a_0)}{b_0},\;\; \omega_1=-k_1^{\,3}, \tag{15}$$

where a_0, b_0, b_1, k_1 and l_1 are constants and a_1, ω_1 are defined by (15). Substituting Eq. (15) into Eq. (14), we obtain the one-wave solutions of Eq. (10). By the scale transformation $x\to -x$, we can obtain the one-wave solutions of Eq. (9).

We have plotted the one-wave solution with $a_0=1$, $b_0=2$, $b_1=2$, $k_1=1$, $l_1=1$, $m_1=1$, $z=1$ and $t=1$, in Figure 1.

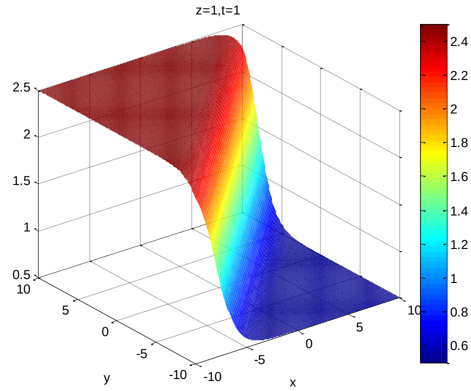

Figure 1. The one-wave solution with $a_0=1, b_0=2, b_1=2, k_1=1, l_1=1, m_1=1, z=1$ and $t=1$.

3.2 Two-wave Solutions

Similarly, we require the linear conditions

$$\xi_{i,x}=k_i\xi_i,\quad \xi_{i,y}=l_i\xi_i,\quad \xi_{i,z}=m_i\xi_i,\quad \xi_{i,t}=-\omega_i\xi_i,\quad 1\le i\le 2, \tag{16}$$

where k_i, l_i, m_i and ω_i, $1\le i\le 2$, are constants. Based on the Eq. (3), we can obtain the exponential function solutions

$$\xi_i=c_ie^{k_ix+l_iy+m_iz-\omega_it},\quad 1\le i\le 2. \tag{17}$$

We then suppose the polynomials,

$$\begin{cases} p(\xi_1,\xi_2)=2[k_1\xi_1+k_2\xi_2+a_{12}(k_1+k_2)\xi_1\xi_2], \\ q(\xi_1,\xi_2)=1+\xi_1+\xi_1+a_{12}\xi_1\xi_1, \end{cases} \tag{18}$$

where a_{12} are constants to be determined. Substituting Eq. (17) and Eq. (18) into Eq. (4), we obtain the equation

$$\begin{cases} u(x,y,z,t)=\dfrac{2[k_1c_1e^{\eta_1}+k_2c_2e^{\eta_2}+a_{12}(k_1+k_2)c_1e^{\eta_1}c_2e^{\eta_2}]}{1+c_1e^{\eta_1}+c_2e^{\eta_2}+a_{12}c_1e^{\eta_1}c_2e^{\eta_2}}, \\ \eta_i=k_ix+l_iy+m_iz-\omega_it,\quad i=1,2. \end{cases} \tag{19}$$

By the multiple exp-function method and the Eq. (16), we can get the following solutions

$$\begin{cases} a_{12}=\dfrac{(k_1-k_2)(m_1+l_1-l_2-m_2)}{(k_1+k_2)(m_1+l_1+l_2+m_2)}, \\ \omega_i=-k_i^3,\quad 1\le i\le 2. \end{cases} \tag{20}$$

where k_i, l_i and m_i, $1\le i\le 2$, are constants. Substituting Eq. (20) into Eq. (19), we obtain the two-wave solutions of Eq. (10). By the scale transformation $x\to -x$, we can obtain the two-wave solutions of Eq. (9).

We have plotted the two-wave solution with some special values of its parameters $c_1=1, c_2=2, k_1=1$, $k_2=3, l_1=1, l_2=3, m_1=1, m_2=3, z=1$ and $t=1$ in Figure 2.

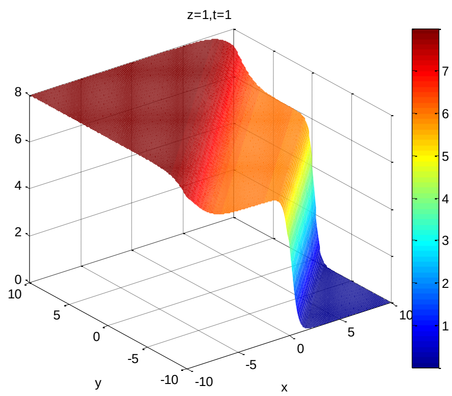

Figure 2. The two-wave solution with $c_1 = 1$, $c_2 = 2$, $k_1 = 1$, $k_2 = 3$, $l_1 = 1$, $l_2 = 3$, $m_1 = 1$, $m_2 = 3$, $z = 1$ and $t = 1$.

3.3 Three-wave Solutions

Again similarly, let's consider the linear conditions

$$\xi_{i,x} = k_i \xi_i, \quad \xi_{i,y} = l_i \xi_i, \quad \xi_{i,z} = m_i \xi_i, \quad \xi_{i,t} = -\omega_i \xi_i, \quad 1 \le i \le 3, \tag{21}$$

where k_i, l_i, m_i and ω_i, $1 \le i \le 3$, are constants. Based on the Eq. (3), we can obtain the exponential function solutions

$$\xi_i = c_i e^{k_i x + l_i y + m_i z - \omega_i t}, \quad 1 \le i \le 3. \tag{22}$$

In this condition, we set the polynomials as

$$\begin{cases} p(\xi_1,\xi_2,\xi_3) = 2[k_1\xi_1 + k_2\xi_2 + k_3\xi_3 + a_{12}(k_1+k_2)\xi_1\xi_2 + a_{13}(k_1+k_3)\xi_1\xi_3 \\ \qquad + a_{23}(k_2+k_3)\xi_2\xi_3 + a_{12}a_{13}a_{23}(k_1+k_2+k_3)\xi_1\xi_2\xi_3], \\ q(\xi_1,\xi_2,\xi_3) = 1 + \xi_1 + \xi_2 + \xi_3 + a_{12}\xi_1\xi_2 + a_{13}\xi_1\xi_3 + a_{23}\xi_2\xi_3 + a_{12}a_{13}a_{23}\xi_1\xi_2\xi_3. \end{cases} \tag{23}$$

where a_{12}, a_{13} and a_{23} are constants to be determined. Substituting Eq. (22) and Eq. (23) into Eq. (4), we obtain the equation

$$\begin{cases} u(x,y,z,t) = 2[k_1c_1e^{\eta_1} + k_2c_2e^{\eta_2} + k_3c_3e^{\eta_3} + a_{12}(k_1+k_2)c_1e^{\eta_1}c_2e^{\eta_2} \\ \qquad + a_{13}(k_1+k_3)c_1e^{\eta_1}c_3e^{\eta_3} + a_{23}(k_2+k_3)c_2e^{\eta_2}c_3e^{\eta_3} \\ \qquad + a_{12}a_{13}a_{23}(k_1+k_2+k_3)c_1e^{\eta_1}c_2e^{\eta_2}c_3e^{\eta_3}]/(1+c_1e^{\eta_1} \\ \qquad + c_2e^{\eta_2} + c_3e^{\eta_3} + a_{12}c_1e^{\eta_1}c_2e^{\eta_2} + a_{13}c_1e^{\eta_1}c_3e^{\eta_3} \\ \qquad + a_{23}c_2e^{\eta_2}c_3e^{\eta_3} + a_{12}a_{13}a_{23}c_1e^{\eta_1}c_2e^{\eta_2}c_3e^{\eta_3}), \\ \eta_i = k_i x + l_i y + m_i z - \omega_i t, \qquad i = 1,2,3. \end{cases} \tag{24}$$

Solving the above polynomials with Eq. (21), we obtain the following relations for the algebraic equations.

$$\begin{cases} a_{ij} = \dfrac{(k_i - k_j)(l_i - l_j + m_i - m_j)}{(k_i + k_j)(l_i + l_j + m_i + m_j)}, \quad 1 \le i \le 3, \quad 1 \le j \le 3, \\ \omega_i = -k_i^3, \end{cases} \tag{25}$$

where k_i, l_i and m_i, $1 \le i \le 3$, are constants. Substituting Eq. (25) into Eq. (24), we can get the three-wave solutions for Eq. (10). By the scale transformation $x \to -x$, we solve the solutions to Eq. (9). The determination of three-wave solutions confirms the fact that N-wave solutions exist for any order. The higher level wave solutions, for $N \ge 4$, can be obtained in a parallel manner and do not contain any new free parameters other than a_{ij} derived for the three-wave solutions.

We have plotted the three-wave solution with $c_1 = 1, c_2 = 2, c_3 = 2, k_1 = 1, k_2 = 3, k_3 = 5, l_1 = 1, l_2 = 3$, $l_3 = 5, m_1 = 1, m_2 = 3, m_3 = 5, z = 1$ and $t = 1$, in Figure 3.

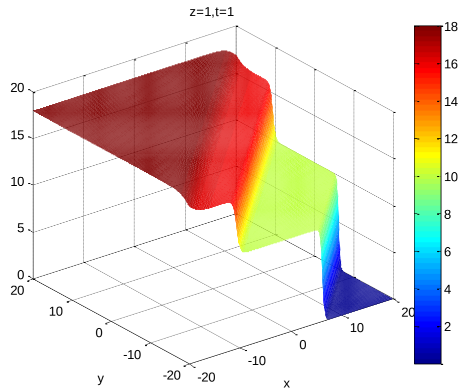

Figure 3. The three-wave solution with $c_1 = 1, c_2 = 2, c_3 = 2, k_1 = 1, k_2 = 3, k_3 = 5, l_1 = 1, l_2 = 3, l_3 = 5$, $m_1 = 1, m_2 = 3, m_3 = 5, z = 1$ and $t = 1$.

It is worth noting that when the two-wave solutions and three-wave solutions were solved by Darvishi, et al. (2012) , they assumed that the coefficients of $p(\xi_1, \xi_2, \xi_3)$ within Eq. (18) and Eq. (23) are -2 and obtained some special relations of k_i, l_i and m_i. However, in this paper, we assume that the coefficients are 2 and obtain general relation of k_i, l_i and m_i such as Eq. (20) and Eq. (25) with the scale transformation, i.e., k_i, l_i and m_i are arbitrary constants. So our results are more general.

As a special example, when we set $l_i = k_i$, $m_i = k_i$ and $\omega_i = -k_i^3$, $1 \le i \le 3$, the one-wave, two-wave and three-wave solutions obtained in this paper correspond to Wronskian determinant solutions for the order $N = 1, 2, 3$ obtained by Ma and Bai (2013). The determination of three-wave solutions confirms the fact that N-wave solutions exist for any order, thus the Wronskian determinant solutions for the arbitrary order N obtained by Ma and Bai (2013) are the special examples of the solutions in this paper.

4. Application to the (3+1)-dimensional Generalized Shallow Water Equation

The (3+1)-dimensional generalized Shallow Water equation

$$u_{xxxy} - 3u_{xx}u_y - 3u_x u_{xy} + u_{yt} - u_{xz} = 0, \tag{26}$$

was investigated by Tian and Gao (1996). Soliton-typed solutions for Eq. (26) were obtained by a generalized tanh algorithm with symbolic computation (Tian & Gao, 1996).

In this paper, in order to use the multiple exp-function method, we firstly make a scale transformation $x \to -x$, then the Eq. (26) is reduced equivalently to

$$u_{xxxy} + 3u_{xx}u_y + 3u_x u_{xy} - u_{yt} - u_{xz} = 0. \tag{27}$$

We use the multiple exp-function approach to solve the Eq. (26), and then substitute $-x$ for x in the obtained solutions. The new solutions have checked in Eq. (26) and all solutions satisfy in the equation. Next, we will solve one-wave, two-wave and three-wave solutions for Eq. (26).

4.1 One-wave Solutions

By using the approach and the Eq. (11) as well as BLMP equation, we obtain that

$$a_1 = \frac{b_1(2b_0k_1 + a_0)}{b_0}, \quad \omega_1 = -\frac{k_1(-m_1 + k_1^2 l_1)}{l_1}, \tag{28}$$

where a_0, b_0, b_1, k_1, l_1 and m_1 are constants. Substituting Eq. (28) into Eq. (14), we obtain the one-wave solutions of Eq. (27). By the scale transformation $x \to -x$, we can obtain the one-wave solutions of Eq. (26).

We have plotted the above result for $a_0 = 1, b_0 = 1, b_1 = 1, k_1 = 1, l_1 = 1, m_1 = 2, z = 1$ and $t = 0$, as shown in Figure 4.

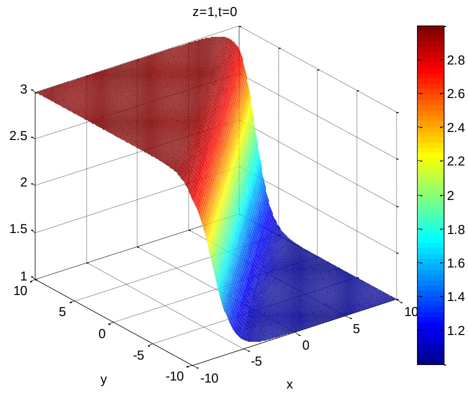

Figure 4. The one-wave solution with $a_0 = 1, b_0 = 1, b_1 = 1, k_1 = 1, l_1 = 1, m_1 = 2, z = 1$ and $t = 0$.

4.2 Two-wave Solutions

Similarly, by applying the method and Eq. (16) as well as BLMP equation, we get that

$$\begin{cases} a_{12} = (-3k_1^2k_2l_2^2l_1 + 3k_1^2l_1^2k_2l_2 - 3k_1k_2^2l_1^2l_2 - l_1l_2m_2k_1 + 3k_1k_2^2l_2^2l_1 + k_1l_2^2m_1 \\ \qquad - m_1l_1l_2k_2 + m_2k_2l_1^2) / (3k_1^2k_2l_2^2l_1 + 3k_1^2l_1^2k_2l_2 + 3k_1k_2^2l_1^2l_2 - l_1l_2m_2k_1 \\ \qquad + 3k_1k_2^2l_2^2l_1 + k_1l_2^2m_1 - m_1l_1l_2k_2 + m_2k_2l_1^2), \\ \omega_1 = -\dfrac{k_1(k_1^2l_1 - m_1)}{l_1}, \omega_2 = -\dfrac{k_2(k_2^2l_2 - m_2)}{l_2}. \end{cases} \tag{29}$$

where k_i, l_i and m_i, $1 \le i \le 2$, are constants. Substituting Eq. (29) into Eq. (19), we obtain the two-wave solutions of Eq. (27). By the scale transformation $x \to -x$, we can obtain the two-wave solutions of Eq. (26).

We have plotted the above result for $c_1 = 1, c_2 = 2, k_1 = 1, k_2 = 3, l_1 = 1, l_2 = 3, m_1 = 2, m_2 = 36, z = 1$ and $t = 0$, as shown in Figure 5.

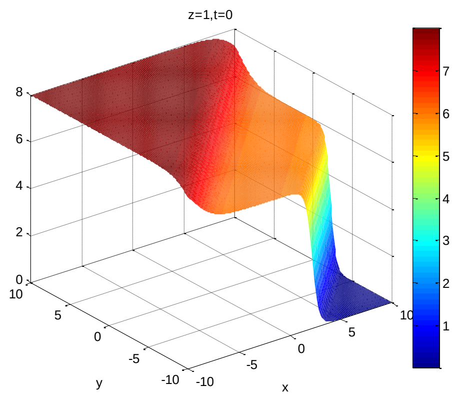

Figure 5. The two-wave solution with $c_1=1, c_2=2, k_1=1, k_2=3, l_1=1, l_2=3, m_1=2, m_2=36, z=1$ and $t=0$.

4.3 Three-wave Solutions

Again similarly, by using the method and Eq. (21), we obtain that

$$\begin{cases} a_{ij}=\dfrac{(k_i-k_j)^2}{(k_i+k_j)^2}, \\ \omega_i=m_i-k_i^3. \end{cases} \qquad 1\le i\le 3, 1\le j\le 3. \tag{30}$$

when $l_i=k_i, 1\le i\le 3$, and

$$\begin{cases} a_{ij}=\dfrac{3k_i^2k_jl_i^2l_j-3k_i^2k_jl_il_j^2+k_i^2l_j^2-2k_ik_jl_il_j-3k_ik_j^2l_i^2l_j+3k_ik_j^2l_il_j^2+l_i^2k_j^2}{3k_i^2k_jl_i^2l_j+3k_i^2k_jl_il_j^2+k_i^2l_j^2-2k_ik_jl_il_j+3k_ik_j^2l_i^2l_j+3k_ik_j^2l_il_j^2+l_i^2k_j^2}, \\ \omega_i=-\dfrac{k_i^2(k_il_i-1)}{l_i}. \end{cases} \tag{31}$$

when $m_i=k_i, 1\le i\le 3$.

Substituting Eq. (30) and (31) into Eq. (24), we obtain the three-wave solutions of Eq. (27). By the scale transformation $x\to -x$, we can obtain the three-wave solutions of Eq. (26). The determination of three-wave solutions confirms the fact that N-wave solutions exist for any order. The higher level wave solutions, for $N\ge 4$, can be obtained in a parallel manner and do not contain any new free parameters other than a_{ij} derived for the three-wave solutions (Wazwaz, 2010).

We have plotted the above result obtained from Eq. (30) for $c_1=1, c_2=2, c_3=2, k_1=1, k_2=2, k_3=3$, $l_1=1, l_2=2, l_3=3, m_1=2, m_2=12, m_3=36$, $z=3$ and $t=3$, as shown in Figure 6.

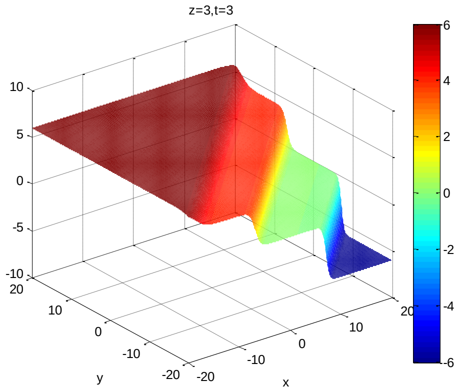

Figure 6. The three-wave solution with $c_1 = 1, c_2 = 2, c_3 = 2, k_1 = 1, k_2 = 2, k_3 = 3, l_1 = 1, l_2 = 2, l_3 = 3,$
$m_1 = 2, m_2 = 12, m_3 = 36, z = 3$ and $t = 3$.

As a special example, when we set $l_i = k_i, \omega_i = k_i^2$, based on the Eq. (30), we obtain $m_i = k_i^2 + k_i^3, 1 \le i \le 3$, the one-wave, two-wave and three-wave solutions obtained in this paper correspond to Wronskian determinant solutions when the order $N = 1,2,3$ obtained by Tang and Su (2012). The determination of three-wave solutions confirms the fact that N-wave solutions exist for any order, thus Wronskian determinant solutions for the arbitrary order N obtained by Tang and Su (2012) are the special examples of the solutions in this paper.

5. Concluding Remarks

The multiple exp-function method offers a simple and straightforward way to study exact solutions to NLPDEs. The method has been applied to the (3+1)-dimensional BLMP equation and a generalized Shallow Water Equation. The one-wave, two-wave and three-wave solutions have been obtained in this paper. For some special values of parameters, we obtain some exact solutions. In addition, by comparing the solutions obtained in this paper with those solved in the references, we find that our results are more general. The multiple exp-function method can be applied to obtain the multi-soliton solutions of other nonlinear partial differential equations, because the multiple exp-function method is effective in generating exact solutions of NLPDEs.

It is worth pointing out that a few NLPDEs including the (3+1)-dimensional YTSF equation, generalized KP equation, generalized BKP equation and BLMP equation have been solved in (Ma et al., 2010, Su et al., 2013, Ma & Zhu, 2012, Darvishi, et al., 2012), when the authors solved the two-wave solutions and three-wave solutions for the first three equations, they assumed that the coefficients of $p(\xi_1, \xi_2, \xi_3)$ within $u(x, y, z, t)$ were 2. However, the author assumed that the coefficients were -2 for BLMP equation and obtained some special relations among k_i, l_i and m_i. In this paper, if, for two-wave and three-wave solutions, we set that the coefficients of $p(\xi_1, \xi_2, \xi_3)$ within $u(x, y, z, t)$ are 2, we will meet contradictions in the resulting algebraic system. If we suppose the coefficients of $p(\xi_1, \xi_2, \xi_3)$ within $u(x, y, z, t)$ are -2, we will not get the more general relations such as Eq. (20) and Eq. (23). By computation and analysis, if we associate the scale transformation with the multiple exp method, we will obtain the general solutions for BLMP equation. Any general form of two-waves and three-waves, which does not involve any relation among the angular wave numbers k_i, l_i and m_i, would be more significant and diverting.

Acknowledgements

The work was sponsored by the Seed Foundation of Innovation and Creation for Graduate Students in Northwestern Polytechnical University and the National Natural Science Foundation of China (Grant No. 11202161).

References

Darvishi, M. T., Najafi, M., Kavitha, L., & Venkatesh M. (2012). Stair and Step Soliton Solutions of the Integrable (2+1) and (3+1)-Dimensional Boiti-Leon-Manna-Pempinelli Equations. *Commun. Theor. Phys. 58*, 785-794. http://dx.doi.org/10.1088/0253-6102/58/6/01

Fan, E. G. (1998). A note on the homogenous balance method. *Phys. Lett. A 246*, 403-406. http://dx.doi.org/10.1016/S0375-9601(98)00547-7

Fan, E. G. (2000). Extended tanh-function method and its applications to nonlinear equations. *Phys. Lett. A 277*, 212-218. http://dx.doi.org/10.1016/S0375-9601(00)00725-8

Gilson, C. R., Nimmo, J. J. C. & Willox R. (1993). A (2+1)-dimensional generalization of the AKNS shallow water wave equation. *Phys. Lett. A 180*, 337-345 (1993). http://dx.doi.org/10.1016/0375-9601(93)91187-A

Gu, C. H., & Zhou, Z. X. (1994). On Darboux transformations for soliton equations in high-dimensional spacetime. *Letters in mathematical physics. 32*, 1-10. http://dx.doi.org/10.1007/BF00761119

Hong, L. I. (2010). Painlevé analysis and Darboux transformation for a variable-coefficient Boussinesq system in fluid dynamics with symbolic computation. Commun. *Theor. Phys. 53,* 831-836. http://dx.doi.org/10.1088/0253-6102/53/5/08

Hu, H. C., Tang, X. Y., Lou, S. Y., & Liu Q. P. (2004). Variable separation solutions obtained from Darboux transformations for the asymmetric Nizhnik–Novikov–Veselov system. *Chaos Solitons Fract. 22,* 327-334. http://dx.doi.org/10.1016/j.chaos.2004.02.002

Ma, H. C., & Bai, Y. B. (2013). Wronskian Determinant Solutions for the (3+1)-Dimensional Boiti-Leon-Manna-Pempinelli Equation. *Journal of Applied Mathematics and Physics. 1,* 18-24. http://dx.doi.org/10.4236/jamp.2013.15004

Ma, S. H., & Fang, J. P. (2009). Multi dromion-solitoff and fractal excitations for (2+1)-dimensional Boiti–Leon–Manna–Pempinelli system. *Commun. Theor. Phys. 52,* 641-645. http://www.cnki.com.cn/Article/CJFDTotal-CITP200910021.htm

Ma, W. X. (2004). Wronskians, generalized Wronskians and solutions to the Korteweg–de Vries equation. *Chaos Solitons Fract. 19*, 163-170. http://dx.doi.org/10.1016/S0960-0779(03)00087-0

Ma, W. X., & Zhu, Z. N. (2012). Solving the (3+1)-dimensional generalized KP and BKP equations by the multiple exp-function algorithm. *Appl. Math. Comput. 218,* 11871-11879. http://dx.doi.org/ 10.1016/j.amc.2012.05.049

Ma, W. X., Huang, T. W., & Zhang, Y. (2010). A multiple exp-function method for nonlinear differential equations and its application. *Phys. Scr. 82,* 065003. http://dx.doi.org/10.1088/0031-8949/82/06/065003

Ma, W. X., Zhou, R. G., & Gao, L. (2009). Exact one-periodic and two-periodic wave solutions to Hirota bilinear equations in (2+1) dimensions. *Mod. Phys. Lett. A 24,* 1677-1688. http://dx.doi.org/10.1142/S0217732309030096

Nimmo, J. J. C. (1983). A bilinear Backlund transformation for the nonlinear Schrodinger equation. *Phys. Lett. A 99*, 279-280. http://dx.doi.org/10.1016/0375-9601(83)90884-8

Su, P. P., Tang, Y. N., & Zhao, Y. (2013). Multiple exp-function Method for Two Kinds of (3+1)-dimensional Generalized BKP Equations. *Chinese Journal of Engineering Mathematics. 30,* 890-900. http://www.cqvip.com/qk/91079x/201306/48091642.html

Tang, Y. N., & Su, P. P. (2012). Two Different Classes of Wronskian Conditions to a (3+1)-Dimensional Generalized Shallow Water Equations. International Scholarly Research Notices. 2012, 384906. http://dx.doi.org/10.5402/2012/384906

Tian, B., & Gao, Y. T. (1996). Beyond travelling waves: a new algorithm for solving nonlinear evolution equations. *Comput. Phys. Commun. 95,* 139-142. http://dx.doi.org/10.1016/0010-4655(96)00014-8

Vakhnenko, V. O., Parkes, E. J. & Morrison, A. J. (2003). A Bäcklund transformation and the inverse scattering transform method for the generalised Vakhnenko equation. *Chaos Solitons Fract. 17,* 683-692. http://dx.doi.org/10.1016/S0960-0779(02)00483-6

Wazwaz, A. M. (2010). Multiple-Soliton Solutions for Extended Shallow Water Wave Equations. *Studies in Mathematical Sciences. 1,* 21–29. http://dx.doi.org/10.3968%2Fj.sms.1923845220120101.003

Yan, C. T. (1996). A simple transformation for nonlinear waves. *Phys. Lett. A 224,* 77–84 (1996). http://dx.doi.org/ 10.1016/S0375-9601(96)00770-0