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Relationships between Ordered Compositions and Fibonacci Numbers

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Abstract

A sequence of four compositions of 3 is: 1 + 1 + 1, 1 + 2, 2 + 1, 3. By the replacement of the plus signs (+) and commas (,) by the multiplication dots (·) and plus signs (+) respectively, the sequence becomes the summation series: 1·1·1 + 1·2 + 2·1 + 3, which is equal to 8 or 6th number in the famous Fibonacci sequence. It is a curious fact that the sum of a positive integer n and the products of summands corresponding to the compositions of n is equal to (2n)-th Fibonacci number. We establish the proposition after obtaining a special order of the compositions of n; and then obtain some other results. The results show that Fibonacci sequence has close connection with the special order of the compositions of n. Two Fibonacci identities, which we derive from a special recurrence relation, are useful to prove two theorems. The relationships are stated first in the theorems and are then shown in the consequences of the theorems.

Keywords: composition, summand, Fibonacci number, recurrence, sequence

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1. Introduction

Partitions of a positive integer n including permutations of the parts or summands are called compositions of n. For example, eight compositions of 4 are: 4, 3 + 1, 1 + 3, 2 + 2, 2 + 1 + 1, 1 + 2 + 1, 1 + 1 + 2, 1 + 1 + 1 + 1. On the other hand, the famous Fibonacci sequence is defined by a linear recurrence relation: $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$ with the initial conditions: $F_0 = 0$ and $F_1 = 1$. Fibonacci sequence is the following sequence of integers: 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, …. There exists a definite order of the compositions of n, which has close connection with the Fibonacci sequence. We use the following simple notations for compositions of n to establish the relationship.

1. Compositions of n = $C(n)$.

2. All $C(n)$ = $\{C(n)\}$. Only for the mathematical representations, we use the notation: $\{C(n)\}$ to mean all $C(n)$. Otherwise we use an adjective to specify $C(n)$.

3. Number of all $C(n)$ = $NC(n)$. We know: $NC(n) = 2^{n-1}$.

4. $x_1 + \ldots + x_{r-1} + \{C(x_r)\}$ for $x_1 + \ldots + x_r = n$ denotes some $C(n)$ under which the compositions start with the common summands: $x_1, \ldots, x_{r-1}$ in succession.

We use the symbol of equivalence (≡) between $\{C(n)\}$ and its implication; and similarly between $x_1 + \ldots + x_{r-1} + \{C(x_r)\}$ and its implication.

Examples: $\{C(3)\} \equiv 1 + 1 + 1, 1 + 2, 2 + 1, 3$.

$NC(3) = 4$.

$2 + 5 + \{C(3)\}$ denotes some $C(10)$ such that $2 + 5 + \{C(3)\} \equiv 2 + 5 + 1 + 1 + 1, 2 + 5 + 1 + 2, 2 + 5 + 2 + 1, 2 + 5 + 3$.

Main Results:

(a) We obtain a 'significant order of the compositions of n' or in brief '$SOC(n)$'. The rule for $SOC(n)$ is stated below.

Under $SOC(n)$ for $n \geq 2$, the summands of 1st $C(n)$ are all 1; the last $C(n)$ is n itself; and if any k^{th} $C(n)$ is $x_1 + \ldots + x_r$ then $(k+1)^{th}$ $C(n)$ is $x_1 + \ldots + x_{r-2} + (x_{r-1} + 1)$ + the sum of $x_r - 1$ summands which are all 1. The number of summands of k^{th} and $(k+1)^{th}$ $C(n)$ under $SOC(n)$ are r and $r + x_r - 2$ respectively.

Example: We use the symbol of equivalence (≡) between $SOC(n)$ and its implication.

$SOC(5)$ ≡ 1 + 1 + 1 + 1 + 1, 1 + 1 + 1 + 2, 1 + 1 + 2 + 1, 1 + 1 + 3, 1 + 2 + 1 + 1, 1 + 2 + 2, 1 + 3 + 1, 1 + 4, 2 + 1 + 1 + 1, 2 + 1 + 2, 2 + 2 + 1, 2 + 3, 3 + 1 + 1, 3 + 2, 4 + 1, 5

(b) By the replacement of the plus signs (+) and commas (,) by the multiplication dots (·) and plus signs (+) respectively in $SOC(n)$, we can find a summation series of 2^{n-1} terms. Denoting the summation series by $\prod C(n)$, we show:

(i) $\prod C(n) = F_{2n}$ such that the sum of the terms of odd places and the sum of the terms of even places of $\prod C(n)$ are F_{2n-2} and F_{2n-1} respectively. For example, when $SOC(4)$ ≡ 1 + 1 + 1 + 1, 1 + 1 + 2, 1 + 2 + 1, 1 + 3, 2 + 1 + 1, 2 + 2, 3 + 1, 4 then (i) $\prod C(4)$ = 1 ·1 ·1 ·1 + 1 ·1 ·2 + 1 ·2 ·1 + 1 ·3 + 2 ·1 ·1 + 2 ·2 + 3 ·1 + 4 = 21 or F_8; (ii) the sum of the terms of odd places of $\prod C(4)$ = 1 ·1 ·1 ·1 + 1 ·2 ·1 + 2 ·1 ·1 + 3 ·1 = 8 or F_6; and (iii) the sum of the terms of even places of $\prod C(4)$ = 1 ·1 ·2 + 1 ·3 + 2 ·2 + 4 = 13 or F_7.

(ii) The 1st term, the sums of 1st 2, 1st 2^2, … 1st 2^{n-2} and all 2^{n-1} terms of $\prod C(n)$ for $n \geq 2$ are F_2, F_4, F_6 , … , F_{2n-2} and F_{2n} ; and the 2nd term, the sums of 2nd 2, 2nd 2^2, …, 2nd 2^{n-2} terms of the series are $F_3, F_5, \ldots, F_{2n-1}$ in succession.

(c) Two Fibonacci identities:

$$F_{2n} = \sum_{i=0}^{n-1} i\, F_{2(n-i)} + n$$

and

$$F_{2n-1} = \sum_{i=0}^{n-1} i\, F_{2(n-i)-1} + 1$$

have the important roles to establish the relationships. We obtain the identities from a special recurrence relation.

In the description of the relationships, the basic results are stated first in two theorems; other results are the consequences of the theorems.

2. Ordered Compositions

By the notations, we can write for $n \geq 2$,

$$\{C(n)\} \equiv 1 + \{C(n-1)\},\ 2 + \{C(n-2)\},\ \ldots,\ (n-1) + \{C(1)\},\quad n \tag{1}$$

(1) is an initial sequential arrangement for all $C(n)$ such that the arrangement is composed of n sets of $C(n)$. The last two sets of $C(n)$ are $(n-1) + \{C(1)\}$ and n. Obviously these are two particular $C(n)$ as: $(n-1) + 1$ and n respectively. When $n = 2$ then $\{C(2)\} \equiv 1 + 1, 2$; and when $n \geq 3$ then besides the last two sets as two $C(n)$, we can further obtain the sequential arrangements for other sets of $C(n)$ of the type: $a + \{C(n-a)\}$ where these arrangements have the same form as that of the initial sequential arrangement for all $C(n)$. For instance,

$$2 + \{C(n-2)\} \equiv 2 + 1 + \{C(n-3)\},\ 2 + 2 + \{C(n-4)\},\ \ldots,\ 2 + (n-4) + \{C(2)\},$$
$$2 + (n-3) + 1,\quad 2 + (n-2);$$

where $C(n)$ in the 1st set on the right have 1st two common summands 2 and 1; $C(n)$ in the 2nd set have 1st two common summands 2 and 2; and so on provided that the last two sets are two $C(n)$ of 3 and 2 summands as: $2 + (n-3) + 1$ and $2 + (n-2)$ respectively. Subsequently we can obtain the sequential arrangements for the sets of $C(n)$ of the type: $a + b + \{C(n-a-b)\}$ in like manner. Thus carrying out the operations of obtaining the sequential arrangements for the successive sets of $C(n)$ recursively where the last two sets in each arrangement are found as two $C(n)$, finally we can find a definite order of all $C(n)$. For instance,

$\{C(5)\} \equiv 1 + \{C(4)\}, 2 + \{C(3)\}, 3 + \{C(2)\}, 4 + 1, 5$

$\equiv 1 + 1 + \{C(3)\}, 1 + 2 + \{C(2)\}, 1 + 3 + 1, 1 + 4, 2 + 1 + \{C(2)\}, 2 + 2 + 1, 2 + 3,$

$3 + 1 + 1, 3 + 2, 4 + 1, 5$

$\equiv 1 + 1 + 1 + \{C(2)\}$, 1 + 1 + 2 + 1, 1 + 1 + 3, 1 + 2 + 1 + 1, 1 + 2 + 2, 1 + 3 + 1, 1 + 4, 2 + 1 + 1 + 1, 2 + 1 + 2, 2 + 2 + 1, 2 + 3, 3 + 1 + 1, 3 + 2, 4 + 1, 5

≡ 1 + 1 + 1 + 1 + 1, 1 + 1 + 1 + 2, 1 + 1 + 2 + 1, 1 + 1 + 3, 1 + 2 + 1 + 1, 1 + 2 + 2, 1 + 3 + 1, 1 + 4, 2 + 1 + 1 + 1, 2 + 1 + 2, 2 + 2 + 1, 2 + 3, 3 + 1 + 1, 3 + 2, 4 + 1, 5.

For convenience, we name 5 sets of $C(5)$, which is shown in the 1st step, as the 'basic' or 'initial exposition' of all $C(5)$; and name the sixteen $C(5)$, which are yielded in a definite order in the last step, as the 'final exposition' of all $C(5)$. Thus the expression on the right of (1), which is composed of n sets of $C(n)$, is the initial exposition of all $C(n)$; and $2^{n-1}\, C(n)$ in a particular order, which can be yielded finally by recursive exposition, is the final exposition of all $C(n)$. The particular order of all $C(n)$ is named as 'the significant order of compositions of n' or in brief $SOC(n)$. $SOC(n)$ is the final exposition of all $C(n)$.

We can use also the phrases: 'initial exposition' and 'final exposition' in the expositions of a set of $C(n)$. In the process of recursive exposition to find $SOC(n)$ starting with (1), we find in general a set of $C(n)$ in the form: $x_1 + \ldots + x_{r-1} + \{C(x_r)\}$ for $x_1 + \ldots + x_r = n$ such that the initial exposition of this set of $C(n)$ is:

$x_1 + \ldots + x_{r-1} + 1 + \{C(x_r - 1)\}$, $x_1 + \ldots + x_{r-1} + 2 + \{C(x_r - 2)\}$, …, …, $x_1 + \ldots + x_{r-1} + (x_r - 2) + \{C(2)\}$, $x_1 + \ldots + x_{r-1} + (x_r - 1) + 1$, $x_1 + \ldots + x_{r-1} + x_r$.

Following (1), we can write:

$$1 + \{C(n-1)\} \equiv 1 + 1 + \{C(n-2)\},\quad 1 + 2 + \{C(n-3)\},\ \ldots \tag{1.1}$$

$$1 + 1 + \{C(n-2)\} \equiv 1 + 1 + 1 + \{C(n-3)\},\quad 1 + 1 + 2 + \{C(n-4)\},\ \ldots \tag{1.2}$$

$$1 + 1 + 1 + \{C(n-3)\} \equiv 1 + 1 + 1 + 1 + \{C(n-4)\},\ 1 + 1 + 1 + 2 + \{C(n-5)\},\ \ldots \tag{1.3}$$

… …

(1.1) is the initial exposition of $1 + \{C(n-1)\}$. The exposition starts with $1 + 1 + \{C(n-2)\}$ of which the initial exposition (1.2) starts with $1 + 1 + 1 + \{C(n-3)\}$ of which the initial exposition (1.3) starts with $1 + 1 + 1 + 1 + \{C(n-4)\}$; and so on. It follows that $SOC(n)$ starts with the longest $C(n)$ of which the summands are all 1. n as a $C(n)$ is written at the last of (1). Consequently n as a $C(n)$ appears last in $SOC(n)$. Hence the sum of n summands, which are all 1, is the first $C(n)$; and n itself is the last $C(n)$ in $SOC(n)$. For example, the first and last $C(5)$ in $SOC(5)$ are $1 + 1 + 1 + 1 + 1$ and 5 respectively. Similarly $x_1 + \ldots + x_{r-1} + \{C(x_r)\}$ for $x_1 + \ldots + x_r = n$, which represents a set of $C(n)$, has the final exposition with the first $C(n)$ as: $x_1 + \ldots + x_{r-1}$ + the sum of x_r summands which are all 1 and the last $C(n)$ as: $x_1 + \ldots + x_r$.

In the process to find $SOC(n)$, when a set of $C(n)$ appears in the form: $x_1 + \ldots + x_k + a + \{C(b)\}$ for $x_1 + \ldots + x_k + a + b = n$, then the next set of $C(n)$ appears in the form: $x_1 + \ldots + x_k + (a + 1) + \{C(b - 1)\}$. Let these two sets of $C(n)$ be denoted by S_1 and S_2 respectively. Since S_1 and S_2 are the consecutive sets of $C(n)$, it follows that the last $C(n)$ in the final exposition of S_1 and the first $C(n)$ in the final exposition of S_2 are two consecutive $C(n)$ under $SOC(n)$. The last $C(n)$ in the final exposition of S_1 is: $x_1 + \ldots + x_k + a + b$; and the first $C(n)$ in the final exposition of S_2 is $x_1 + \ldots + x_k + (a + 1)$ + the sum of $b - 1$ summands which are all 1. Hence under $SOC(n)$, when a $C(n)$ appears in the form: $x_1 + \ldots + x_k + a + b$, then the next $C(n)$ appears in the form: $x_1 + \ldots + x_k + (a + 1)$ + the sum of $b - 1$ summands which are all 1, where these two consecutive $C(n)$ are composed of $k + 2$ and $k + b$ summands respectively. The forms of S_1 and S_2 can be: $a + \{C(b)\}$ and $(a + 1) + \{C(b - 1)\}$ respectively for $a + b = n$ so that the last $C(n)$ in the final exposition of S_1 is $a + b$ and the first $C(n)$ in the final exposition of S_2 is $(a + 1)$ + the sum of $b - 1$ summands which are all 1. Hence under $SOC(n)$, the forms of two consecutive $C(n)$ can be: (i) $a + b$ and (ii) $(a + 1)$ + the sum of $b - 1$ summands which are all 1, where these two consecutive $C(n)$ are composed of 2 and b summands respectively. The rule for appearances of successive $C(n)$ under $SOC(n)$ is clear from the above demonstration.

Rule for *SOC*(*n*): *Under SOC(n), the summands of the* 1^{st} *C(n) are all 1; the last C(n) is n itself; and for* $n \geq r \geq 2$, *if any* k^{th} *C(n) is:* $x_1 + \ldots + x_r$ *then* $(k + 1)^{th}$ *C(n) is:* $x_1 + \ldots + x_{r-2} + (x_{r-1} + 1)$ *+ the sum of* $x_r - 1$ *summands which are all 1 such that if* $r \geq 3$ *then the first* $r - 2$ *summands of* k^{th} *C(n) appear also in* $(k + 1)^{th}$ *C(n) in the same order, but if* $r = 2$ *then such common summands of* k^{th} *C(n) and* $(k + 1)^{th}$ *C(n) cannot exist. The number of summands of* k^{th} *C(n) and* $(k + 1)^{th}$ *C(n) under SOC(n) are r and* $r + x_r - 2$ *respectively.*

Remark1. The number of the compositions of n is a familiar result. It is easy to obtain the result from (1) also.

From (1), we get: For $n \geq 2$, $NC(n) = NC(n - 1) + NC(n - 2) + \ldots + NC(1) + 1$

Then we have the successive results as shown.

$\{C(1)\} \equiv 1$. Hence $NC(1) = 1$.

$\{C(2)\} \equiv 1 + \{C(1)\}, 2$. Hence $NC(2) = 2$.

$\{C(3)\} \equiv 1 + \{C(2)\}, 2 + \{C(1)\}, 3$.

Hence $NC(3) = NC(2) + NC(1) + 1 = 2 + 1 + 1 = 2^2$.

Similarly $NC(4) = NC(3) + NC(2) + NC(1) + 1 = 2^2 + 2 + 1 + 1 = 2^3$.

Proceeding thus, we get: For $n \geq 2$, $NC(n) = 2^{n-1}$.

Furthermore $NC(1) = 1 = 2^0$. Hence for $n \geq 1$, $NC(n) = 2^{n-1}$.

3. A Recurrence Relation and Fibonacci Identities

For some numbers: *a*, *b* and S_0, we define an n^{th} order recurrence function S_n by a linear recurrence relation:

$$S_n = \sum_{i=0}^{n-1} (a+ib)\, S_{n-1-i} + (a+nb) \tag{2}$$

(2) can generate some Fibonacci identities. We show here the generation of two particular identities which are needed to establish two theorems that involve with the required relationships. The following Fibonacci formula is useful to obtain the identities.

$$F_{n+m} = L_m F_n + (-1)^{m+1} F_{n-m} \tag{3}$$

(3) is the formula (15a) in 'List of formulae' in the book, Fibonacci and Lucas Numbers, and the Golden Section: Theory and Applications by Steven Vajda. (3) involves with Lucas number L_m. Lucas sequence is similar to Fibonacci sequence and defined by a linear recurrence relation: $L_{n+1} = L_n + L_{n-1}$ for $n \geq 1$ with the initial conditions: $L_0 = 2$ and $L_1 = 1$.

From (2), we get:

$$S_{n+1} = a\,S_n + (a+b)\,S_{n-1} + \ldots + (a+nb)\,S_0 + (a+(n+1)b)$$

$$= a\,S_n + a\,S_{n-1} + \ldots + (a+(n-1)b)\,S_0 + (a+nb) + b\,(S_{n-1} + \ldots + S_0 + 1)$$

$$\Rightarrow S_{n+1} = a\,S_n + S_n + b\,(S_{n-1} + \ldots + S_0 + 1) \tag{4.1}$$

Similarly

$$S_{n+2} = a\,S_{n+1} + S_{n+1} + b\,(S_n + \ldots + S_0 + 1) \tag{4.2}$$

From (4.2) and (4.1), we get:

$$b\,(S_{n-1} + \ldots + S_0 + 1) =$$

$$S_{n+2} - a\,S_{n+1} - S_{n+1} - b\,S_n = S_{n+1} - a\,S_n - S_n$$

$$\Rightarrow S_{n+2} = (a+2)\,S_{n+1} + \left(b - a - 1\right) S_n \tag{5}$$

(5) is a second order linear recurrence relation.

Case1: When $a = b = S_0 = 1$,

(i) from (2), we get: $S_1 = 3 = F_4$ and $S_2 = 8 = F_6$;

(ii) from (5), we get: $S_{n+2} = 3\,S_{n+1} - S_n$.

Then $S_3 = 3S_2 - S_1 = 3F_6 - F_4 = F_8$;

$S_4 = 3S_3 - S_2 = 3F_8 - F_6 = F_{10}$;

… …

In general

$$S_n = F_{2n+2} \tag{6}$$

In the solution, we use a Fibonacci formula:

$$F_{2n+4} = 3F_{2n+2} - F_{2n} \tag{7}$$

(7) is a particular Fibonacci relation from (3) obtained by the substitutions of $2n + 2$ and 2 for *n* and *m* respectively. From (6) and (2), we get:

$$F_{2n+2} = F_{2n} + 2F_{2n-2} + \ldots + nF_2 + (n+1)$$

$$\Rightarrow F_{2n} = \sum_{i=0}^{n-1} i\,F_{2(n-i)} + n \tag{8}$$

Case 2: When the triplet (a, b, S_0) is (1, 1, –1), then from (2) and (5), we get:

$S_1 = 1 = F_1$; $S_2 = 2 = F_3$; and $S_{n+2} = 3\,S_{n+1} - S_n$. From these results, we get the solution for S_n as:

$$S_n = F_{2n-1} \tag{9}$$

by the application of the Fibonacci formula:

$$F_{2n+3} = 3F_{2n+1} - F_{2n-1} \tag{10}$$

(10) is another Fibonacci formula from (3) obtained by the substitution of $2n + 1$ and 2 for *n* and *m* respectively.

From (2) and (9), we get

$$F_{2n-1} = \sum_{i=0}^{n-1} i\, F_{2(n-i)-1} + 1 \tag{11}$$

(8) and (11) are the required identities. (2) can yield some more Fibonacci identities and also some Lucas identities for the particular values of the triplet (a, b, S_0).

4. Ordered Fibonacci Numbers from the Summation Series Involving Ordered Compositions

4.1 Two Theorems

Theorem 1. *If $\prod C(n)$ for n ≥ 2 denotes the summation series such that $\prod C(n)$ = product of the summands of 1st C(n) in SOC(n) + product of the summands of 2nd C(n) in SOC(n) + ... + product of the summands of $(2^{n-1}-1)^{th}$ C(n) in SOC(n) + n, then $\prod C(n) = F_{2n}$. If the initial condition is defined as: $\prod C(1) = 1$, then the result in general is: for n ≥ 1, $\prod C(n) = F_{2n}$.*

Proof: It follows from (1) and the definition of $\prod C(n)$ that for $n \geq 2$,

$$\prod C(n) = \prod C(n-1) + 2\prod C(n-2) + \dots + (n-1)\prod C(1) + n \tag{12}$$

The initial condition is: $\prod C(1) = 1 = F_2$. Then from (12), we get:

$$\prod C(2) = \prod C(1) + 2 = 1 + 2 = 3 = F_4 \;;$$

$$\prod C(3) = \prod C(2) + 2\prod C(1) + 3 = 3 + 2 + 3 = 8 = F_6 \;.$$

We assume that the theorem is true for the first *n* natural numbers for any given *n*. Then from (12) and (8), we deduce that

$$\prod C(n+1) = \prod C(n) + 2\prod C(n-1) + \dots + n\prod C(1) + n + 1$$

$$= F_{2n} + 2F_{2n-2} + \dots + nF_2 + n + 1 = F_{2n+2}$$

Hence we have the theorem by induction. ▮

Theorem 2. *If $\prod C(n)^{\pm}$ for n ≥ 2 denotes the summation series obtained by changing the same sign of the series $\prod C(n)$ for n ≥ 2 by alternating signs starting with + sign, then $\prod C(n)^{\pm} = -F_{2n-3}$.*

Proof: From the definition of $\prod C(n)^{\pm}$, we have

$\prod C(2)^{\pm} = 1\cdot 1 - 2;$

$\prod C(3)^{\pm} = 1\cdot 1\cdot 1 - 1\cdot 2 + 2\cdot 1 - 3 = \prod C(2)^{\pm} + 2\cdot 1 - 3;$

$\prod C(4)^{\pm} = \left(1\cdot 1\cdot 1\cdot 1 - 1\cdot 1\cdot 2 + 1\cdot 2\cdot 1 - 1\cdot 3\right) + \left(2\cdot 1\cdot 1 - 2\cdot 2\right) + 3\cdot 1 - 4$

$= \prod C(3)^{\pm} + 2\prod C(2)^{\pm} + 3\cdot 1 - 4$; and so on.

In general for $n \geq 3$, $\prod C(n)^{\pm} =$

$$\prod C(n-1)^{\pm} + 2\prod C(n-2)^{\pm} + \dots + (n-2)\prod C(2)^{\pm} + (n-1)\cdot 1 - n$$

$$\Rightarrow \prod C(n)^{\pm} = \prod C(n-1)^{\pm} + 2\prod C(n-2)^{\pm} + \dots + (n-2)\prod C(2)^{\pm} - 1 \tag{13}$$

Now we follow the rules of induction.

(1) $\prod C(3)^{\pm} = -2 = -F_3$

(2) Assume the theorem is true for all $n \in \mathbb{N}$ with $3 \leq n \leq m$. Then from (13) and (11), we deduce that

$\prod C(m+1)^{\pm} = \prod C(m)^{\pm} + 2\prod C(m-1)^{\pm} + \dots + (m-1)\prod C(2)^{\pm} - 1$

$= -F_{2m-3} + 2\cdot -F_{2m-5} + \dots + (m-1)\cdot -F_1 - 1$

$= -F_{2m-1}$

Hence by inductive process, we establish:

$$\text{for } n \geq 3,\quad \prod C(n)^{\pm} = -F_{2n-3}.$$

Again we have $\prod C(2)^{\pm} = -1 = -F_1$.

The results together prove the theorem. ▮

4.2 Consequences of the Theorems

Theorem1 and Theorem 2 lead to find some relationships among $\prod C(n)$, $\prod C(n)^{\pm}$ and Fibonacci sequence.

Consequence1. Let the successive 2^{n-1} terms of $\prod C(n)$ for $n \geq 2$ be denoted by $i_1, i_2, \dots, i_r$ where $r = 2^{n-1}$. Then for $n \geq 2$,

$$\prod C(n) + \prod C(n)^{\pm} = 2(i_1 + i_3 + \ldots + i_{r-1})$$
$$= F_{2n} - F_{2n-3} = 2F_{2n-2}$$
$$\prod C(n) - \prod C(n)^{\pm} = 2(i_2 + i_4 + \ldots + i_r)$$
$$= F_{2n} + F_{2n-3} = 2F_{2n-1}$$

Hence

$$i_1 + i_3 + \ldots + i_{r-1} = F_{2n-2} \tag{14.1}$$
$$i_2 + i_4 + \ldots + i_r = F_{2n-1} \tag{14.2}$$

(14.1) and (14.2) imply that the sum of the terms of odd places and this of even places of $\prod C(n)$ for $n \geq 2$ are F_{2n-2} and F_{2n-1} respectively.

Consequence 2. By Theorem1, $\prod C(n) = F_{2n}$ and has an expansion of 2^{n-1} terms. Again from (12), we get:

$\prod C(n) = 1 \cdot \prod C(n-1) + \ldots$;

$1 \cdot \prod C(n-1) = 1^2 \cdot \prod C(n-2) + \ldots$;

$1^2 \cdot \prod C(n-2) = 1^3 \cdot \prod C(n-3) + \ldots$; and so on.

It follows that the sum of all 2^{n-2} terms of $\prod C(n-1) = F_{2n-2}$ = the sum of 1st 2^{n-2} terms of $\prod C(n)$; the sum of all 2^{n-3} terms of $\prod C(n-2) = F_{2n-4}$ = the sum of 1st 2^{n-3} terms of $\prod C(n)$; and in this way, the initial condition is: the sum of 2 terms of $\prod C(2) = F_4$ = the sum of 1st 2 terms of $\prod C(n)$. In general from Theorem 1 and (12), we get: for $n - 2 \geq k \geq 1$, the sum of all 2^k terms of $\prod C(k+1) = F_{2(k+1)}$ = the sum of 1st 2^k terms of $\prod C(n)$. Furthermore the first term of $\prod C(n) = 1^n = 1 = F_2$. In other words the 1st term, the sums of 1st 2, 1st 2^2, … 1st 2^{n-2} and all 2^{n-1} terms of $\prod C(n)$ for $n \geq 2$ are $F_2, F_4, F_6, \ldots, F_{2n-2}$ and F_{2n} in succession. Consequently the 2nd term, the sums of 2nd 2, 2nd 2^2, …, 2nd 2^{n-2} terms of $\prod C(n)$ for $n \geq 2$ are $(F_4 - F_2, F_6 - F_4, F_8 - F_6, \ldots, F_{2n} - F_{2n-2})$ or $(F_3, F_5, \ldots, F_{2n-1})$ in succession.

In like manner from Theorem2, we can find that the sums of 1st 2, 1st 2^2, …, 1st 2^{n-2}, all 2^{n-1} terms of $\prod C(n)^{\pm}$ for $n \geq 3$ are $-F_1, -F_3, \ldots -F_{2n-5}, -F_{2n-3}$ and the sums of 2nd 2, 2nd 2^2, …, 2nd 2^{n-2} terms of $\prod C(n)^{\pm}$ for $n \geq 3$ are $-F_2, -F_4, \ldots, -F_{2n-4}$ successively.

Consequence 3. Let $A = i_1 + i_3 + \ldots + i_{r-1}$ and $B = i_2 + i_4 + \ldots + i_r$.

From (12) and (13), we get:

$\prod C(n) + \prod C(n)^{\pm} = 1 \cdot [\prod C(n-1) + \prod C(n-1)^{\pm}] + \ldots$

Then we have

$1 \cdot [\prod C(n-1) + \prod C(n-1)^{\pm}] = 1^2 \cdot [\prod C(n-2) + \prod C(n-2)^{\pm}] + \ldots$

$1^2 \cdot [\prod C(n-2) + \prod C(n-2)^{\pm}] = 1^3 \cdot [\prod C(n-3) + \prod C(n-3)^{\pm}] + \ldots$

… …

$1^{n-3} \cdot [\prod C(3) + \prod C(3)^{\pm}] = 1^{n-2} \cdot [\prod C(2) + \prod C(2)^{\pm}] + \ldots$

Again from Consequence 1,

$A = \frac{1}{2}(\prod C(n) + \prod C(n)^{\pm}) = i_1 + i_3 + \ldots + i_{r-1} = F_{2n-2}$ where $= 2^{n-1}$.

Then

$\frac{1}{2}(\prod C(n-1) + \prod C(n-1)^{\pm}) = i_1 + i_3 + \ldots + i_{s-1} = F_{2n-4}$ where $s = 2^{n-2}$;

… …

$\frac{1}{2}(\prod C(4) + \prod C(4)^{\pm}) = i_1 + i_3 + i_5 + i_7 = F_6$;

$\frac{1}{2}(\prod C(3) + \prod C(3)^{\pm}) = i_1 + i_3 = F_4$;

$i_1 = 1^n = 1 = F_2$.

This implies that the 1st term, the sums of 1st 2, 1st 2^2, …, 1st 2^{n-2}, all 2^{n-1} terms of A are $F_2, F_4, F_6, \ldots, F_{2n-4}, F_{2n-2}$; and then the 2nd term, the sums of 2nd 2, 2nd 2^2 , …, 2nd 2^{n-2} terms of A are $F_3, F_5, F_7, \ldots, F_{2n-3}$ successively.

Similarly the 1st term, the sums of 1st 2, 1st 2^2, …, 1st 2^{n-2}, all 2^{n-1} terms of B are $F_3, F_5, F_7, \ldots, F_{2n-3}, F_{2n-1}$; and the 2nd term, the sums of 2nd2, 2nd 2^2, …, 2nd 2^{n-2} terms of B are F_4, F_6, F_8, …, F_{2n-2} successively.

From the consequences of the theorems, it follows that both $\prod \mathrm{C}(n)$ and $\prod C(n)^{\pm}$ have the sets of terms in the definite orders such that the sums of these sets of terms represent ordered Fibonacci numbers with some repetitions. Thus it is a remarkable fact that there exists a special order of the compositions of a natural number *n*, which has very close connection with the famous Fibonacci sequence.

References

Arthur, T. B., & Jennifer, Q. (2003). Proofs That Really Count: The Art of Combinatorial Proof, MAA.

Richard, S. (2011). Enumerative Combinatorics, volume 1, second edition, Cambridge University Press, Exercise 35 of Chapter 1.

Vajda, S. (2008). Fibonacci and Lucas Numbers, and the Golden Section: Theory and Applications, Dover Publications, Inc. New York, Formula (15a).

Annexure

SOC(*n*) for *n* = 6 and for *n* = 7 are listed below. The list can be useful to examine some important results that are established in the paper.

SOC(6) ≡ 1 + 1 + 1 + 1 + 1 + 1, 1 + 1 + 1 + 1 + 2, 1 + 1 + 1 + 2 + 1, 1 + 1 + 1 + 3, 1 + 1 + 2 + 1 + 1, 1 + 1 + 2 + 2, 1 + 1 + 3 + 1, 1 + 1 + 4, 1 + 2 + 1 + 1 + 1, 1 + 2 + 1 + 2, 1 + 2 + 2 + 1, 1 + 2 + 3, 1 + 3 + 1 + 1, 1 + 3 + 2, 1 + 4 + 1, 1 + 5, 2 + 1 + 1 + 1 + 1, 2 + 1 + 1 + 2, 2 + 1 + 2 + 1, 2 + 1 + 3, 2 + 2 + 1 + 1, 2 + 2 + 2, 2 + 3 + 1, 2 + 4, 3 + 1 + 1 + 1, 3 + 1 + 2, 3 + 2 + 1, 3 + 3, 4 + 1 + 1, 4 + 2, 5 + 1, 6.

SOC(7) ≡ 1 + 1 + 1 + 1 + 1 + 1 + 1, 1 + 1 + 1 + 1 + 1 + 2, 1 + 1 + 1 + 1 + 2 + 1, 1 + 1 + 1 + 1 + 3, 1 + 1 + 1 + 2 + 1 + 1, 1 + 1 + 1 + 2 + 2, 1 + 1 + 1 + 3 + 1, 1 + 1 + 1 + 4, 1 + 1 + 2 + 1 + 1 + 1, 1 + 1 + 2 + 1 + 2, 1 + 1 + 2 + 2 + 1, 1 + 1 + 2 + 3, 1 + 1 + 3 + 1 + 1, 1 + 1 + 3 + 2, 1 + 1 + 4 + 1, 1 + 1 + 5, 1 + 2 + 1 + 1 + 1 + 1, 1 + 2 + 1 + 1 + 2, 1 + 2 + 1 + 2 + 1, 1 + 2 + 1 + 3, 1 + 2 + 2 + 1 + 1, 1 + 2 + 2 + 2, 1 + 2 + 3 + 1, 1 + 2 + 4, 1 + 3 + 1 + 1 + 1, 1 + 3 + 1 + 2, 1 + 3 + 2 + 1, 1 + 3 + 3, 1 + 4 + 1 + 1, 1 + 4 + 2, 1 + 5 + 1, 1 + 6, 2 + 1 + 1 + 1 + 1 + 1, 2 + 1 + 1 + 1 + 2, 2 + 1 + 1 + 2 + 1, 2 + 1 + 1 + 3, 2 + 1 + 2 + 1 + 1, 2 + 1 + 2 + 2, 2 + 1 + 3 + 1, 2 + 1 + 4, 2 + 2 + 1 + 1 + 1, 2 + 2 + 1 + 2, 2 + 2 + 2 + 1, 2 + 2 + 3, 2 + 3 + 1 + 1, 2 + 3 + 2, 2 + 4 + 1, 2 + 5, 3 + 1 + 1 + 1 + 1, 3 + 1 + 1 + 2, 3 + 1 + 2 + 1, 3 + 1 + 3, 3 + 2 + 1 + 1, 3 + 2 + 2, 3 + 3 + 1, 3 + 4, 4 + 1 + 1 + 1, 4 + 1 + 2, 4 + 2 + 1, 4 + 3, 5 + 1 + 1, 5 + 2, 6 + 1, 7.

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