Characterizations of Intra-Regular Left Almost Semigroups by Their Fuzzy Ideals

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Abstract

In this paper we have introduced fuzzy quasi-ideal and fuzzy left(right, two-sided) ideals in LA-semigroup. We have proved some results related to fuzzy quasi-ideals and fuzzy left(right, two-sided) ideals of an LA-semigroup. Further we characterize an intra-regular LA-semigroup by the properties of their fuzzy ideals.

Keywords: LA-semigroup, Left invertive law, Medial law, Fuzzy ideals

1. Introduction

The fuzzy set theory and its applications has been growing rapidly now a days. These applications are used in various fields such as computer science, artificial intelligence, operation research, management science, control engineering, robotics, expert systems and many others.

Fuzzification functions in fuzzy image processing, fuzzy data bases, fuzzy decision making and fuzzy linear programming. It has a wide range of applications in engineering such as civil engineering, mechanical engineering, industrial engineering computer engineering, rehability theory and robotic. Moreover the usage of fuzzification can be found in mechanics, economics, fuzzy systems and genetic algorithms.

In J. N. Mordeson (2003) has discovered the grand exploration of fuzzy semigroups, where theoretical explanation of fuzzy semigroups and their applications are used in fuzzy coding, fuzzy finite state mechanics and fuzzy languages. The use of fuzzification in automata and formal language has already been shown.

People take semigroup theory more seriously due to the vital link between semigroup theory and the theoretical computer science.

A fuzzy subset f of a given set S is described as an arbitrary function $f : S \longrightarrow [0, 1]$, where $[0, 1]$ is the usual closed interval of real numbers. This fundamental concept of fuzzy sets, was first introduced by L. A. Zadeh in his paper in 1965. The concept of fuzzy ideals in semigroups was first developed by N. Kuroki (1979). (Rosenfeld, A. 1971) was first who considered the case when S is a group. Here in this paper we have introduced the concept of fuzzy theory in a non-associative structure known as left almost semigroup. The preliminaries about this structure are given below.

A Left almost semigroup (LA-semigroup) or Abel-Grassmann's groupoid (AG-groupoid) (Protic, P. V. 1995), is a groupoid S holding left invertive law

$$(ab)c = (cb)a, \text{ for all } a, b, c \in S. \tag{1}$$

In an LA-semigroup medial law (Kazim, M. A. 1972),

$$(ab)(cd) = (ac)(bd), \text{ holds for all } a, b, c, d \in S. \tag{2}$$

The left identity in an LA-semigroup if it exists is unique (Mushtaq, Q. 1978). In an LA-semigroup S with left identity the paramedial law

$$(ab)(cd) = (db)(ca), \text{ holds, for all } a, b, c, d \in S. \tag{3}$$

An LA-semigroup is a non-associative algebraic structure mid way between a groupoid and a commutative semigroup with wide applications in the theory of flocks. A groupoid satisfying (2) and (3) is called medial and paramedial groupoid explored extensively in (Jezek, J. 1993) and (Jezek, J. 2002). It is interesting to note that an LA-semigroup S with left identity becomes medial and paramedial. However LA-semigroup with right identity becomes a commutative semigroup with identity (Mushtaq, Q. 1978). An LA-semigroup S is said to be locally associative LA-semigroup if for all $a \in S$, $(aa)a = a(aa)$ holds. If an LA-semigroup holds the following law,

$$a(bc) = b(ac), \text{ for all } a, b, c \in S. \tag{4}$$

Then it is called an AG**-groupoid.

Q. Iqbal and Q. Mushtaq proved in (Mushtaq, Q. 1990) that in a locally associative LA-semigroup S, $(ab)^n = a^n b^n$ holds for all $a, b \in S$. Also it has been proved in (Mushtaq, Q. 2009) that if S is a locally associative AG**-groupoid then $a^n b^m = b^m a^n$ holds for all $a, b \in S$ and $m, n \geq 2$, while these identities holds in commutative semigroups.

P. Holgate called the AG-groupoids simple invertive groupoids(Holgate, 1992). A commutative inverse semigroup $(S, \circ)$ becomes an LA-semigroup $(S, \cdot)$ under the following relation $a \cdot b = b \circ a^{-1}$ (Mushtaq, Q. 1988). A binary operation "$\circ$" on an LA-semigroup S has been defined as, if for all $x, y \in S$, there exists a such that $x \circ y = (xa)y$, (Stevanović, N. 2004).

If S contains left identity, then "$\circ$" become associative. Using (4), we get

$$\begin{aligned} (x \circ y) \circ z &= (((xa)y)a)z = (za)((xa)y) = (xa)((za)y) \\ &= (xa)((ya)z) = x \circ (y \circ z). \end{aligned}$$

Hence $(S, \circ)$ is a semigroup. Connection discussed above make this non-associative structure interesting and useful.

LA-semigroup has vast applications in collaboration with semigroup like other branches of mathematics.

Here we begin with an example of an LA-semigroup.

Example 1 Let $S = \{1, 2, 3, 4\}$. The following multiplication table shows that S is an LA-semigroup with left identity 3.

$\cdot$	1	2	3	4
1	3	4	1	2
2	2	3	4	1
3	1	2	3	4
4	4	1	2	3

It is easy to note that if an LA-semigroup S contains left identity then $eS = Se = S^2 = S$.

Let f and g be any fuzzy subsets of an LA-semigroup S, then the product $f \circ g$ is defined by

$$(f \circ g)(a) = \begin{cases} \bigvee\limits_{a=bc} \{f(b) \wedge g(c)\}, & \text{if there exist } a, b \in S, \text{ such that } a = bc \\ 0, & \text{otherwise.} \end{cases}$$

A fuzzy subset f of an LA-semigroup S is called a fuzzy LA-subsemigroup if $f(xy) \geq f(x) \wedge f(y)$ for all $x, y \in S$.

A fuzzy subset f of an LA-semigroup S is called fuzzy ideal if $f(xy) \geq f(x) \vee f(y)$ for all $x, y \in S$.

A fuzzy LA-subsemigroup f of an LA-semigroup S is called fuzzy bi-ideal if $f((xy)z) \geq f(x) \wedge f(z)$, for all x, y and $z \in S$.

A fuzzy LA-subsemigroup f of an LA-semigroup S is called fuzzy interior ideal if $f((xa)z) \geq f(a)$, for all x, a and $y \in S$.

A fuzzy subset f of an LA-semigroup S is a fuzzy quasi-ideal of S if $(f \circ S) \cap (S \circ f) \subseteq f$.

The proof of the following two lemma's are given in (Khan, M. N. A. 2009).

Lemma 1. *Let S be an LA-semigroup, then the following properties hold in S.*

(i) $(f \circ g) \circ h = (h \circ g) \circ f$ for all fuzzy subsets f, g and h of S.

(ii) $(f \circ g) \circ (h \circ k) = (f \circ h) \circ (g \circ k)$ for all fuzzy subsets f, g, h and k of S.

Lemma 2. *Let S be an LA-semigroup with left identity, then the following properties hold in S.*

(i) $f \circ (g \circ h) = g \circ (f \circ h)$ for all fuzzy subsets f, g and h of S.

(ii) $(f \circ g) \circ (h \circ k) = (k \circ h) \circ (g \circ f)$ for all fuzzy subsets f, g, h and k of S.

Let $F(S)$ denote the collection of all fuzzy subsets of an LA-semigroup S with left identity, then $(F(S), \circ)$ becomes an LA-semigroup with left identity S, that is $(F(S), \circ)$ satisfy all properties (1), (2), (3) and (4).

The proof of the following two lemma's are same as in (Mordeson, J. N., 2003).

Lemma 3. *Let f be a fuzzy subset of an LA-semigroup S. Then the following properties hold.*

(i) f is a fuzzy LA-subsemigroup of S if and only if $f \circ f \subseteq f$.

(ii) f is a fuzzy left(right) ideal of S if and only if $S \circ f \subseteq f(f \circ S \subseteq f)$.

(iii) f is a fuzzy two-sided ideal of S if and only if $S \circ f \subseteq f$ and $f \circ S \subseteq f$.

Lemma 4. *Let f be a fuzzy LA-subsemigroup of an LA-semigroup S. Then f is a fuzzy bi-ideal of S if and only if $(f \circ S) \circ f \subseteq f$.*

Note that an LA-semigroup S with left identity can be considered as a fuzzy subset of S itself, then we can write $S \circ S = S$ and $S = C_S$ for all $x \in S$.

The principal left, right and two-sided ideals of an LA-semigroup S generated by a^2 is denoted by $L[a^2]$, $R[a^2]$ and $J[a^2]$. Note that the principal left, right and two-sided ideals generated by a^2 are equals, that is,

$$L[a^2] = R[a^2] = J[a^2] = Sa^2 = a^2S = Sa^2S = \{sa^2 : s \in S\}.$$

The characteristic function C_A for a subset A of an LA-semigroup S is defined by

$$C_A(x) = \begin{cases} 1, \text{ if } x \in A, \\ 0, \text{ if } x \notin A. \end{cases}$$

The proof of the following three lemma's are same as in (Khan, M. N. A. 2009).

Lemma 5. *Let A be a non-empty subset of an LA-semigroup S. Then the following properties holds.*

(i) A is an LA-subsemigroup if and only if C_A is a fuzzy LA-subsemigroup of S.

(ii) A is a left(right, two-sided) ideal of S if and only if C_A is a fuzzy left(right, two-sided) of S.

Lemma 6. *Let A be a non-empty subset of an LA-semigroup S. Then A is a bi-ideal of S if and only if C_A is a fuzzy bi-ideal of S.*

Lemma 7. *Let f be a fuzzy left ideal of an LA-semigroup S with left identity then $S \circ f = f$.*

Lemma 8. *Let A be a non-empty subset of an LA-semigroup S, then A is a quasi-ideal of S if and only if the characteristic function C_A is a fuzzy quasi-ideal of S.*

proof. It is same as in (Mordeson, J. N., 2003).

Lemma 9. *Let f be any fuzzy right ideal and g be any fuzzy left ideals of S, then $f \cap g$ is a fuzzy quasi-ideal of S.*

proof. It is easy to observe the following,

$$((f \cap g) \circ S) \cap (S \circ (f \cap g)) \subseteq (f \circ S) \cap (S \circ g) \subseteq f \cap g.$$

Lemma 10. *Every fuzzy quasi-ideal of an LA-semigroup S is a fuzzy LA-subsemigroup of S.*

proof. Let f be any fuzzy quasi-ideal of S, then $f \circ f \subseteq f \circ S$, and $f \circ f \subseteq S \circ f$, therefore

$$f \circ f \subseteq f \circ S \cap S \circ f \subseteq f.$$

Hence f is a fuzzy LA-subsemigroup of S.

A fuzzy subset f of an LA-semigroup S is called idempotent, if $f \circ f = f$, shortly $f^2 = f$.

Lemma 11. *In an LA-semigroup S, every idempotent fuzzy quasi-ideal is a fuzzy bi-ideal of S.*

proof. Let f be any fuzzy quasi-ideal of S, then by lemma 10, f is a fuzzy LA-subsemigroup. Now by using (2) we have

$$(f \circ S) \circ f \subseteq (S \circ S) \circ f \subseteq S \circ f \text{ and}$$

$$(f \circ S) \circ f = (f \circ S) \circ (f \circ f) = (f \circ f) \circ (S \circ f) \subseteq f \circ (S \circ S) \subseteq f \circ S,$$

which implies that $(f \circ S) \circ f \subseteq (f \circ S) \cap (S \circ f) \subseteq f$.

Hence by lemma 4, f is a fuzzy bi-ideal of S.

Lemma 12. *Let f or g is an idempotent fuzzy quasi-ideal of an LA-semigroup S with left identity, then $f \circ g$ or $g \circ f$ is a fuzzy bi-ideal of S.*

proof. Clearly $f \circ g$ is a fuzzy LA-subsemigroup. Now using lemma 4 and 11, (1), (3) and (2), we have

$$\begin{aligned}((f \circ g) \circ S) \circ (f \circ g) &= ((S \circ g) \circ f) \circ (f \circ g) \subseteq ((S \circ S) \circ f) \circ (f \circ g) \\ &= (S \circ f) \circ (f \circ g) = (g \circ f) \circ (f \circ S) \\ &= ((f \circ S) \circ f) \circ g \subseteq (f \circ g).\end{aligned}$$

Similarly we show that $g \circ f$ is a fuzzy bi-ideal of S.

Lemma 13. *In an LA-semigroup S, each one sided fuzzy (left,right) ideal is a fuzzy quasi-ideal of S.*

proof. It is obvious.

Corollary 1. *In an LA-semigroup S, every fuzzy two sided ideal of S is a fuzzy quasi-ideal of S.*

Lemma 14. *The intersection of any set of fuzzy quasi-ideals of S is either empty or a fuzzy quasi-ideal of S.*

proof. It is same as in (Mordeson, J. N., 2003).

Lemma 15. *The product of two fuzzy left(right) ideals of an LA-semigroup S with left identity is a fuzzy left(right) ideal of S.*

proof. Let f and g be any two fuzzy left ideals of S, then by using (4), we have

$$S \circ (f \circ g) = f \circ (S \circ g) \subseteq f \circ g.$$

Let f and g be any two fuzzy right ideals of S, then by using (2), we have

$$(f \circ g) \circ S = (f \circ g) \circ (S \circ S) = (f \circ S) \circ (g \circ S) \subseteq f \circ g.$$

Lemma 16. *If g is a fuzzy right ideal of an LA-semigroup S with left identity, then $g \circ S = g$.*

proof. Let $a \in S$, then by using (1), we have $a = ea = (ee)a = (ae)e$. Thus

$$(g \circ S)(a) = \bigvee_{a=(ae)e} \{g(ae) \wedge S(e)\} \geq \{g(ae) \wedge S(e)\} \geq \{g(a) \wedge 1\} = g(a).$$

Lemma 17. *In an LA-semigroup S, each one sided fuzzy (left,right) ideal is a fuzzy generalized bi-ideal of S.*

proof. Assume that f be any fuzzy left ideal of S and $a, b, c \in S$, then by using (1) we have $f((ab)c) \geq f((cb)a) \geq f(a)$ and $f((ab)c) \geq f(c)$. Thus $f((ab)c) \geq f(a) \wedge f(c)$. Similarly in case of right ideal.

An element a of an LA-semigroup S is called intra-regular if there exists elements $x, y \in S$ such that $a = (xa^2)y$ and S is called intra-regular if every element of S is intra-regular.

Note that in an intra-regular LA-semigroup S, we can write $S \circ S = S$.

Example 2 Let $S = \{1, 2, 3, 4, 5\}$ be an LA-semigroup with left identity 5 with the following multiplication table.

$\cdot$	1	2	3	4	5
1	5	1	2	3	4
2	4	5	1	2	3
3	3	4	5	1	2
4	2	3	4	5	1
5	1	2	3	4	5

Clearly S is an intra-regular because $(2 \cdot 1^2) \cdot 4 = 1$, $(3 \cdot 2^2) \cdot 4 = 2$, $(5 \cdot 3^2) \cdot 3 = 3$, $(5 \cdot 4^2) \cdot 4 = 4$, $(2 \cdot 5^2) \cdot 3 = 5$.

Lemma 18. *A fuzzy subset f of an intra-regular LA-semigroup S is a fuzzy right ideal if and only if it is a fuzzy left ideal.*

proof. Assume that f is a fuzzy right ideal of S. Since S is intra-regular, so for each $a \in S$ there exist $x, y \in S$ such that $a = (xa^2)y$. So by using (1), we have

$$f(ab) = f(((xa^2)y)b) = f((by)(xa^2)) \geq f(by) \geq f(b).$$

Conversely, assume that f is a fuzzy left ideal of S, then using (1), we have

$$f(ab) = f(((xa^2)y)b) = f((by)(xa^2)) \geq f(xa^2) \geq f(a^2) \geq f(a).$$

Lemma 19. *Every fuzzy two-sided ideal of an intra-regular LA-semigroup S with left identity is idempotent.*

proof. Assume that f is a fuzzy two-sided ideal of S, then clearly $f \circ f \subseteq f \circ S \subseteq f$. Since S is intra-regular, so for each $a \in S$ there exist $x, y \in S$ such that $a = (xa^2)y$, so by using (4) and (1) we have

$$a = (xa^2)y = (x(aa))y = (a(xa))y = (y(xa))a.$$

Thus we have,

$$\begin{aligned}(f \circ f)(a) &= \bigvee_{a=(y(xa))a} \{f(y(xa)) \wedge f(a)\} \geq f(y(xa)) \wedge f(a) \\ &\geq f(a) \wedge f(a) = f(a).\end{aligned}$$

Hence $f \circ f = f$.

Theorem 1. *For a fuzzy subset f of an intra-regular LA-semigroup S with left identity, the following conditions are equivalent.*

(i) f is a fuzzy bi-ideal ideal of S.

(ii) f is a fuzzy generalized bi-ideal of S.

proof. $(i) \Longrightarrow (ii)$

Let f be any fuzzy bi-ideal of S, then obviously f is a fuzzy generalized bi-ideal of S.

Now $(ii) \Rightarrow (i)$

Let f be any fuzzy generalized bi-ideal of S, and a, $b \in S$. Then, since S is an intra-regular, so for each $a \in S$ there exist x, $y \in S$ such that $a = (xa^2)y$. So by using (3), (2) and (4), we have

$$\begin{aligned}f(ab) &= f(((xa^2)y)b) = f(((xa^2)(ey))b) = f(((ye)(a^2x))b) = f((a^2((ye)x))b) \\ &= f(((aa)((ye)x))b) = f(((x(ye))(aa))b) = f((a((x(ye))a))b) \geq f(a) \wedge f(b).\end{aligned}$$

Therefore f is a fuzzy bi-ideal of S.

Theorem 2. *For a fuzzy subset f of an intra-regular LA-semigroup S with left identity, the following conditions are equivalent.*

(i) f is a fuzzy two sided ideal of S.

(ii) f is a fuzzy bi-ideal ideal of S.

proof. $(i) \Longrightarrow (ii)$

Let f be any fuzzy two sided ideal of S, then obviously f is a fuzzy bi-ideal of S.

Now $(ii) \Rightarrow (i)$

Let f be any fuzzy bi-ideal of S and for any $a, b \in S$. Since S is intra-regular, so there exists x, y and u, $v \in S$ such that

$a = (xa^2)y$ and $b = (ub^2)v$. Therefore by using (1), (3), (2) and (4), we have

$$\begin{aligned}
f(ab) &= f(((xa^2)y)b) = f((by)(xa^2)) = f((a^2x)(yb)) = f(((yb)x)a^2) \\
&= f(((yb)x)(aa)) = f((aa)(x(yb))) = f(((x(yb))a)a) \\
&= f(((x(yb))((xa^2)y))a) = f(((xa^2)((x(yb))y))a) \\
&= f(((y(x(yb)))(a^2x))a) = f((a^2((y(x(yb)))x))a) \\
&= f(((aa)((y(x(yb)))x))a) = f(((x(y(x(yb))))(aa))a) \\
&= f((a((x(y(x(yb))))a))a) \geq f(a) \wedge f(a) = f(a). \text{ And} \\
f(ab) &= f(a((ub^2)v)) = f((ub^2)(av)) = f((va)(b^2u)) \\
&= f(b^2((va)u)) = f((bb)((va)u)) = f((((va)u)b)b) \\
&= f\Big(\Big(((va)u)((ub^2)v)\Big)b\Big) = f\Big(\Big((ub^2)(((va)u)v)\Big)b\Big) \\
&= f\Big(\Big((v((va)u))(b^2u)\Big)b\Big) = f\Big(\Big(b^2((v((va)u))u)\Big)b\Big) \\
&= f(((bb)((v((va)u))u))b) = f(((u(v((va)u)))(bb))b) \\
&= f((b((u(v((va)u)))b))b) \geq f(b) \wedge f(b) = f(b)
\end{aligned}$$

Corollary 2. *A fuzzy right ideal of an LA-semigroup S with left identity is a bi-ideal ideal of S.*

Theorem 3. *For a fuzzy subset f of an intra-regular LA-semigroup S with left identity, the following conditions are equivalent.*

(i) f is a fuzzy two sided ideal of S.

(ii) f is a fuzzy interior ideal of S.

proof. $(i) \Rightarrow (ii)$

Let f be any fuzzy two sided ideal of S, then obviously f is a fuzzy interior ideal of S.

Now $(ii) \Rightarrow (i)$

Let f be any fuzzy interior ideal of S and $a, b \in S$. Since S is intra-regular LA-semigroup, so there exist x, y and $u, v \in S$ such that $a = \left(xa^2\right)y$ and $b = \left(ub^2\right)v$, thus by using (1), (4) and (2), we have

$$\begin{aligned}
f(ab) &= f\left(\left(\left(xa^2\right)y\right)b\right) = f\left((by)\left(xa^2\right)\right) = f((by)(x(aa))) \\
&= f((by)(a(xa))) = f((ba)(y(xa))) \geq f(a).
\end{aligned}$$

Also by using (4), (3) and (2) we have

$$\begin{aligned}
f(ab) &= f\left(a\left(\left(ub^2\right)v\right)\right) = f\left(\left(ub^2\right)(av)\right) = f((b(ub))(av)) \\
&= f((va)((ub)b)) = f((ub)((va)b)) \geq f(b).
\end{aligned}$$

Hence f is a fuzzy two sided ideal of S.

Corollary 3. *A fuzzy right ideal of an LA-semigroup S with left identity is a fuzzy interior ideal of S.*

Theorem 4. *A fuzzy subset f of an intra-regular LA-semigroup S with left identity is fuzzy two-sided ideal if and only if it is quasi-ideal.*

proof. Let f be any fuzzy two-sided ideal of S, then obviously f is a fuzzy quasi-ideal of S.

Conversely, assume that f is a fuzzy quasi-ideal of S. Let a be an arbitrary element of S. Then, since S is intra-regular, so there exist element x and y in S such that $a = (xa^2)y$, by using (2) and (3), we have

$$\begin{aligned}
a &= (xa^2)y = (xa^2)(ey) = (xe)(a^2y) = a^2((xe)y) = (aa)((xe)y) \\
&= (ya)((xe)a) = (y(xe))(aa) = a((y(xe))a).
\end{aligned}$$

Also, we have

$$S \circ f = (S \circ S) \circ f = (f \circ S) \circ S.$$

Therefore,

$$\begin{aligned}(S \circ f)(a) &= ((f \circ S) \circ S)(a) \\ &= \bigvee_{a=a((y(xe))a)} \{(f \circ S)(a) \wedge S((y(xe))a)\} \\ &\geq (f \circ S)(a)\end{aligned}$$

$$f \circ S \subseteq (f \circ S) \cap (S \circ f) \subseteq f,$$

which shows that f is a fuzzy right ideal of S and by lemma 18, f is a fuzzy left ideal of S. Hence f is a fuzzy two-sided ideal of S.

Theorems 2, 1, 3 and 4 shows that fuzzy two-sided ideal, fuzzy bi-ideal, fuzzy generalized bi-ideal, fuzzy interior ideals and fuzzy quasi-ideal coincide in an intra-regular LA-semigroup with left identity.

Lemma 20. *Let A and B be any non-empty subsets of an LA-semigroup S, then the following properties hold.*

(i) $C_A \cap C_B = C_{A \cap B}$.

(ii) $C_A \circ C_B = C_{AB}$.

proof. It is same as in (Mordeson, J. N., 2003).

A subset A of an LA-semigroup S is called semiprime if, $a^2 \in A$ implies $a \in A$ for all a in S.

A fuzzy subset f of an LA-semigroup S is called fuzzy semiprime if, $f(a) \geq f(a^2)$ for all a in S.

Theorem 5. *A non-empty subset A of an LA-semigroup S is semiprime if and only the characteristic function C_A of A is fuzzy semiprime.*

proof. It is same as in (Mordeson, J. N., 2003).

Theorem 6. *For an LA-semigroup S with left identity, the following conditions are equivalent.*

(i) S is intra-regular.

(ii) $L \cap R \subseteq LR$, for every left ideal L and every right R of S and right ideal R is semiprime.

(iii) $f \cap g \subseteq f \circ g$, for every fuzzy left ideal f and every fuzzy right ideal g of S and fuzzy right ideal g is fuzzy semiprime.

proof. $(i) \Longrightarrow (iii)$

Assume that S is intra-regular. Let f and g be any fuzzy left and fuzzy right ideal of S. Since S is intra-regular, so for each $a \in S$ there exist $x, y \in S$ such that $a = (xa^2)y$, then by using (4), (1) and (2) we have

$$\begin{aligned}a &= (xa^2)y = (a(xa))y = (y(xa))a = (y(xa))(ea) = (ye)((xa)a) \\ &= (xa)((ye)a) = (xa)((ae)y).\end{aligned}$$

Thus we have

$$(f \circ g)(a) = \bigvee_{a=(xa)((ae)y)} \{f(xa) \wedge g((ae)y)\} \geq f(a) \wedge g(a) = (f \cap g)(a).$$

Therefore $f \cap g \subseteq f \circ g$, and by using (3), (2) and (4) we have

$$\begin{aligned}g(a) &= g((xa^2)y) = g((xa^2)(ey)) = g((ye)(a^2x)) \\ &= g(a^2((ye)(x))) \geq g(a^2). \text{ Hence } g \text{ is fuzzy semiprime.}\end{aligned}$$

$(iii) \Longrightarrow (ii)$

Assume that L and R be any left and right ideal of S, then by lemma 5 (ii), C_L and C_R are fuzzy left and fuzzy right ideals of S. Let $a \in L \cap R$ which implies that $a \in L$ and $a \in R$. So by lemma 20 and assumption (iii), we have

$$1 = C_{L \cap R}(a) = (C_L \cap C_R)(a) \geqslant (C_L \circ C_R)(a) = (C_{LR})(a). \text{ Hence } L \cap R \subseteq LR.$$

By assumption (iii) characteristic function C_R of a right ideal R is fuzzy semiprime, then by theorem 5, right ideal R is semiprime.

$(ii) \Longrightarrow (i)$

Let $a \in S$, then clearly $a \in Sa$, where Sa is a left ideal of S and $a^2 \in a^2S$, where a^2S is a right ideal of S. By assumption (ii) right ideal of S is semiprime, which implies that $a \in a^2S$, thus by using (2) we have

$$a \in Sa \cap a^2S \subseteq (Sa)(a^2S) = (Sa^2)(aS) \subseteq (Sa^2)(SS) = (Sa^2)S.$$

Therefore there exist $x, y \in S$ such that $a = (xa^2)y$. Hence S is intra-regular.

Theorem 7. *In an LA-semigroup S with left identity the following conditions are equivalent.*

(i) S is intra-regular.

(ii) $R \cap L = RL$ for every right ideal R and every left ideal L of S and right ideal R is semiprime.

proof. It is simple.

Theorem 8. *In an LA-semigroup S with left identity the following conditions are equivalent.*

(i) S is intra-regular.

(ii) $f \cap g = f \circ g$ for every fuzzy right ideal f and every fuzzy left ideal g of S and fuzzy right ideal f is fuzzy semiprime.

proof. $(i) \Longrightarrow (ii)$

Assume that S is intra-regular. Let f and g be any fuzzy right and fuzzy left ideal of S, then we have

$$f \circ g \subseteq f \circ S \subseteq f \text{ and } f \circ g \subseteq S \circ g \subseteq g \text{ which implies that } f \circ g \subseteq f \cap g.$$

Since S is intra-regular, so for each $a \in S$ there exist $x, y \in S$ such that $a = (xa^2)y$, now by using (4), (1), (3) and (2) we have

$$a = (xa^2)y = (x(aa))y = (a(xa))y = (y(xa))a = ((ey)(xa))a = ((ax)(ye))a.$$

Therefore we have

$$\begin{aligned}(f \circ g)(a) &= \bigvee_{a=((ax)(ye))a} \{f((ax)(ye)) \wedge g(a)\} \geq f((ax)(ye)) \wedge g(a) \\ &\geq f(a) \wedge g(a) = (f \cap g)(a).\end{aligned}$$

Thus $f \circ g = f \cap g$. Now by using (3), (2) and (4) we have

$$\begin{aligned}f(a) &= f((xa^2)y) = f((xa^2)(ey)) = f((ye)(a^2x)) \\ &= f(a^2((ye)(x))) \geq f(a^2). \text{ Hence } f \text{ is fuzzy semiprime.}\end{aligned}$$

$(ii) \Longrightarrow (i)$

Assume that R and L be any right and left ideal of S, then by lemma 5 (ii), characteristic function C_R and C_L are fuzzy right and fuzzy left ideal of S. Let $a \in R \cap L$ which implies that $a \in R$ and $a \in L$. So by lemma 20 and assumption (ii), we have

$$1 = C_{R\cap L}(a) = (C_R \cap C_L)(a) = (C_R \circ C_L)(a) = (C_{RL})(a),$$

which implies that $R \cap L \subseteq RL$, and clearly $RL \subseteq R \cap L$, so we obtained $R \cap L = RL$. Now by assumption (ii) characteristic function C_R of a right ideal R is fuzzy semiprime, then by theorem 5, R is semiprime. Hence by theorem 7, S is intra-regular.

Theorem 9. *For an LA-semigroup S with left identity, the following conditions are equivalent.*

(i) S is intra-regular.

(ii) $f \cap g \subseteq (f \circ g) \circ f$, for every fuzzy right ideal f and every fuzzy left ideal g of S and fuzzy right ideal f is fuzzy semiprime.

(iii) $f \cap g \subseteq (f \circ g) \circ f$, for every fuzzy right ideal f and every fuzzy generalized bi-ideal(fuzzy quasi-ideal) g of S and fuzzy right ideal f is fuzzy semiprime.

proof. $(i) \Longrightarrow (iii)$

Assume that S is intra-regular. Let f and g be any fuzzy right and fuzzy generalized bi-ideal(fuzzy quasi-ideal) of S. Since S is intra-regular, so for each $a \in S$ there exist $x, y \in S$ such that $a = (xa^2)y$, then by using (4), (1), (3) and (2) we have

$$\begin{aligned} a &= (xa^2)y = (a(xa))y = (y(xa))a = (y(x((xa^2)y)))a = (y((xa^2)(xy)))a \\ &= ((xa^2)(y(xy)))a = ((x(aa))(y(xy)))a = ((a(xa))(y(xy)))a \\ &= (((y(xy))(xa))a)a = (((ax)((xy)y))a)a. \end{aligned}$$

Thus we have

$$\begin{aligned} ((f \circ g) \circ f)(a) &= \bigvee_{a=(((ax)((xy)y))a)a} \{(f((ax)((xy)y)) \wedge g(a)) \wedge f(a)\} \\ &\geq (f((ax)((xy)y)) \wedge g(a)) \wedge f(a) \\ &\geq (f(a) \wedge g(a)) \wedge f(a) = (f \cap g)(a). \end{aligned}$$

Therefore $f \cap g \subseteq (f \circ g) \circ f$, and by using (3), (2) and (4) we have

$$\begin{aligned} f(a) &= f((xa^2)y) = f((xa^2)(ey)) = f((ye)(a^2x)) \\ &= f(a^2((ye)(x))) \geq f(a^2). \end{aligned}$$

Hence g is fuzzy semiprime. Clearly $(iii) \Longrightarrow (ii)$.

Now $(ii) \Longrightarrow (i)$

Assume that f and g be any fuzzy right and fuzzy left ideal of S, then by assumption (ii) we have

$$g \cap f = f \cap g \subseteq (f \circ g) \circ f \subseteq (S \circ g) \circ f \subseteq g \circ f.$$

Hence by theorem 6, S is intra-regular.

Theorem 10. *For an LA-semigroup S with left identity, the following conditions are equivalent.*

(i) S is intra-regular.

(ii) $f \cap g \subseteq f \circ g$, for every fuzzy right ideal f and every fuzzy left ideal g of S and fuzzy right ideal f is fuzzy semiprime.

(iii) $f \cap g \subseteq f \circ g$, for every fuzzy right ideal f and every fuzzy generalized bi-ideal(quasi-ideal) g of S and fuzzy right ideal g is fuzzy semiprime.

proof. $(i) \Longrightarrow (iii)$ and $(iii) \Longrightarrow (ii)$ are obvious.

Now $(ii) \Longrightarrow (i)$

It follows from theorem 8.

Theorem 11. *The set of fuzzy ideals of an intra-regular LA-semigroup S with left identity forms a semilattice structure with identity S.*

proof. Let $\mathbb{I}_S$ be the set of fuzzy ideals of an intra-regular LA-semigroup S and f, g and $h \in \mathbb{I}_S$, then clearly $\mathbb{I}_S$ is closed and by lemma 20 and theorem 31, we have $f = f^2$ and $f \circ g = f \cap g$, where f and g are fuzzy ideals of S. Clearly $f \circ g = g \circ f$, and now by using (1), we get $(f \circ g) \circ h = (h \circ g) \circ f = f \circ (g \circ h)$. Also by using (i) and lemma 7, we have $f \circ S = (f \circ f) \circ S = (S \circ f) \circ f = f \circ f = f$.

Note that in an intr-regular LA-semigroup S with left identity every fuzzy ideal becomes fuzzy semiprime.

A fuzzy ideal f of an LA-semigroup S is said to be strongly irreducible if and only if for fuzzy ideals g and h of S, $g \cap h \subseteq f$ implies that $g \subseteq f$ or $h \subseteq f$.

The set of fuzzy ideals of an LA-semigroup S is called totally ordered under inclusion if for any fuzzy ideals f and g of S either $f \subseteq g$ or $g \subseteq f$.

A fuzzy ideal h of an LA-semigroup S is called fuzzy prime ideal of S, if for any fuzzy ideals f and g of S, $f \circ g \subseteq h$, implies that $f \subseteq h$ or $g \subseteq h$.

Theorem 12. *In an intra-regular LA-semigroup S with left identity, a fuzzy ideal is strongly irreducible if and only if it is fuzzy prime.*

proof. It follows from theorem 8.

Theorem 13. *Every fuzzy ideal of an intra-regular LA-semigroup S with left identity is fuzzy prime if and only if the set of fuzzy ideals of S is totally ordered under inclusion.*

proof. It follows from theorem 8.

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