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Calmness for Closed Multifunctions over Constraint Sets in Banach Spaces

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Abstract

In this paper, we mainly study calmness and strong calmness of closed multifunctions over constraint sets in Banach spaces. In terms of tangent cones, normal cones and coderivatives, we provide some dual necessary/sufficient conditions ensuring calmness over constraint sets. In particular we proved a dual characterization for strong calmness of a closed multifunction over constraint closed sets with mild assumptions.

Keywords: calmness, L-subsmoothness, coderivative, normal cone, tangent derivative

Mathematics Subject Classification (2000) 90C31 90C25 49J52 46B20

1. Introduction

Calmness is a well-known concept in mathematical programming and optimization. Recall that a multifunction $M : Y \rightrightarrows X$ between Banach spaces Y and X is said to be calm at $(\bar{y}, \bar{x}) \in \mathrm{gph}(M)$(graph of M) if there exists $\tau \in (0, +\infty)$ such that

$$d(x, M(\bar{y})) \le \tau\|y - \bar{y}\| \text{ for all } (y, x) \in \mathrm{gph}(M) \text{ close to } (\bar{y}, \bar{x}). \tag{1}$$

Since calmness is closely related to many issues from mathematical programming and optimization, it has been extensively studied by many authors (see Poliquin, Rockafellar & Thibault, 2000; Dontchev & Rockafellar, 2004; Henrion, 2001; Henrion & Jourani, 2002; Herion, Jourani & Outrata, 2002; Herion & Outrata, 2005; Wei, Yao & Zheng, 2014; Zheng & Ng, 2007; Zheng & Ng, 2008; Zheng & Ng, 2009; Zheng & Ng, 2010 and references therein). The aim of this paper is to study calmness of multifunctions over constraint subsets. Let A be a closed subset of X. Recall that generally M is said to be calm at $(\bar{y}, \bar{x}) \in \mathrm{gph}(M)$ over A if there exist $\tau \in (0, +\infty)$ such that

$$d(x, M(\bar{y}) \cap A) \le \tau(\|y - \bar{y}\| + d(x, A)) \text{ for all } (y, x) \in \mathrm{gph}(M) \text{ close to } (\bar{y}, \bar{x}). \tag{2}$$

When we take $A := X$, (2) reduces to (1). Hence it is more general to study calmness with constraint subsets as (2).

Calmness is essentially equivalent to metric subregularity which is another well-known and important concept in mathematical programming and optimization. If we take $F(x) := \{y \in Y : x \in M(y)\}$ for all $x \in X$, then (2) is equivalent to

$$d(x, F^{-1}(\bar{y}) \cap A) \le \tau(d(\bar{y}, F(x)) + d(x, A)) \text{ for all } x \text{ close to } \bar{x}, \tag{3}$$

and (3) means that the generalized equation with constraint: $\bar{y} \in F(x)$ *subject to* $x \in A$ is metrically subregular at $\bar{x} \in F^{-1}(\bar{y}) \cap A$. The metric subregularity can be used to estimate the distance of a candidate x to the solution set of generalized equation.

Calmness is known to be a weakened version of the Aubin pseudo-Lipschitz property and closely relates to the upper Lipschitz property of multifunctions. Several subdifferential conditions ensuring calmness for multifunctions in finite-dimensional spaces have been developed. Reader are invited to consult (Henrion, 2001; Henrion & Jourani, 2002; Herion, Jourani & Outrata, 2002; Herion & Outrata, 2005; Mordukhovich, 1995) and references therein for more details. It is noted that Zheng and Ng studied calmness of convex closed multifunctions in Banach spaces and provided its dual characterizations in terms of normal cones and coderivative (see Zheng & Ng, 2007). Subsequently, in (Zheng & Ng 2009; Zheng & Ng, 2010), they further consider calmness of closed (not convex necessarily) multifunctions and gave several necessary and/or sufficient dual conditions for calmness. Motivated by (Zheng & Ng 2007; Zheng & Ng 2009; Zheng & Ng, 2010), we mainly discuss calmness of closed multifunctions over constraint subsets in this paper and aim to establish several subdifferential conditions ensuring calmness over constraint subsets via normal cones and coderivatives.

This paper is organized as follows. Several preliminaries and known results will be given in Section 2. Section 3 is devoted to main results on sufficient and/or necessary conditions for calmness and strong calmness over constraint subsets which are established by using some results in Section 2 and in terms of normal cone and coderivative. Applications of main results to calmness of one special multifunction are also given therein. The conclusion of this paper is presented in Section 4.

2. Preliminaries

Let X, Y be Banach spaces with the closed unit balls denoted by B_X and B_Y, and let X^*, Y^* denote the dual spaces of X and Y respectively. Let A be a closed subset of X and $a \in A$. Denote $T_c(A, a)$ the Clarke tangent cone of A at a which is defined as

$$T_c(A, a) = \operatorname*{Liminf}_{x \xrightarrow{A} a, t \to 0^+} \frac{A - x}{t},$$

where $x \xrightarrow{A} a$ means that $x \to a$ with $x \in A$. Therefore, $v \in T_c(A, a)$ if and only if for any $a_n \xrightarrow{A} a$ and any $t_n \to 0^+$, there exists $v_n \to v$ such that $a_n + t_n v_n \in A$ for all n.

We denote by $N_c(A, a)$ the Clarke normal cone of A at a which is defined by

$$N_c(A, a) := \{x^* \in X^* : \langle x^*, h \rangle \leq 0 \text{ for all } h \in T_c(A, a)\}.$$

Let $\hat{N}(A, a)$ denote the Fréchet normal cone of A at a which is defined by

$$\hat{N}(A, a) := \left\{ x^* \in X^* : \limsup_{y \xrightarrow{A} a} \frac{\langle x^*, y - a \rangle}{\|y - a\|} \leq 0 \right\},$$

and let $N(A, a)$ denote the Mordukhovich (limiting/basic) normal cone of A at a which is defined by

$$N(A, a) := \operatorname*{Limsup}_{x \xrightarrow{A} a, \varepsilon \downarrow 0} \hat{N}_\varepsilon(A, x),$$

where $\hat{N}_\varepsilon(A, x)$ is the set of ε-normal to A at x and defined as

$$\hat{N}_\varepsilon(A, x) := \left\{ x^* \in X^* : \limsup_{y \xrightarrow{A} x} \frac{\langle x^*, y - x \rangle}{\|y - x\|} \leq \varepsilon \right\}.$$

This means that $x^* \in N(A, a)$ if and only if there exist $x_n \xrightarrow{A} a, \varepsilon_n \to 0^+$ and $x_n^* \xrightarrow{w^*} x^*$ such that $x_n^* \in \hat{N}_{\varepsilon_n}(A, x_n)$ for all n.

For the case when X is an Asplund space (see Phelps, 1989 for definitions and their equivalences), it has been proved in (Mordukhovich & Shao, 1996) that

$$N_c(A, a) = \overline{\text{co}}^{w^*}(N(A, a)) \text{ and } N(A, a) = \operatorname*{Limsup}_{x \xrightarrow{A} a} \hat{N}(A, x) \tag{4}$$

where $\overline{\text{co}}^{w^*}$ denotes the weak* closed convex hull. Thus, $x^* \in N(A, a)$ if and only if there exist $x_n \xrightarrow{A} a$ and $x_n^* \xrightarrow{w^*} x^*$ such that $x_n^* \in \hat{N}(A, x_n)$ for all n.

It is known from (Mordukhovich, 2006) that

$$\hat{N}(A, a) \subset N(A, a) \subset N_c(A, a).$$

If A is convex, all normal cones coincide and reduce to the normal cone in the sense of convex analysis; that is

$$N_c(A, a) = N(A, a) = \hat{N}(A, a) = \{x^* \in X^* : \langle x^*, x - a \rangle \leq 0 \text{ for all } x \in A\}.$$

The following result for a general closed set is derived from (Zheng & Ng, 2008) which will be used in the proofs of our main results.

Lemma 2.1. *Let X be a Banach space and A be a nonempty closed subset of X. Let* $\gamma \in (0,\ 1)$. *Then for any* $x \notin A$ *there exist* $a \in A$ *and* $a^* \in N_c(A, a)$ *with* $\|a^*\| = 1$ *such that*

$$\gamma\|x - a\| < \min\{d(x, A),\ \langle a^*, x - a\rangle\}.$$

Let $F : X \rightrightarrows Y$ be a multifunction between X and Y. We define the graph of F by

$$\mathrm{gph}(F) := \{(x, y) \in X \times Y : y \in F(x)\}.$$

Recall that F is said to be closed if gph(F) is a closed subset of $X \times Y$. Let $(x, y) \in \mathrm{gph}(F)$. Recall that the Clarke tangent derivative $D_cF(x, y) : X \rightrightarrows Y$ of F at (x, y) is defined by

$$D_cF(x, y)(u) := \{v \in Y : (u, v) \in T_c(\mathrm{gph}(F), (x, y))\} \text{ for all } u \in X.$$

Let $\hat{D}^*F(x, y), D^*F(x, y), D^*_cF(x, y) : Y^* \rightrightarrows X^*$ denote Fréchet, Mordukhovich and Clarke coderivatives of F at (x, y) respectively and they are defined as

$$\begin{aligned}\hat{D}^*F(x, y)(y^*) &:= \{x^* \in X^* : (x^*, -y^*) \in \hat{N}(\mathrm{gph}(F), (x, y))\},\\ D^*F(x, y)(y^*) &:= \{x^* \in X^* : (x^*, -y^*) \in N(\mathrm{gph}(F), (x, y))\},\\ D^*_cF(x, y)(y^*) &:= \{x^* \in X^* : (x^*, -y^*) \in N_c(\mathrm{gph}(F), (x, y))\}.\end{aligned}$$

It is known that prox-regularity is an important extension of convexity, and prox-regularity of a set express a variational behavior of "order two". Recall that A is said to be prox-regular at $a \in A$ if there exist $\sigma, \delta > 0$ such that

$$\langle x^* - u^*, x - u\rangle \geq -\sigma\|x - u\|^2$$

whenever $x, u \in B(a, \delta) \cap A$, $x^* \in N_c(A, x) \cap B_{X^*}$ and $u^* \in N_c(A, u) \cap B_{X^*}$. (see Clarke, Stern & Wolenski, 1995; Poliquin & Rockafellar, 1996; Rockafelar & Wets, 1998).

In 2005, Aussel, Daniilidis and Thibault introduced the concept of subsmoothness which is the extension of prox-regularity and smoothness (see Aussel, Daniilidis & Thibault, 2005). This concept expresses a variational behavior of "order one".

Let A be a closed subset of X and $a \in A$. *Recall that*

(i) A is said to be subsmooth at a if for any $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\langle x^* - u^*, x - u\rangle \geq -\varepsilon\|x - u\|$$

whenever $x, u \in B(a, \delta) \cap A$, $x^* \in N_c(A, x) \cap B_{X^*}$ *and* $u^* \in N_c(A, u) \cap B_{X^*}$.

(ii) A is said to satisfy Condition (S) at a if for any $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\langle a^*, x - a\rangle \leq \varepsilon\|x - a\|$$

holds for all $x \in B(a, \delta) \cap A$ *and all* $a^* \in N_c(A, a) \cap B_{X^*}$.

It is easy to verify that A is subsmooth at a if and only if for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\langle u^*, x - u\rangle \leq \varepsilon\|x - u\|$$

whenever $x, u \in B(a, \delta) \cap A$ and $u^* \in N_c(A, u) \cap B_{X^*}$.

In 2008, Zheng and Ng further studied the concept of subsmooth and provide a characterization for this concept; that is, A is subsmooth at a if and only if for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\langle u^*, x - u\rangle \leq d(x, A) + \varepsilon\|x - u\| \text{ for all } x \in B(a, \delta)$$

holds for all $u \in B(a,\delta) \cap A$ and $u^* \in N_c(A,u) \cap B_{X^*}$ (see Zheng & Ng, 2008). Further, they considered a weakened notion which is called L-subsmooth, and studied calmness for closed multifunctions with L-subsmooth assumptions (see Zheng & Ng, 2009).

Let $M : Y \rightrightarrows X$ be a closed multifunction and $(\bar{y},\bar{x}) \in \mathrm{gph}(M)$. Recall that

(i) M is said to subsmooth (resp. satisfy Condition (S)) at $(\bar{y},\bar{x})$ if $\mathrm{gph}(M)$ is subsmooth (resp. satisfies Condition (S)) at $(\bar{y},\bar{x})$;

(ii) M is said to be L-subsmooth at $(\bar{y},\bar{x})$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $u \in M(\bar{y}) \cap B(\bar{x},\delta)$, one has

$$\langle v^*, y-\bar{y}\rangle + \langle u^*, x-u\rangle \le \varepsilon(\|y-\bar{y}\| + \|x-u\|) \tag{5}$$

holds for all $(v^*,u^*) \in N_c(\mathrm{gph}(M),(\bar{y},u)) \cap (B_{Y^*} \times B_{X^*})$ and $(y,x) \in \mathrm{gph}(M) \cap (B(\bar{y},\delta) \times B(\bar{x},\delta))$.

It is easy to verify that L-smoothness is weaker than subsmoothness but stronger than Condition (S). We refer readers to (Zheng & Ng, 2009) for more properties and examples with respect to L-subsmoothness.

For a proper lower semicontinuous convex function $\psi : X \to \mathbb{R} \cup \{+\infty\}$, recall that the subdifferential of ψ at $\bar{x} \in \mathrm{dom}(\psi) := \{x \in X : \psi(x) < +\infty\}$ is defined as

$$\partial\psi(\bar{x}) := \{x^* \in X^* : \langle x^*, x-\bar{x}\rangle \le \psi(x) - \psi(\bar{x}) \ \text{ for all } x \in X\}.$$

We close this section with the following result which is a cornerstone in convex analysis and convex optimization. Readers could consulte Theorem 3.16 in (Phelps, 1989).

Lemma 2.2. *Let $\psi_1, \psi_2 : X \to \mathbb{R} \cup \{+\infty\}$ be proper lower semicontinuous convex functions and $x \in \mathrm{dom}(\psi_1) \cap \mathrm{dom}(\psi_2)$ be such that ψ_1 is continuous at x. Then*

$$\partial(\psi_1 + \psi_2)(x) = \partial\psi_1(x) + \partial\psi_2(x).$$

3. Main Results

In this section, we main study calmness and strong calmness of closed multifunctions over constraint subsets, and aim to provide sufficient conditions for calmness and strong calmness. We begin with the definition of calmness over constraint subset.

Let $M : Y \rightrightarrows X$ be a closed multifunction and A be a closed subset of X. Let $\bar{y} \in Y$ and $\bar{x} \in M(\bar{y}) \cap A$. Recall that M is said to be calm at $(\bar{y},\bar{x})$ over A if there exist $\tau,\delta \in (0,+\infty)$ such that

$$d(x, M(\bar{y}) \cap A) \le \tau(\|y-\bar{y}\| + d(x,A)) \ \ \forall (y,x) \in \mathrm{gph}(M) \cap (B(\bar{y},\delta) \times B(\bar{x},\delta)). \tag{6}$$

Noting that $d(x,\emptyset) = +\infty$ and $d(x, M(\bar{y})) \le \|x-\bar{x}\|$, it follows that M is calm at $(\bar{y},\bar{x})$ if and only if there exist $\tau,\delta \in (0,+\infty)$ such that

$$d(x, M(\bar{y}) \cap A) \le \tau(d(\bar{y}, M^{-1}(x)) + d(x,A)) \ \ \forall x \in B(\bar{x},\delta). \tag{7}$$

The following proposition is on necessary conditions for calmness of closed multifunctions over constraint subsets. The proof can be obtained by using Proposition 4.1 and Theorem 4.2 in (Huang, He & Wei, 2014) and Theorem 4.2 in (Zheng & Ng, 2009).

Proposition 3.1. *Let $M : Y \rightrightarrows X$ be a closed multifunction, A be a closed subset of X and let $\bar{y} \in Y$ and $\bar{x} \in M(\bar{y}) \cap A$. If M is calm at $(\bar{y},\bar{x})$ over A, then there exist $\tau,\delta \in (0,+\infty)$ such that*

$$\hat{N}(M(\bar{y}) \cap A, u) \cap B_{X^*} \subset \tau(D_c^* M^{-1}(u,\bar{y})(B_{Y^*}) + N_c(A,u) \cap B_{X^*}) \tag{8}$$

holds for all $u \in M(\bar{y}) \cap A \cap B(\bar{x},\delta)$.

Assume further that X is finite-dimensional and Y is an Asplund space. If M is calm at $(\bar{y},\bar{x})$ over A, then there exist $\tau,\delta \in (0,+\infty)$ such that

$$N(M(\bar{y}) \cap A, u) \cap B_{X^*} \subset \tau(D^* M^{-1}(u,\bar{y})(B_{Y^*}) + N(A,u) \cap B_{X^*}) \tag{9}$$

holds for all $u \in M(\bar{y}) \cap A \cap B(\bar{x},\delta)$.

Let $F := M^{-1}$ and consider the following generalized equation with constraint:

$$\bar{y} \in F(x) \text{ subject to } x \in A. \tag{10}$$

By using (7), one has that M is calm at $(\bar{y}, \bar{x})$ over A if and only if the generalized equation with constraint of (10) is metrically subregular at $\bar{x} \in F^{-1}(\bar{y}) \cap A$. Thus, the proof of Proposition 3.1 can be obtained by Proposition 4.1 and Theorem 4.2 in (Huang, He & Wei, 2014).

The following proposition provides a sufficient condition for calmness of closed multifunctions over constraint subset with the help of subsmooth assumptions. We give its proof for the sake of completeness.

Proposition 3.2. *Let $M : Y \rightrightarrows X$ be a closed multifunction, A be a closed subset of X and let $\bar{y} \in Y$ and $\bar{x} \in M(\bar{y}) \cap A$. Suppose that M is L-subsmooth at $(\bar{y}, \bar{x})$, A is subsmooth at $\bar{x}$ and there exist $\tau, \delta \in (0, +\infty)$ such that*

$$N_c(M(\bar{y}) \cap A, u) \cap B_{X^*} \subset \tau(D_c^* M^{-1}(u, \bar{y})(B_{Y^*}) + N_c(A, u) \cap B_{X^*}) \tag{11}$$

holds for all $u \in M(\bar{y}) \cap A \cap B(\bar{x}, \delta)$. Then M is calm at $(\bar{y}, \bar{x})$ over A. More precise, for any $\varepsilon \in (0, \frac{1}{2\tau+1})$ there exist $\delta_1 \in (0, +\infty)$ such that

$$d(x, M(\bar{y}) \cap A) \leq \tau_1(d(\bar{y}, M^{-1}(x)) + d(x, A)) \text{ for all } x \in B(\bar{x}, \delta_1) \tag{12}$$

holds with constant $\tau_1 := \frac{(\tau+1)\varepsilon+\tau}{1-(2\tau+1)\varepsilon} > 0$.

Proof. Let $\varepsilon \in (0, \frac{1}{2\tau+1})$. Since M is L-subsmooth at $(\bar{y}, \bar{x})$ and A is subsmooth at $\bar{x}$, there exists $r \in (0, \delta)$ such that whenever $u \in M(\bar{y}) \cap B(\bar{x}, r)$, for any $(v^*, u^*) \in N_c(\text{gph(M)}, (\bar{y}, u)) \cap (B_{Y^*} \times B_{X^*})$ and any $(y, x) \in \text{gph}(M) \cap (B(\bar{y}, r) \times B(u, r))$, one has

$$\langle (v^*, u^*), (y - \bar{y}, x - u) \rangle \leq \varepsilon(\|y - \bar{y}\| + \|x - u\|), \tag{13}$$

and

$$\langle z_1^*, z_2 - z_1 \rangle \leq d(z_2, A) + \varepsilon \|z_2 - z_1\| \quad \forall z_2 \in B(\bar{x}, r) \tag{14}$$

holds for any $z_1 \in A \cap B(\bar{x}, r)$ and $z_1^* \in N_c(A, z_1) \cap B_{X^*}$.

Take $\delta_1 \in (0, \frac{r}{2})$ such that $\delta_1 < \frac{(\tau+1)\varepsilon+\tau}{1-(2\tau+1)\varepsilon} r$. Let $x \in B(\bar{x}, \delta_1) \backslash (M(\bar{y}) \cap A)$. Then $d(x, M(\bar{y}) \cap A) \leq \|x - \bar{x}\| < \delta_1$. Choose arbitrary number $\gamma \in (0, 1)$ such that $\gamma > \max\{(2\tau + 1)\varepsilon, \frac{d(x, M(\bar{y}) \cap A)}{\delta_1}\}$. By virtue of Lemma 2.1, there exist $u \in M(\bar{y}) \cap A$ and $u^* \in N_c(M(\bar{y}) \cap A, u)$ with $\|u^*\| = 1$ such that

$$\gamma \|x - u\| < \min\{d(x, M(\bar{y}) \cap A), \langle u^*, x - u \rangle\}. \tag{15}$$

Since $\|u - \bar{x}\| \leq \|u - x\| + \|x - \bar{x}\| < \frac{d(x, M(\bar{y}) \cap A)}{\gamma} + \delta_1 < r$, it follows from (11) that there exist $x_1^* \in D_c^* M^{-1}(u, \bar{y})(y_1^*)$ for some $y_1^* \in B_{Y^*}$ and $x_2^* \in N_c(A, u) \cap B_{X^*}$ such that

$$u^* = \tau(x_1^* + x_2^*). \tag{16}$$

Noting that $\|x_1^*\| \leq \|\frac{1}{\tau} u^*\| + \|x_2^*\| \leq \frac{\tau+1}{\tau}$, it follows that

$$\frac{\tau}{\tau + 1}(x_1^*, -y_1^*) \in N_c(\text{gph}(M^{-1}), (u, \bar{y})) \cap (B_{X^*} \times B_{Y^*}).$$

By using (13) and (14), one has

$$\langle x_1^*, \tilde{x} - u \rangle - \langle y_1^*, \tilde{y} - \bar{y} \rangle \leq \frac{\tau + 1}{\tau} \varepsilon(\|\tilde{x} - u\| + \|\tilde{y} - \bar{y}\|) \tag{17}$$

holds for all $(\tilde{y}, \tilde{x}) \in \text{gph}(M) \cap (B(\bar{y}, r) \times B(u, r))$ and

$$\langle x_2^*, x - u \rangle \leq d(x, A) + \varepsilon \|x - u\|. \tag{18}$$

If $M^{-1}(x) \cap B(\bar{y}, r) = \emptyset$, one has $d(\bar{y}, M^{-1}(x)) \geq r$ and thus

$$d(x, M(\bar{y}) \cap A) \leq \|x - \bar{x}\| < \delta_1 \leq \frac{(\tau + 1)\varepsilon + \tau}{1 - (2\tau + 1)\varepsilon}(d(\bar{y}, M^{-1}(x)) + d(x, A)) \tag{19}$$

(thanks to the choice of δ_1). Next, we assume that $M^{-1}(x)\cap B(\bar{y},r)\neq\emptyset$. Then, it is easy to verify that

$$d(\bar{y},M^{-1}(x))=d(\bar{y},M^{-1}(x)\cap B(\bar{y},r)). \tag{20}$$

By virtue of (17), one has

$$\langle x_1^*,x-u\rangle\ \leq\|y-\bar{y}\|+\frac{1+\tau}{\tau}\varepsilon(\|y-\bar{y}\|+\|x-u\|)\ \ \forall y\in M^{-1}(x)\cap B(\bar{y},r).$$

This and (20) imply that

$$\langle x_1^*,x-u\rangle\ \leq\ \frac{\tau+(\tau+1)\varepsilon}{\tau}d(\bar{y},M^{-1}(x))+\frac{\tau+1}{\tau}\varepsilon\|x-u\|.$$

By (15),(16) and (18), one has

$$\gamma\|x-u\|\leq((\tau+1)\varepsilon+\tau)d(\bar{y},M^{-1}(x))+\tau d(x,A)+(2\tau+1)\varepsilon\|x-u\|. \tag{21}$$

Noting that $d(x,M(\bar{y})\cap A)\leq\|x-u\|$, it follows from (21) that

$$d(x,M(\bar{y})\cap A)\leq\frac{(\tau+1)\varepsilon+\tau}{\gamma-(2\tau+1)\varepsilon}(d(\bar{y},M^{-1}(x))+d(x,A)).$$

Taking limits as $\gamma\to 1^-$ and together with (19), one has

$$d(x,M(\bar{y})\cap A)\leq\tau_1(d(\bar{y},M^{-1}(x))+d(x,A))\ \text{ with } \tau_1:=\frac{(\tau+1)\varepsilon+\tau}{1-(2\tau+1)\varepsilon}. \tag{22}$$

Hence (12) holds. This proof is complete. □

It is known from Proposition 3.2 that (11) is a key matter for proving the calmness. Naturally, our attention will be paid to equivalent conditions for ensuring (11). We first prove the following lemma.

Lemma 3.1 *Let $M:Y\rightrightarrows X$ be a closed multifunction and A be a closed subset of X. Suppose that $(\bar{y},\bar{x})\in\mathrm{gph}(M)$ with $\bar{x}\in A$ and $\tau\in(0,+\infty)$. Then*

$$N_c(M(\bar{y})\cap A,\bar{x})\cap B_{X^*}\subset\tau(D_c^*M^{-1}(\bar{x},\bar{y})(B_{Y^*})+N_c(A,\bar{x})\cap B_{X^*}) \tag{23}$$

if and only if

$$d(x,T_c(M(\bar{y})\cap A,\bar{x}))\leq\tau(\|y\|+d(x,T_c(A,\bar{x}))) \tag{24}$$

holds for all $y\in Y$ and $x\in D_cM(\bar{y},\bar{x})(y)$.

Proof. The sufficiency part. Let $\delta_{\mathrm{gph}(D_cM(\bar{y},\bar{x}))}$ denote the indicator function of $\mathrm{gph}(D_cM(\bar{y},\bar{x}))$. Take any $x^*\in N_c(M(\bar{y})\cap A,\bar{x})\cap B_{X^*}$. Note that

$$N_c(M(\bar{y})\cap A,\bar{x})=N(T_c(M(\bar{y})\cap A,\bar{x}),0)$$

and thus $x^*\in\partial d(\cdot,T_c(M(\bar{y})\cap A,\bar{x}))(0)=N(T_c(M(\bar{y})\cap A,\bar{x}),0)\cap B_{X^*}$. Then by (24), one has

$$\langle x^*,x\rangle\leq d(x,T_c(M(\bar{y})\cap A,\bar{x}))\leq\tau\|y\|+\tau d(x,T_c(A,\bar{x}))$$

holds for all $(y,x)\in\mathrm{gph}(D_cM(\bar{y},\bar{x}))$ and consequently $(0,0)$ is a global minimizer of function $\phi+\delta_{\mathrm{gph}(D_cM(\bar{y},\bar{x}))}$, where $\phi(y,x):=-\langle x^*,x\rangle+\tau\|y\|+\tau d(x,T_c(A,\bar{x}))$ for any $(y,x)\in Y\times X$. This and Lemma 2.2 imply that

$$\begin{aligned}(0,0)\ &\in\ \partial(\phi+\delta_{\mathrm{gph}(D_cM(\bar{y},\bar{x}))})(0,0)=\partial\phi(0,0)+\partial\delta_{\mathrm{gph}(D_cM(\bar{y},\bar{x}))}(0,0)\\ &=\ (\tau B_{Y^*}\times\{-x^*\})+(\{0\}\times\tau\partial d(\cdot,T_c(A,\bar{x}))(0))+N(\mathrm{gph}(D_cM(\bar{y},\bar{x})),(0,0))\\ &=\ (\tau B_{Y^*}\times\{-x^*\})+(\{0\}\times\tau\partial d(\cdot,T_c(A,\bar{x}))(0))+N_c(\mathrm{gph}(M),(\bar{y},\bar{x})).\end{aligned}$$

Noting that $\partial d(\cdot,T_c(A,\bar{x}))(0)=N(T_c(A,\bar{x}),0)\cap B_{X^*}=N_c(A,\bar{x})\cap B_{X^*}$, it follows that there exist $y^*\in B_{Y^*}$ and $x_1^*\in N_c(A,\bar{x})\cap B_{X^*}$ such that

$$(-\tau y^*,x^*-\tau x_1^*)\in N_c(\mathrm{gph}(M),(\bar{y},\bar{x})).$$

This means that $x^* - \tau x_1^* \in D_c^* M^{-1}(\bar{x}, \bar{y})(\tau y^*)$ and thus

$$x^* \in \tau(D_c^* M^{-1}(\bar{x}, \bar{y})(B_{Y^*}) + N_c(A, \bar{x}) \cap B_{X^*}).$$

Hence (23) holds.

The necessity part. Let $y \in Y$ and $x \in D_c M(\bar{y}, \bar{x})(y) \backslash T_c(M(\bar{y}) \cap A, \bar{x})$. Take any $\gamma \in (0, 1)$. Applying Lemma 2.1, there exist $z \in T_c(M(\bar{y}) \cap A, \bar{x})$ and $z^* \in N(T_c(M(\bar{y}) \cap A, \bar{x}), z)$ with $\|z^*\| = 1$ such that

$$\gamma\|x - z\| \leq \langle z^*, x - z\rangle. \tag{25}$$

Since $T_c(M(\bar{y}) \cap A, \bar{x})$ is a closed and convex cone, then one has that $\langle z^*, z\rangle = 0$ by using $z^* \in N(T_c(M(\bar{y}) \cap A, \bar{x}), z)$ and consequently

$$z^* \in N(T_c(M(\bar{y}) \cap A, \bar{x}), 0) = N_c(M(\bar{y}) \cap A, \bar{x}).$$

By (23), there exist $y_1^* \in B_{Y^*}$, $x_1^* \in D_c M^{-1}(\bar{x}, \bar{y})(y_1^*)$ and $x_2^* \in N_c(A, \bar{x}) \cap B_{X^*}$ such that

$$z^* = \tau(x_1^* + x_2^*). \tag{26}$$

Noting that $x_2^* \in N_c(A, \bar{x}) \cap B_{X^*} = N(T_c(A, \bar{x}), 0) \cap B_{X^*} = \partial d(\cdot, T_c(A, \bar{x}))(0)$, it follows that

$$\langle x_2^*, h\rangle \leq d(h, T_c(A, \bar{x})) \ \ \forall h \in X.$$

This with (25) and (26) implies that

$$\begin{aligned} \gamma\|x - z\| &\leq \langle z^*, x - z\rangle = \langle z^*, x\rangle = \tau\langle x_1^*, x\rangle + \tau\langle x_2^*, x\rangle \\ &\leq \tau\langle y_1^*, y\rangle + \tau d(x, T_c(A, \bar{x})) \\ &\leq \tau(\|y\| + d(x, T_c(A, \bar{x}))) \end{aligned}$$

as $(x, y) \in T_c(\mathrm{gph}(M^{-1}), (\bar{x}, \bar{y}))$ and $(x_1^*, -y_1^*) \in N_c(\mathrm{gph}(M^{-1}), (\bar{x}, \bar{y}))$. Taking limits as $\gamma \to 1^-$, one has

$$d(x, T_c(M(\bar{y}) \cap A, \bar{x})) \leq \tau(\|y\| + d(x, T_c(A, \bar{x})))$$

(thanks to $z \in T_c(M(\bar{y}) \cap A, \bar{x})$. This means that (24) holds for all $y \in Y$ and $x \in D_c M(\bar{y}, \bar{x})(y)$. The proof is complete. □

The following theorem presents one sufficient condition for calmness of closed multifunctions over constraint closed subset under subsmooth assumptions. The proof is immediate from Proposition 3.2 and Lemma 3.1.

Theorem 3.1. *Let $M : Y \rightrightarrows X$ be a closed multifunction, A be a closed subset of X and let $\bar{y} \in Y$ and $\bar{x} \in M(\bar{y}) \cap A$. Suppose that M is L-subsmooth at $(\bar{y}, \bar{x})$, A is subsmooth at $\bar{x}$ and there exist $\tau, \delta \in (0, +\infty)$ such that for any $u \in M(\bar{y}) \cap A \cap B(\bar{x}, \delta)$, one has*

$$d(x, T_c(M(\bar{y}) \cap A, u)) \leq \tau(\|y\| + d(x, T_c(A, u)))$$

holds for all $y \in Y$ and $x \in D_c M(\bar{y}, u)(y)$. Then for any $\varepsilon \in (0, \frac{1}{2\tau+1})$, M is calm at $(\bar{y}, \bar{x})$ over A with constant $\frac{(\tau+1)\varepsilon+\tau}{1-(2\tau+1)\varepsilon} > 0$.

Next, we study the strong calmness for close multifunctions over constraint subsets. Recall that M is said to be strongly calm at $(\bar{y}, \bar{x})$ over A if there exist $\tau, \delta > 0$ such that

$$\|x - \bar{x}\| \leq \tau(\|y - \bar{y}\| + d(x, A)) \ \ \forall\, y \in B(\bar{y}, \delta) \text{ and } \forall x \in M(y) \cap B(\bar{x}, \delta). \tag{27}$$

It is easy to verify that M is strongly calm at $(\bar{y}, \bar{x})$ over A if and only if M is calm at $(\bar{y}, \bar{x})$ over A and $M(\bar{y}) \cap A \cap B(\bar{x}, \delta_0) = \{\bar{x}\}$ for some $\delta_0 > 0$.

Under the assumption of Condition (S), we provide a characterization for strong calmness of closed multifunctions over constraint closed subsets through the following theorem.

Theorem 3.2. *Let $M : Y \rightrightarrows X$ be a closed multifunction, A be a closed subset of X and let $\bar{y} \in Y$ and $\bar{x} \in M(\bar{y}) \cap A$. Suppose that M satisfies Condition (S) at $(\bar{y}, \bar{x})$ and A satisfies Condition (S) at $\bar{x}$. Then M is strongly calm at $(\bar{y}, \bar{x})$ over A if and only if there exists $\eta \in (0, +\infty)$ such that*

$$\eta B_{X^*} \subset D_c^* M^{-1}(\bar{x}, \bar{y})(B_{Y^*}) + N_c(A, \bar{x}) \cap B_{X^*}. \tag{28}$$

Proof. The necessity part. Suppose that M is strongly calm at $(\bar{y}, \bar{x})$ over A. Then there exist $\tau, \delta > 0$ such that M is calm at $(\bar{y}, \bar{x})$ over A with constant $\tau > 0$ and $M(\bar{y}) \cap A \cap B(\bar{x}, \delta) = \{\bar{x}\}$. By virtue of Proposition 3.1, one has

$$\hat{N}(M(\bar{y}) \cap A, \bar{x}) \cap B_{X^*} \subset \tau(D_c^* M^{-1}(z, \bar{y})(B_{Y^*}) + N_c(A, \bar{x}) \cap B_{X^*}).$$

Noting that $\hat{N}(M(\bar{y}) \cap A, \bar{x}) = X^*$ as $M(\bar{y}) \cap A \cap B(\bar{x}, \delta) = \{\bar{x}\}$, it follows that (28) holds with $\eta := \frac{1}{\tau} > 0$.

The sufficiency part. Let $\varepsilon \in (0, \frac{\eta}{\eta+2})$. Since M satisfies Condition (S) at $(\bar{y}, \bar{x})$ and A satisfies Condition (S) at $\bar{x}$, there exists $\delta \in (0, +\infty)$ such that whenever $x^* \in N_c(A, \bar{x}) \cap B_{X^*}$ and $v^* \in D_c^* M^{-1}(\bar{x}, \bar{y})(B_{Y^*}) \cap B_{X^*}$, one has

$$\langle v^*, x - \bar{x} \rangle \leq \varepsilon \|x - \bar{x}\| \quad \forall x \in M(\bar{y}) \cap B(\bar{x}, \delta), \tag{29}$$

and

$$\langle x^*, u - \bar{x} \rangle \leq d(u, A) + \varepsilon \|u - \bar{x}\| \quad \forall u \in B(\bar{x}, \delta). \tag{30}$$

We claim that

$$M(\bar{y}) \cap A \cap B(\bar{x}, \delta) = \{\bar{x}\}. \tag{31}$$

Granting this, it follows from (28) that (11) holds with $\tau := \frac{1}{\eta}$ for all $u \in M(\bar{y}) \cap A \cap B(\bar{x}, \delta)$, and thus M is calm at $(\bar{y}, \bar{x})$ over A by Proposition 3.2.

Let $x \in M(\bar{y}) \cap A \cap B(\bar{x}, \delta)$. By the Hahn-Banach Theorem, there exists $u^* \in B_{X^*}$ such that $\langle u^*, x - \bar{x} \rangle = \|x - \bar{x}\|$. By virtue of (28), there exist $x_1^* \in D_c^* M^{-1}(\bar{x}, \bar{y})(y^*)$ for some $y^* \in B_{Y^*}$ and $x_2^* \in N_c(A, \bar{x}) \cap B_{X^*}$ such that

$$\eta u^* = x_1^* + x_2^*. \tag{32}$$

Noting that $(\frac{x_1^*}{\eta+1}, -\frac{y^*}{\eta+1}) \in N_c(\mathrm{gph}(M^{-1}), (\bar{x}, \bar{y})) \cap (B_{X^*} \times B_{Y^*})$, it follows from (29) and (30) that

$$\langle x_1^*, x - \bar{x} \rangle \leq (\eta + 1)\varepsilon \|x - \bar{x}\| \text{ and } \langle x_2^*, x - \bar{x} \rangle \leq \varepsilon \|x - \bar{x}\|.$$

This and (32) imply that

$$\begin{aligned} \eta\|x - \bar{x}\| &= \eta\langle u^*, x - \bar{x} \rangle = \langle x_1^* + x_2^*, x - \bar{x} \rangle \\ &\leq (\eta + 1)\varepsilon\|x - \bar{x}\| + (\varepsilon + 1)\|x - \bar{x}\| \\ &= (\eta + 2)\varepsilon\|x - \bar{x}\|. \end{aligned}$$

Thus $x = \bar{x}$ (thanks to $\varepsilon < \frac{\eta}{\eta+2}$). The proof is complete. □

Remark 3.1 Let $\bar{y} \in Y$ and $\bar{x} \in M(\bar{y}) \cap A$. We define

$$\tau(M, A; \bar{y}, \bar{x}) := \inf\{\tau > 0 : \text{there exists } \delta > 0 \text{ such that (27) holds}\}$$

and

$$\eta(M, A; \bar{y}, \bar{x}) := \sup\{\eta > 0 : \text{(28) holds}\}.$$

Under the assumptions that M satisfies Condition (S) at $(\bar{y}, \bar{x})$ and A satisfies Condition (S) at $\bar{x}$, by using the proof of Theorem 3.2, it is easy to verify that

$$\frac{1}{\tau(M, A; \bar{y}, \bar{x})} = \eta(M, A; \bar{y}, \bar{x}),$$

here we use the convention that the infimum over the empty set is $+\infty$ and the supremum over the empty set is 0.

As applications of main results in this paper, we are now in a position to consider calmness and strong calmness of the following multifunction (see Herion, 2001; Herion, Jourani & Outrata, 2002):

$$G(y) := \{x \in A : g(x) + y \in D\} \text{ for all } y \in Y, \tag{33}$$

where A is a closed subset of X, $g : X \to Y$ and D is a closed subset of Y.

Let $\bar{y} \in Y$ and $\bar{x} \in G(\bar{y})$. We set $M(y) := g^{-1}(D - y)$ for any $y \in Y$. Then it is easy to verify that M is calm (resp. strongly calm) at $(\bar{y}, \bar{x})$ over A implies the calmness (resp. strong calmness) of G at $(\bar{y}, \bar{x})$.

The following proposition provides one sufficient condition for calmness of G in (33).

Proposition 3.3. *Let G be as* (33) *and* $(\bar{y}, \bar{x}) \in \mathrm{gph}(G)$*. Suppose that g is smooth, A is subsmooth at* $\bar{x}$*, D is subsmooth at* $g(\bar{x}) + \bar{y}$ *and there exist* $\tau, \delta \in (0, +\infty)$ *such that*

$$N_c(G(\bar{y}), u) \cap B_{X^*} \subset \tau(\nabla g(u)^*(N_c(D, g(u) + \bar{y}) \cap B_{Y^*}) + N_c(A, u) \cap B_{X^*}) \tag{34}$$

holds for all $u \in G(\bar{y}) \cap B(\bar{x}, \delta)$*. Then G is calm at* $(\bar{y}, \bar{x})$*.*

Proof. Let $M(y) := g^{-1}(D - y)$ for any $y \in Y$. Then $M^{-1}(x) = D - g(x)$ for all $x \in X$. Since D is subsmooth at $g(\bar{x}) + \bar{y}$, by using Proposition 3.3 in (Zheng & Ng, 2009), one has that M is subsmooth at $(\bar{y}, \bar{x})$ and thus M is L-subsmooth at $(\bar{y}, \bar{x})$.

Let $u \in G(\bar{y}) \cap B(\bar{x}, \delta)$. It only suffices to prove that

$$D_c^* M^{-1}(u, \bar{y})(B_{Y^*}) = \nabla g(u)^*(N_c(D, g(u) + \bar{y}) \cap B_{Y^*}). \tag{35}$$

Granting this, by virtue of Proposition 3.2 and (34), it yields that M is calm at $(\bar{y}, \bar{x})$ over A and consequently G is calm at $(\bar{y}, \bar{x})$.

Indeed, since $M^{-1}(x) = D - g(x)$ for any $x \in X$ and g is smooth, by using Proposition 3.3 in (Zheng & Ng, 2009), one has

$$N_c(\mathrm{gph}(M^{-1})(u, \bar{y})) = \{(\nabla g(u)^*(y^*), y^*) : y^* \in N_c(D, g(u) + \bar{y})\}. \tag{36}$$

Therefore, for any $x^* \in D_c^* M^{-1}(u, \bar{y})(B_{Y^*})$, there exists $y^* \in B_{Y^*}$ such that

$$(x^*, -y^*) \in N_c(\mathrm{gph}(M^{-1}), (u, \bar{y})),$$

and it follows from (36) that $-y^* \in N_c(D, g(u) + \bar{y})$ and $x^* = \nabla g(u)^*(-y^*)$.

Conversely, for any $y^* \in N_c(D, g(u) + \bar{y}) \cap B_{Y^*}$, one has

$$(\nabla g(u)^*(y^*), y^*) \in N_c(\mathrm{gph}(M^{-1}), (u, \bar{y}))$$

by (36) and thus

$$\nabla g(u)^*(y^*) \in D_c^* M^{-1}(u, \bar{y})(-y^*) \subset D_c^* M^{-1}(u, \bar{y})(B_{Y^*}).$$

This implies that (35) holds. The proof is complete. □

By Proposition 3.3 and Theorem 3.2, we present one sufficient condition for ensuring strong calmness of G in (33) via the following Proposition.

Proposition 3.4. *Let G be as* (33) *and* $(\bar{y}, \bar{x}) \in \mathrm{gph}(G)$*. Suppose that g is smooth, A satisfies Condition (S) at* $\bar{x}$*, D is subsmooth at* $g(\bar{x}) + \bar{y}$*, and there exists* $\eta \in (0, +\infty)$ *such that*

$$\eta B_{X^*} \subset (\nabla g(\bar{x})^*(N_c(D, g(\bar{x}) + \bar{y}) \cap B_{Y^*}) + N_c(A, \bar{x}) \cap B_{X^*}). \tag{37}$$

Then G is strongly calm at $(\bar{y}, \bar{x})$*.*

4. Conclusions

This paper is devoted to calmness of closed multifunctions over constraint closed sets. Some necessary or sufficient conditions for calmness have been provided in terms of tangent cones, normal cones and coderivatives with the help of subsmooth assumptions. For a closed multifunction and closed set satisfying Condition (S), a dual characterization for strong calmness has been proved. Applications of obtained results are also given.

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