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Exact Traveling Wave Solutions for the Modified Double Sine-Gordon Equation

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Abstract

With some elementary methods, a number of new travelling solutions of the modified double Sine-Gordon (SG) equation are obtained, including different types of exact solion solutions and exact periodic solutions.

Keywords: modified double SG equation, exact solion solution, exact periodic solution

1. Introduction

The equation

$$u_{tt} - u_{xx} = \alpha_1 \sin u + \alpha_2 \sin 2u \tag{1}$$

is commonly called the double SG equation, which describes weak higher anisotropy in an easy-plane ferromagnetic field, and

$$u_{tt} - u_{xx} = \alpha_1 \sin u + \alpha_2 \cos \frac{u}{2} \tag{2}$$

may be called the modified double SG equation because of $\cos \frac{u}{2} = \sin(\frac{\pi}{2} - \frac{u}{2})$, where α_1 and α_2 are arbitrary constants (Khater, 2013, Webb, 1988, Kivshar, 1989). In fact, when α_2 is a small dissipative coefficient, equation (2) is a perturbed SG equation, and an external magnetic field perpendicular both to the x axis and to the magnetization vector is described by the perturbed term $\alpha_2 \cos \frac{u}{2}$ (Kivshar, 1989).

As a typical model of fluxion dynamics in Josephson junctions, equation (1) and its other forms always attract much attention. For example, a new group of traveling wave solutions with Jacobian amplitude function for the generalized form of the double SG equation were presented (Sun, 2015), by using the approach of dynamical systems to the travelling wave solutions, all possible explicit exact travelling wave solutions to the (n + 1)-dimensional double SG equation were obtained (Geng, 2007), some complex hyperbolic functions were proposed to derive travelling wave solutions to equation (1) (Bin, 2007), exact solutions to the double SG equation were studied by F-expansion method (Wang, 2006), the tanh method and a variable separated ODE method were used for solving the double SG equation (Wazwaz, 2006). However, the relative references for equation (2) are very few, the method of bifurcation theory of dynamical systems was used for the exact solutions to the (N + 1)-dimensional sine-cosine-Gordon equations (Tang, 2010), the solutions of the combined sine-cosine-Gordon equation were studied by the variable separated ODE method (Kuo, 2009). In the paper, we first make the travelling wave transformation for equation (2), and take some elementary methods to solve its exact traveling wave solutions. Most of all, we consider the periodic characteristic of $\alpha_1 \sin u + \alpha_2 \cos \frac{u}{2}$, this is mainly different from the references mentioned above.

Set

$$u = u(\xi),\ \xi = \frac{\beta_1 t + \beta_2 x}{\sqrt{\beta_1^2 - \beta_2^2}} + c, \tag{3}$$

where β_1, β_2 and c are arbitrary constants, but $\beta_1^2 - \beta_2^2 > 0$. Substituting $u = u(\xi)$ into equation (2) gets

$$u_{\xi\xi} = \alpha_1 \sin u + \alpha_2 \cos \frac{u}{2}. \tag{4}$$

Since every solution to equation (4) is a particular solution of equation (2), we only discuss the exact solutions for ordinary differential equation (4).

2. First Group of Solutions to Equation (2)

Multiplying (4) by u_ξ and integrating with respect to ξ both sides leads to

$$u_\xi^2 = 4(\alpha_1 \sin^2 \frac{u}{2} + \alpha_2 \sin \frac{u}{2} + A), \tag{5}$$

where A is an integral constant. Especially, we fix $\alpha_1 > 0$, if $\alpha_1 < 0$, making another travelling transformation $u = u(\eta),\ \eta = \frac{\beta_1 x+\beta_2 t}{\sqrt{\beta_1^2-\beta_2^2}} + c$ for equation (2) leads to

$$u_\eta^2 = 4(-\alpha_1 \sin^2 \frac{u}{2} - \alpha_2 \sin \frac{u}{2} + A),$$

so we only consider the solutions to (5) in the case of $\alpha_1 > 0$.

Next, we make the substitution $u = v + n\pi, n \in \boldsymbol{N}$ for equation (5), and further get

$$v_\xi^2 = \begin{cases} 4[\alpha_1 \sin^2 \frac{v}{2} + \alpha_2(-1)^k \sin \frac{v}{2} + A], & n = 2k, \quad (6a)\\ 4[\alpha_1 \cos^2 \frac{v}{2} + \alpha_2(-1)^k \cos \frac{v}{2} + A]), & n = 2k+1, \quad (6b)\end{cases}$$

where $k \in \boldsymbol{N}$.

(1) When $A = \frac{\alpha_2^2}{4\alpha_1}$, (6) is simplified as

$$v_\xi = \begin{cases} \pm 2\sqrt{\alpha_1}[(-1)^k \sin \frac{v}{2} + \frac{\alpha_2}{2\alpha_1}], & n = 2k, \quad (7a)\\ \pm 2\sqrt{\alpha_1}[(-1)^k \cos \frac{v}{2} + \frac{\alpha_2}{2\alpha_1}], & n = 2k+1. \quad (7b)\end{cases}$$

As for (7a), separating the variables and integrating both sides, we finally obtain periodic solutions

$$u_1 = 2k\pi + 4\arctan[\frac{2\alpha_1(-1)^{k+1}}{\alpha_2} \pm \frac{\sqrt{\alpha_2^2-4\alpha_1^2}}{\alpha_2}\tan\frac{\sqrt{\alpha_2^2-4\alpha_1^2}}{4\sqrt{\alpha_1}}\xi],$$

$$u_2 = 2k\pi + 4\arctan[\frac{2\alpha_1(-1)^{k+1}}{\alpha_2} \pm \frac{\sqrt{\alpha_2^2-4\alpha_1^2}}{\alpha_2}\cot\frac{\sqrt{\alpha_2^2-4\alpha_1^2}}{4\sqrt{\alpha_1}}\xi],$$

and soliton solutions

$$u_3 = 2k\pi + 4\arctan[\frac{2\alpha_1(-1)^{k+1}}{\alpha_2} \pm \frac{\sqrt{4\alpha_1^2-\alpha_2^2}}{\alpha_2}\tanh\frac{\sqrt{4\alpha_1^2-\alpha_2^2}}{4\sqrt{\alpha_1}}\xi],$$

$$u_4 = 2k\pi + 4\arctan[\frac{2\alpha_1(-1)^{k+1}}{\alpha_2} \pm \frac{\sqrt{4\alpha_1^2-\alpha_2^2}}{\alpha_2}\coth\frac{\sqrt{4\alpha_1^2-\alpha_2^2}}{4\sqrt{\alpha_1}}\xi].$$

For (7b), repeating the same procedure, we can get particular solutions to equation (2)

$$u_5 = (2k+1)\pi \pm 4\arctan[\frac{2\alpha_1(-1)^k+\alpha_2}{\sqrt{\alpha_2^2-4\alpha_1^2}}\tan\frac{\sqrt{\alpha_2^2-4\alpha_1^2}}{4\sqrt{\alpha_1}}\xi],$$

$$u_6 = (2k+1)\pi \pm 4\arctan[\frac{2\alpha_1(-1)^k+\alpha_2}{\sqrt{\alpha_2^2-4\alpha_1^2}}\cot\frac{\sqrt{\alpha_2^2-4\alpha_1^2}}{4\sqrt{\alpha_1}}\xi],$$

$$u_7 = (2k+1)\pi \pm 4\arctan[\frac{2\alpha_1(-1)^k+\alpha_2}{\sqrt{4\alpha_1^2-\alpha_2^2}}\tanh\frac{\sqrt{4\alpha_1^2-\alpha_2^2}}{4\sqrt{\alpha_1}}\xi]$$

and

$$u_8 = (2k+1)\pi \pm 4\arctan[\frac{2\alpha_1(-1)^k + \alpha_2}{\sqrt{4\alpha_1^2 - \alpha_2^2}} \coth \frac{\sqrt{4\alpha_1^2 - \alpha_2^2}}{4\sqrt{\alpha_1}}\xi].$$

Note that, in u_1, u_2, u_3 and u_4, $\alpha_2^2 - 4\alpha_1^2 > 0$, in u_3, u_4, u_7 and u_8, $4\alpha_1^2 - \alpha_2^2 > 0$.

(2) When $A = \alpha_1 > 0, \alpha_2 = \pm 2\alpha_1$, (6) is transformed into

$$v_\xi = \begin{cases} \pm 2\sqrt{\alpha_1}[(-1)^k \sin\frac{v}{2} \pm 1], & n = 2k, \quad (8a) \\ \pm 2\sqrt{\alpha_1}[(-1)^k \cos\frac{v}{2} \pm 1], & n = 2k+1. \quad (8b) \end{cases}$$

By the method of separating variables, some soliton solutions are obtained from (8)

$$u_9 = 2k\pi + (-1)^k\pi \pm 4\arctan\sqrt{\alpha_1}\xi,$$

$$u_{10} = 2k\pi + (-1)^{k+1}\pi \pm 4\arctan\sqrt{\alpha_1}\xi,$$

$$u_{11} = (2k+1)\pi \pm 4\arctan\sqrt{\alpha_1}\xi$$

and

$$u_{12} = (2k+1)\pi \pm 4\text{arccot}\sqrt{\alpha_1}\xi.$$

(3) When $A = -\alpha_1 + \alpha_2(-1)^{k+1}$, (6) is rewritten as

$$v_\xi^2 = \begin{cases} 4(1-\sin\frac{v}{2})[-\alpha_1(1+\sin\frac{v}{2}) + \alpha_2(-1)^{k+1}], & n = 2k, \quad (9a) \\ 4(1-\cos\frac{v}{2})[-\alpha_1(1+\cos\frac{v}{2}) + \alpha_2(-1)^{k+1}], & n = 2k+1. \quad (9b) \end{cases}$$

In equation (9a), making transformation $\varphi = \sin\frac{v}{2}$ yields

$$\varphi_\xi = \pm(1-\varphi)\sqrt{(1+\varphi)[-\alpha_1(1+\varphi) + \alpha_2(-1)^{k+1}]},$$

this is a separated variables differential equation, solving it gets a solution in the form

$$u_{13} = 2k\pi + 2\arcsin\frac{4\alpha_1 - 3\alpha_2(-1)^{k+1} \pm \alpha_2 \sin\sqrt{4\alpha_1 + 2\alpha_2(-1)^k}\,\xi}{-4\alpha_1 + \alpha_2(-1)^{k+1} \pm \alpha_2 \sin\sqrt{4\alpha_1 + 2\alpha_2(-1)^k}\,\xi},$$

where $2\alpha_1 > \alpha_2(-1)^{k+1}$. By the same way, from equation (9b), we can obtain another solution

$$u_{14} = (2k+1)\pi + 2\arccos\frac{4\alpha_1 - 3\alpha_2(-1)^{k+1} \pm \alpha_2 \sin\sqrt{4\alpha_1 + 2\alpha_2(-1)^k}\,\xi}{-4\alpha_1 + \alpha_2(-1)^{k+1} \pm \alpha_2 \sin\sqrt{4\alpha_1 + 2\alpha_2(-1)^k}\,\xi}.$$

(4) When $A = -\alpha_1 + \alpha_2(-1)^k$, (6) becomes

$$v_\xi^2 = \begin{cases} 4(1+\sin\frac{v}{2})[\alpha_1(\sin\frac{v}{2} - 1) + \alpha_2(-1)^k], & n = 2k, \quad (10a) \\ 4(1+\cos\frac{v}{2})[\alpha_1(\cos\frac{v}{2} - 1) + \alpha_2(-1)^k], & n = 2k+1. \quad (10b) \end{cases}$$

Making transformation $\varphi = \sin\frac{v}{2}$ for (10a) gets

$$\varphi_\xi = \pm(1+\varphi)\sqrt{(1-\varphi)[\alpha_1(\varphi - 1) + \alpha_2(-1)^k]},$$

with the method of separating variables, we can get a solution from the above equation

$$u_{15} = 2k\pi + 2\arcsin\frac{-4\alpha_1 + 3\alpha_2(-1)^k \mp \alpha_2 \sin\sqrt{4\alpha_1 + 2\alpha_2(-1)^{k+1}}\,\xi}{-4\alpha_1 + \alpha_2(-1)^k \pm \alpha_2 \sin\sqrt{4\alpha_1 + 2\alpha_2(-1)^{k+1}}\,\xi},$$

where $2\alpha_1 > \alpha_2(-1)^k$. Similarly, solving (10b) gives a solution to equation (2) in the form

$$u_{16} = (2k+1)\pi + 2\arccos\frac{-4\alpha_1 + 3\alpha_2(-1)^k \mp \alpha_2 \sin\sqrt{4\alpha_1 + 2\alpha_2(-1)^{k+1}}\,\xi}{-4\alpha_1 + \alpha_2(-1)^k \pm \alpha_2 \sin\sqrt{4\alpha_1 + 2\alpha_2(-1)^{k+1}}\,\xi}.$$

3. Second Group of Solutions to Equation (2)

In order to find more exact solutions to equation (2), equation (4) is transformed into

$$u_\xi^2 = -2\alpha_1 \cos u + 4\alpha_2 \sin\frac{u}{2} + B, \tag{11}$$

where B is an integral constant, then making the substitution $u = n\pi + f(\xi)$ for equation (11) yields

$$f_\xi^2 = \begin{cases} -2\alpha_1 \cos f + 4\alpha_2(-1)^k \sin\frac{f}{2} + B, & n = 2k, \quad (12a) \\ 2\alpha_1 \cos f + 4\alpha_2(-1)^k \cos\frac{f}{2} + B, & n = 2k+1. \quad (12b) \end{cases}$$

Again setting $f = 4\text{arccot}g$ for (12), we get

$$16g_\xi^2 = \begin{cases} (B-2\alpha_1)(1+g^2)^2 + 16\alpha_1 g^2 + 8\alpha_2(-1)^k g(1+g^2), & n = 2k, \quad (13a) \\ (B+2\alpha_1)(1+g^2)^2 - 16\alpha_1 g^2 + 4\alpha_2(-1)^k(g^4-1), & n = 2k+1. \quad (13b) \end{cases}$$

(1) Take $B = 2\alpha_1, \alpha_2(-1)^k = \pm\alpha_1$ in (13a), we get

$$2g_\xi^2 = \begin{cases} \alpha_1 g(1+g)^2, & (14a) \\ -\alpha_1 g(1-g)^2, & (14b) \end{cases}$$

solving the above equations, some solutions to equation (2) are finally given by

$$u_{17} = 2k\pi \pm 4\text{arccot}\tan^2\sqrt{\frac{\alpha_1}{2}}\xi,\ \ u_{18} = 2k\pi \pm 4\text{arccot}\cot^2\sqrt{\frac{\alpha_1}{2}}\xi,\ \alpha_1 > 0,$$

and

$$u_{19} = 2k\pi \pm 4\text{arccot}\tanh^2\sqrt{\frac{-\alpha_1}{2}}\xi,\ \ u_{20} = 2k\pi \pm 4\text{arccot}\coth^2\sqrt{\frac{-\alpha_1}{2}}\xi,\ \alpha_1 < 0.$$

(2) When $B = 6\alpha_1, \alpha_2(-1)^k = \pm 2\alpha_1, \alpha_1 > 0$, (13a) is simplified as $4g_\xi^2 = \alpha_1(1 \pm g)^4$, solve it, we obtain

$$u_{21} = 2k\pi + 4\text{arccot}(-1 \pm \frac{2}{\sqrt{\alpha_1}\xi}),\ \ u_{22} = 2k\pi + 4\text{arccot}(1 \pm \frac{2}{\sqrt{\alpha_1}\xi}).$$

(3) When $B = 2\alpha_1, \alpha_2(-1)^k = \alpha_1$, (13b) is reduced to $2g_\xi^2 = \alpha_1 g^2(g^2-1)$, which leads to solutions to equation (2)

$$u_{23} = (2k+1)\pi + 4\text{arccot}\sec\sqrt{\frac{\alpha_1}{2}}\xi,\ \ u_{24} = (2k+1)\pi \pm 4\text{arccot}\csc\sqrt{\frac{\alpha_1}{2}}\xi,\ \alpha_1 > 0,$$

and

$$u_{25} = (2k+1)\pi + 4\text{arccot}\ \text{sech}\sqrt{-\frac{\alpha_1}{2}}\xi, \alpha_1 < 0.$$

(4) Take $B = 2\alpha_1, \alpha_2(-1)^{k+1} = \alpha_1$ in (13b), we get $2g_\xi^2 = \alpha_1(1-g^2)$, solve it and the relative solutions are given by

$$u_{26} = (2k+1)\pi \pm 4\text{arccot}\sin\sqrt{\frac{\alpha_1}{2}}\xi,\ \ u_{27} = (2k+1)\pi + 4\text{arccot}\cos\sqrt{\frac{\alpha_1}{2}}\xi, \alpha_1 > 0$$

and

$$u_{28} = (2k+1)\pi + 4\mathrm{arccot}\cosh\sqrt{-\frac{\alpha_1}{2}}\xi,\ \alpha_1 < 0.$$

4. Conclusions

As we know, the travelling transformation method, the substitution method, the reduced order method and the separable variables method are elementary and straightforward methods to find the exact travelling wave solutions of nonlinear evolution equations, an advantage of these methods is that they avoid tedious algebra and guesswork. In the paper, by fully considering the periodicity of the solution to equation (2), we obtain different kinds of explicit exact solutions with the aid of some elementary methods. Actually, making linear transformations $u = \pi + 2p, t' = \frac{t}{\sqrt{2}}, x' = \frac{x}{\sqrt{2}}$ for equation (2), we can get $p_{x'x'} - p_{t't'} = \alpha_1 \sin 2p + \alpha_2 \sin p$, this is just another form of equation (1). However, we noticed that, this type of solutions such as $u_{17} - u_{20}$ did not appear in relative references (Sun, 2015, Geng, 2007, Wang, 2006, Bin, 2007, Wazwaz, 2006, Tang, 2010, Kuo, 2009).

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