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# On the Irreducibility of Artin's Group of Graphs

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**Abstract**

We consider the graph $E_{3,1}$ with three generators $\sigma_1, \sigma_2, \delta$, where $\sigma_1$ has an edge with each of $\sigma_2$ and $\delta$. We then define the Artin group of the graph $E_{3,1}$ and consider its reduced Perron representation of degree three. After we specialize the indeterminates used in defining the representation to non-zero complex numbers, we obtain a necessary and sufficient condition that guarantees the irreducibility of the representation.

**Keywords:** Artin representation, braid group, Burau representation, graph, irreducibility

## 1. Introduction

To any undirected simple graph $T$, we introduce the Artin group, $A$, which is defined as an abstract group with vertices of $\Gamma$ as its generators and two relations: $xy = yx$ for vertices x and y that have no edge in common and $xyx = yxy$ if the vertices x and y have a common edge.

Let $A_n$ be the graph having $n$ vertices $\sigma_i$'s $(1 \leq i \leq n)$ in which $\sigma_i$ and $\sigma_{i+1}$ share a comon edge, where $i = 1, 2, ..., n-1$. We notice that the Artin group of $A_n$ is the braid group on $n+1$ strands. That is, $A(A_n) = B_{n+1}$ (J.S.Birman, 1975).

Having defined $A_n$, we consider $E_{n+1,p}$, which is the graph obtained from $A_n$ by adding a vertex $\delta$ and an edge connecting $\sigma_p$ and $\delta$. Here $1 \leq p \leq n$. It is easy to see that the graph $A_n$ embeds in the graph $E_{n+1,p}$. That is, $A(A_n) \subset A(E_{n+1,p})$. This induces an injection on $B_{n+1}$ to $A(E_{n+1,p})$. In other words, a representation of $A(E_{n+1,p})$ yields a representation of $B_{n+1}$.

Knowing the reduced Burau representation of $B_{n+1}$ of degree $n$, Perron extends such a representation to a representation of $B_{n+1}$ of degree $2n$. The representation obtained is referred to as Burau bis representation. Next, Perron constructs for each $\lambda = (\lambda_1, \ldots, \lambda_n)$ a representation $\psi_\lambda : A(E_{n+1,p}) \to GL_{2n}(Q(t, d_1, \ldots, d_n))$, where $t, d_1, \ldots, d_n$ $\lambda_1, \ldots, \lambda_n$ are indeterminates. We specialize $t, d_1, \ldots, d_n$ to non zero complex numbers, and we study this representation explicitly in the case $n = 2$ and $p = 1$. We then reduce the complex specialization of the representation $\psi_\lambda$ to a representation of degree 3, namely $A(E_{3,1}) \to GL_3(\mathbb{C})$. A necessary and sufficient condition which guarantees its irreducibility is obtained in that case.

## 2. Burau bis Representation

Perron's strategy is to begin with the Burau representation of the braid group and extend it to a representation of $A(E_{n+1,p})$. He begins with the reduced Burau representation: $B_{n+1} \to GL_n(\mathbb{Z}[t, t^{-1}])$ defined as follows:

$$\sigma_i \to J_i = \left(\begin{array}{c|ccc|c} I_{i-2} & & 0 & & 0 \\ \hline & 1 & 0 & 0 & \\ 0 & t & -t & 1 & 0 \\ & 0 & 0 & 1 & \\ \hline 0 & & 0 & & I_{n-i-1} \end{array}\right),$$

where $I_k$ stands for the $k \times k$ identity matrix. Here, $i = 2, \ldots, n-1$.

$$\sigma_1 \to J_1 = \left(\begin{array}{cc|c} -t & 1 & \\ 0 & 1 & 0 \\ \hline \multicolumn{2}{c|}{0} & I_{n-2} \end{array}\right)$$

$$\sigma_n \to J_n = \left(\begin{array}{c|cc} I_{n-2} & \multicolumn{2}{c}{0} \\ \hline & 1 & 0 \\ 0 & t & -t \end{array}\right)$$

Knowing that this representation is of degree $n$, Perron extends it to a representation of $B_{n+1}$ of degree $2n$. Let $R_i$ denote an $n \times n$ block of zeros with a $t$ placed in the $(i, i)$ $th$ position, and let $I_n$ denote the $n \times n$ identity matrix. The obtained representation is referred to as the Burau bis representation. It is defined as follows:

$$\psi : B_{n+1} \to Gl_{2n}(\mathbb{Z}[t, t^{-1}])$$

$$\psi(\sigma_i) = \begin{pmatrix} I_n & 0 \\ R_i & J_i \end{pmatrix}, \quad 1 \leq i \leq n$$

For more details, see (T.E.Brendle, 2002, B.Perron, 1999).

## 3. Perron Representation

The Burau bis representation extends to $A(E_{n+1,p})$ for all possible values of $n$ and $p$ in the following way.

Let $b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$, $d = \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix}$, and $\lambda = (\lambda_1, \ldots, \lambda_n)$.

We define the following $n \times n$ matrices:

$$A = (\lambda_1 b, \lambda_2 b, \ldots, \lambda_n b)$$
$$B = (0, \ldots, 0, b, 0, \ldots, 0)$$
$$C = (\lambda_1 d, \lambda_2 d, \ldots, \lambda_n d)$$
$$D = (0, \ldots, 0, d, 0, \ldots, 0),$$

where 0 denotes a column of n zeros.

For each $i = 1, \ldots, n$, we have that $b_i$ satisfies the following conditions

$$tb_i = -td_{i-1} + (1+t)d_i - d_{i+1}, \quad i \neq p,$$
$$tb_p = -td_{p-1} + (1+t)d_p - d_{p+1} + t,$$
$$\sum_{i=1}^{n} \lambda_i b_i = -(1 + d_p + t),$$

setting any undefined $d_j$ equal zero.

For any choice $\lambda = (\lambda_1, \ldots, \lambda_n)$ , we get a linear representation

$$\psi_\lambda : A(E_{n+1,p}) \to Gl_{2n}(R),$$

where R is the field of rational fractions in n+1 indeterminates $\mathbb{Q}(t, d_1, ..., d_n)$.

$$\psi_\lambda(\sigma_i) \to \begin{pmatrix} I_n & 0 \\ R_i & J_i \end{pmatrix},$$

$$\psi_\lambda(\delta) \rightarrow \begin{pmatrix} I_n + A & B \\ C & I_n + D \end{pmatrix}.$$

For more details, see (T.E.Brendle, 2002).

**4. Reducibility of $\psi_\lambda : A(E_{3,1}) \rightarrow GL_4(\mathbb{C})$**

Having defined Perron's representation, we set $n = 2$ and $p = 1$ to get the following vectors. $b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$, $d = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$, and $\lambda = (\lambda_1, \lambda_2)$.

We get the following $2 \times 2$ matrices $A = \begin{pmatrix} \lambda_1 b_1 & \lambda_2 b_1 \\ \lambda_1 b_2 & \lambda_2 b_2 \end{pmatrix}$, $B = \begin{pmatrix} b_1 & 0 \\ b_2 & 0 \end{pmatrix}$, $C = \begin{pmatrix} \lambda_1 d_1 & \lambda_2 d_1 \\ \lambda_1 d_2 & \lambda_2 d_2 \end{pmatrix}$, and $D = \begin{pmatrix} d_1 & 0 \\ d_2 & 0 \end{pmatrix}$.

Simple computations show that the parameters satisfy the following equations:

- $tb_2 = -td_1 + (1 + t)d_2$
- $tb_1 = (1 + t)d_1 - d_2 + t$
- $\lambda_1 b_1 + \lambda_2 b_2 = -(1 + t + d_1)$

Having defined the $2 \times 2$ matrices $A$, $B$, $C$ and $D$, we obtain the multiparameter representation $A(E_{3,1})$. This representation is of degree 4. We specialize the parameters $\lambda_1, \lambda_2, b_1, b_2, d_1, d_2, t$ to values in $\mathbb{C} - \{0\}$. **We further assume that** $t \neq -1$ **and** $d_2 = -t$. The representation $\psi_\lambda : A(E_{3,1}) \rightarrow GL_4(\mathbb{C})$ is defined as follows:

$$\psi_\lambda(\sigma_1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ t & 0 & -t & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\psi_\lambda(\sigma_2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & t & t & -t \end{pmatrix},$$

and

$$\psi_\lambda(\delta) = \begin{pmatrix} 1 + \lambda_1 b_1 & \lambda_2 b_1 & b_1 & 0 \\ \lambda_1 b_2 & \lambda_2 b_2 + 1 & b_2 & 0 \\ \lambda_1 d_1 & \lambda_2 d_1 & 1 + d_1 & 0 \\ -t\lambda_1 & -t\lambda_2 & -t & 1 \end{pmatrix}.$$

The graph $E_{3,1}$ has 3 vertices $\sigma_1, \sigma_2$ and $\delta$. Since $p = 1$, it follows that the vertex $\delta$ has a common edge with $\sigma_p = \sigma_1$. Therefore, the following relations are satisfied.

$$\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \qquad (1)$$

$$\sigma_2\delta = \delta\sigma_2 \qquad (2)$$

$$\sigma_1\delta\sigma_1 = \delta\sigma_1\delta \qquad (3)$$

We note that relation (1) is actually Artin's braid relation of the classical braid group, $B_3$ having $\sigma_1$ and $\sigma_2$ as standard generators. This assures that a representation of $A(E_{3,1})$yields a representation of $B_3$.

**Lemma 1** *The representation* $\psi_\lambda : A(E_{3,1}) \rightarrow GL_4(\mathbb{C})$ *is reducible.*

*Proof.* For simplicity, we write $\sigma_i$ instead of $\psi_\lambda(k)$ , where $k$ is a generator of $A(E_{3,1})$.The subspace $S = \left\langle e_1 + \frac{b_2}{b_1}e_2,\ e_3,\ e_4\right\rangle$ is an invariant subspace of dimension 3. To see this:

1. $\sigma_1(e_1 + \frac{b_2}{b_1}e_2) = e_1 + \frac{b_2}{b_1}e_2 + te_3 \in S$
2. $\sigma_2(e_1 + \frac{b_2}{b_1}e_2) = e_1 + \frac{b_2}{b_1}e_2 + t\frac{b_2}{b_1}e_3 \in S$
3. $\delta(e_1 + \frac{b_2}{b_1}e_2) = (1 + \lambda_1 b_1 + \lambda_2 b_2)e_1 + (\lambda_1 b_2 + \frac{b_2}{b_1}(\lambda_2 b_2 + 1))e_2 +$
$$(\lambda_1 d_1 + \frac{b_2}{b_1}\lambda_2 d_1)e_3 + (-t\lambda_1 + \frac{-tb_2}{b_1}\lambda_2)e_4$$
$$= (1 + \lambda_1 b_1 + \lambda_2 b_2)(e_1 + \frac{b_2}{b_1}e_2) + (\lambda_1 d_1 + \frac{b_2}{b_1}\lambda_2 d_1)e_3 +$$
$$(-t\lambda_1 + \frac{-tb_2}{b_1}\lambda_2)e_4 \in S$$
4. $\sigma_1 e_3 = -te_3 \in S$
5. $\sigma_2 e_3 = e_3 + te_4 \in S$
6. $\delta e_3 = b_1(e_1 + \frac{b_2}{b_1}e_2) + (1 + d_1)e_3 - te_4 \in S$
7. $\sigma_1 e_4 = e_3 + e_4 \in S$
8. $\sigma_2 e_4 = -te_4 \in S$
9. $\delta e_4 = e_4 \in S$

## 5. On the Irreducibility of $\psi'_\lambda : A(E_{3,1}) \rightarrow GL_3(\mathbb{C})$

We consider the representation $\psi_\lambda : A(E_{3,1}) \rightarrow GL_4(\mathbb{C})$ restricted to the basis $e_1, e_1 + \frac{b_2}{b_1}e_2, e_3$, and $e_4$. The matrix of $\sigma_1$ becomes

$$\psi_\lambda(\sigma_1) = \begin{pmatrix} 1 & 0 & t & 0 \\ 0 & 1 & t & 0 \\ 0 & 0 & -t & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Similarly, we determine the matrices of $\sigma_2$ and $\delta$. It is easy to see that the first column of the matrices of all generators is $(1, 0, 0, 0, 0)^T$, where $T$ is the transpose. We thus reduce our representation to a 3-dimensional one by deleting the first row and the first column to get $\psi'_\lambda : A(E_{3,1}) \rightarrow GL_3(\mathbb{C})$. The representation is defined as follows:

$$\psi'_\lambda(\sigma_1) = \begin{pmatrix} 1 & t & 0 \\ 0 & -t & 0 \\ 0 & 1 & 1 \end{pmatrix},$$

$$\psi'_\lambda(\sigma_2) = \begin{pmatrix} 1 & 0 & \frac{tb_2}{b_1} \\ 0 & 1 & t \\ 0 & 0 & -t \end{pmatrix},$$

and

$$\psi'_\lambda(\delta) = \begin{pmatrix} 1 + \lambda_1 b_1 + \lambda_2 b_2 & \lambda_1 d_1 + \frac{b_2}{b_1}\lambda_2 d_1 & -t\lambda_1 + \frac{-tb_2}{b_1}\lambda_2 \\ b_1 & 1 + d_1 & -t \\ 0 & 0 & 1 \end{pmatrix}.$$

We then diagonalize the matrix corresponding to $\psi'_\lambda(\sigma_1)$ by an invertible matrix, say $T$, and conjugate the matrices of $\psi'_\lambda(\sigma_2)$ and $\psi'_\lambda(\delta)$ by the same matrix $T$. The invertible matrix $T$ is given by

$$T = \begin{pmatrix} 0 & 1 & t \\ 0 & 0 & -1-t \\ 1 & 0 & 1 \end{pmatrix}.$$

In fact, a computation shows that

$$T^{-1}\psi'_\lambda(\sigma_1)T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -t \end{pmatrix}.$$

After conjugation, we get

$$T^{-1}\psi'_\lambda(\sigma_2)T = \begin{pmatrix} \frac{-t^2}{1+t} & 0 & \frac{-(1+t+t^2)}{1+t} \\ \frac{t(b_2+b_1t+b_2t)}{b_1(1+t)} & 1 & \frac{t(b_2+b_1t+b_2t)}{b_1(1+t)} \\ \frac{-t}{1+t} & 0 & \frac{1}{1+t} \end{pmatrix},$$

$T^{-1}\psi'_\lambda(\delta)T =$

$$\begin{pmatrix} \frac{1}{1+t} & \frac{b_1}{1+t} & \frac{t}{1+t} \\ -t(\lambda_1 + \frac{b_2\lambda_2}{b_1} + \frac{t}{1+t}) & 1 + \lambda_2 b_2 + b_1(\lambda_1 + \frac{t}{1+t}) & \frac{(-t+b_1t-d_1(1+t))[b_2\lambda_2(1+t)+b_1(\lambda_1+t+\lambda_1 t)]}{b_1(1+t)} \\ \frac{t}{1+t} & \frac{b_1}{1+t} & \frac{1}{1+t} \end{pmatrix}.$$

The entries of the matrices $T^{-1}\psi'_\lambda(\sigma_2)T$ and $T^{-1}\psi'_\lambda(\delta)T$ are well-defined since we assume in our work that $t \neq -1$. For simplicity, we denote $T^{-1}\psi'_\lambda(\sigma_1)T$ by $\psi'_\lambda(\sigma_1)$, $T^{-1}\psi'_\lambda(\sigma_2)T$ by $\psi'_\lambda(\sigma_2)$, and $T^{-1}\psi'_\lambda(\delta)T$ by $\psi'_\lambda(\delta)$.

We now prove some propositions to determine a sufficient condition for irreducibility of $\psi'_\lambda : A(E_{3,1}) \to GL_3(\mathbb{C})$.

**Proposition 2** $t(b_2 + b_1t + b_2t) + (1 + t + t^2)(\lambda_2 b_2(1 + t) + b_1(\lambda_1 + t + \lambda_1 t)) = -(t + 1)^2(t^2 + 1)$

*Proof.* The proof easily follows by considering the following relations:

- $tb_2 = -td_1 - t(1 + t)$
- $tb_1 = (1 + t)d_1 + 2t$
- $\lambda_1 b_1 + \lambda_2 b_2 = -(1 + t + d_1)$

**Proposition 3** *The two expressions* $1 + t + t^2$ *and* $b_1t + b_2t + b_2$ *cannot be both equal to zeros.*

*Proof.* We assume, for contradiction, that both are equal to zeros.
By substituting $tb_2 = -td_1 - t(1+t)$ and $tb_1 = (1+t)d_1 + 2t$ in $b_1t + b_2t + b_2 = 0$, we get $-t(1+t+t^2) = -t^2$. By assuming that $1 + t + t^2 = 0$, we get that $t = 0$, a contradiction.

**Proposition 4** $-t + b_1t - d_1(1 + t) \neq 0$.

*Proof.* Assume, for contradiction, that $-t + b_1t - d_1(1 + t) = 0$. Having that $b_1t = (1 + t)d_1 + 2t$, we get $-t + b_1t - d_1(1 + t) = -t + (1 + t)d_1 + t + t - d_1(1 + t) = t$. This implies that $t = 0$, a contradiction.

We use Proposition 2, Proposition 3 and Proposition 4 to prove the following Lemma. We recall that all the indeterminates used in defining the representations are specialized to non zero complex numbers and, in addition, the complex number associated with $t$ is not equal to $-1$.

**Lemma 5** *If $t \neq \pm i$, then any non zero subspace $S$, which is invariant under the action of the representation $\psi'_\lambda : A(E_{3,1}) \to Gl_3(\mathbb{C})$ containing the standard unit vector $e_3$, must be the whole space $\mathbb{C}^3$.*

*Proof.* We have that $\psi'_\lambda(\sigma_2)(e_3) = \frac{-(1+t+t^2)}{1+t}e_1 + \frac{t(b_2+b_1t+b_2t)}{b_1(1+t)}e_2 + \frac{1}{1+t}e_3 \in S$.

Since $e_3 \in S$, it follows that $\frac{-(1+t+t^2)}{1+t}e_1 + \frac{t(b_2+b_1t+b_2t)}{b_1(1+t)}e_2 \in S$. (1)

Moreover,

$\psi'_\lambda(\delta)(e_3) =$

$\frac{-t+b_1t-d_1(1+t)}{1+t}e_1 + \frac{(-t+b_1t-d_1(1+t))[\lambda_2b_2(1+t)+b_1(\lambda_1+t+\lambda_1t)]}{b_1(1+t)}e_2 + \frac{1+t+d_1(1+t)+t-b_1t}{1+t}e_3 \in S$.

This also implies that

$$\frac{-t+b_1t-d_1(1+t)}{1+t}e_1 + \frac{(-t+b_1t-d_1(1+t))[\lambda_2b_2(1+t)+b_1(\lambda_1+t+\lambda_1t)]}{b_1(1+t)}e_2 \in S. \quad (2)$$

Having proved that $1+t+t^2$ and $b_1t+b_2t+b_2$ can't both be zeros, we consider the following cases:

**Case 1.** $1+t+t^2 = 0$

By Proposition 3 and (1), we get that $e_2 \in S$. By Proposition 4 and (2), we get that $e_1 \in S$. Thus, S is the whole space.

**Case 2.** $1+t+t^2 \neq 0$

Let us multiply (1) by $-t+b_1t-d_1(1+t)$ which is proved not to be zero in Proposition 4. We also multiply (2) by $1+t+t^2 \neq 0$. If we add the obtained equations, we get

$$\frac{-t+b_1t-d_1(1+t)}{b_1(1+t)}[t(b_2+b_1t+b_2t)+(1+t+t^2)(\lambda_2b_2(1+t)+b_1(\lambda_1+t+\lambda_1t))]e_2 \in S. \quad (3)$$

By Proposition 2, we have that $t(b_2+b_1t+b_2t)+(1+t+t^2)(\lambda_2b_2(1+t)+b_1(\lambda_1+t+\lambda_1t)) = -(t+1)^2(t^2+1)$. Assuming that $t \neq -1$ and $t \neq \pm i$, we get $[t(b_2+b_1t+b_2t)+(1+t+t^2)(\lambda_2b_2(1+t)+b_1(\lambda_1+t+\lambda_1t)) \neq 0$.

By Proposition 4 and by (3), we get

$$e_2 \in S.$$

From (1) we conclude that

$$e_1 \in S.$$

Thus, S is the whole space $\mathbb{C}^3$.

Next, we present the following theorem which gives a sufficient condition for irreducibility of $\psi'_\lambda : A(E_{3,1}) \to GL_3(\mathbb{C})$.

**Theorem 6** *If $t \neq \pm i$, then the representation $\psi'_\lambda : A(E_{3,1}) \to GL_3(\mathbb{C})$ is irreducible.*

*Proof.* Let S be a non zero proper subspace of $\mathbb{C}^3$, which is invariant under the action of $\psi'_\lambda$. By Lemma 5, we have that $e_3 \notin S$.

Then S is one of the following subspaces:

- $S = \langle e_1 \rangle$
- $S = \langle e_2 \rangle$
- $S = \langle e_1 + ue_2 \rangle$, where $u \in \mathbb{C}^*$
- $S = \langle e_1, e_2 \rangle$

**Case 1.** $S = \langle e_1 \rangle$. We have that $\psi'_\lambda(\sigma_2)(e_1) \in S$. This implies that $\begin{pmatrix} \frac{-t^2}{1+t} \\ \frac{t(b_2+b_1t+b_2t)}{b_1(1+t)} \\ \frac{-t}{1+t} \end{pmatrix} \in S$. This gives a contradiction because $t \neq 0$.

**Case 2.** $S = \langle e_2\rangle$. We have that $\psi'_\lambda(\delta)(e_2) \in S$. This implies that $\begin{pmatrix} \frac{b_1}{1+t} \\ 1+\lambda_2 b_2 + b_1(\lambda_1 + \frac{t}{1+t}) \\ \frac{b_1}{1+t} \end{pmatrix} \in S$. This gives a contradiction since $b_1 \neq 0$.

**Case 3.** $S = \langle e_1 + ue_2\rangle,\ u \in \mathbb{C}^*$. We have that $\psi'_\lambda(\sigma_2)(e_1 + ue_2) \in S$. This implies that $\begin{pmatrix} \frac{-t^2}{1+t} \\ \frac{t(b_2+b_1t+b_2t)}{b_1(1+t)} + u \\ \frac{-t}{1+t} \end{pmatrix} \in S$. This gives a contradiction since $t \neq 0$.

**Case 4.** $S = \langle e_1, e_2\rangle$.

We have that $\psi'_\lambda(\sigma_2)(e_1) \in S$. This implies that $\begin{pmatrix} \frac{-t^2}{1+t} \\ \frac{t(b_2+b_1t+b_2t)}{b_1(1+t)} \\ \frac{-t}{1+t} \end{pmatrix} \in S$. This gives a contradiction since $t \neq 0$.

Therefore, we conclude that the representation is irreducible because there is no proper non zero invariant subspace under the action of $\psi'_\lambda$.

We now give a necessary condition for irreducibility.

**Theorem 7** *If $t = \pm i$, then the subspace $\langle e_1, e_3\rangle$ is a proper invariant subspace.*

*Proof.*

1. $\psi'_\lambda(\sigma_1)(e_1) = e_1 \in S$.

2. $\psi'_\lambda(\sigma_2)(e_1) = \begin{pmatrix} \frac{-t^2}{1+t} \\ \frac{t(b_2+b_1t+b_2t)}{b_1(1+t)} \\ \frac{-t}{1+t} \end{pmatrix} = ae_1 + be_3$, where $a$ and $b \in \mathbb{C} - \{0\}$.

   Here, we have $a = \frac{-t^2}{1+t}$, $b = \frac{-t}{1+t}$, and $\frac{t(b_2+b_1t+b_2t)}{b_1(1+t)} = 0$. This is true since $t^2 = -1$.

3. $\psi'_\lambda(\delta)(e_1) = \begin{pmatrix} \frac{1}{1+t} \\ -t(\lambda_1 + \frac{b_2\lambda_2}{b_1} + \frac{t}{1+t} \\ \frac{t}{1+t} \end{pmatrix} = ae_1 + be_3$ ,

   where $a$ and $b$ are given by $a = \frac{1}{1+t}$ and $b = \frac{t}{1+t}$.

4. $\psi'_\lambda(\sigma_1)(e_3) = -te_3 \in S$.

5. $\psi'_\lambda(\sigma_2)(e_3) = \begin{pmatrix} \frac{-(1+t+t^2}{1+t} \\ \frac{t(b_2+b_1t+b_2t)}{b_1(1+t)} \\ \frac{1}{1+t} \end{pmatrix} = ae_1 + be_3$,

where $a = \frac{-(1+t+t^2)}{1+t}$ and $b = \frac{1}{1+t}$.

6. $$\psi'_\lambda(\delta)e_3 = \begin{pmatrix} \frac{t}{1+t} \\ \frac{(-t+b_1t-d_1(1+t))(b_2\lambda_2(1+t)+b_1(\lambda_1+t+\lambda_1 t))}{b_1(1+t)} \\ \frac{1}{1+t} \end{pmatrix} = ae_1 + be_3,$$

where $a$ and $b$ are given by $a = \frac{t}{1+t}$ and $b = \frac{1}{1+t}$.

By Proposition 2, we have $\frac{(-t+b_1t-d_1(1+t))(b_2\lambda_2(1+t)+b_1(\lambda_1+t+\lambda_1 t))}{b_1(1+t)} = 0$.

Thus, we have determined a necessary and sufficient condition for irreducibility.

**Theorem 8** *Let $\lambda_1, \lambda_2, b_1, b_2, d_1, t \in \mathbb{C} - \{0\}$ and $t \neq -1$. The representation $\psi'_\lambda : A(E_{3,1}) \rightarrow GL_3(\mathbb{C})$ is irreducible if and only if $t \neq \pm i$.*

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