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Enumerations for Compositions and Complete Homogeneous Symmetric Polynomial

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Abstract

We count the number of occurrences of *t* as the summands (i) in the compositions of a positive integer *n* into *r* parts; and (ii) in all compositions of *n*; and subsequently obtain other results involving compositions. The initial counting further helps to solve the enumeration problems for complete homogeneous symmetric polynomial.

Keywords: composition, summand, sequence, recurrence, binomial coefficient identity, complete homogeneous symmetric polynomial

MS classifications: 05A10; 05A17; 05A19; 11B37

1. Introduction

For each nonnegative integer *r*, complete homogeneous symmetric polynomial $h_r(x_1, \dots, x_k)$ is the sum of all distinct monomials of degree *r* in the variables: $x_1, \dots, x_k$ Formally

$$h_r(x_1, \dots, x_k) = \sum_{1 \le i_1 \le \dots \le i_r \le k} x_{i_1} x_{i_2} \dots x_{i_r}$$

Example:

$$h_4(x_1, x_2, x_3) = x_1^4 + x_2^4 + x_3^4 + x_1^3 x_2 + x_1 x_2^3 + x_1^3 x_3 + x_1 x_3^3 + x_2^3 x_3 + x_2 x_3^3$$

$$+ x_1^2 x_2^2 + x_1^2 x_3^2 + x_2^2 x_3^2 + x_1^2 x_2 x_3 + x_1 x_2^2 x_3 + x_1 x_2 x_3^2$$

The number of terms of the polynomial is 15; the numbers of occurrences of 1, 2, 3 and 4 as the exponents in different terms among 15 terms are 12, 9, 6 and 3 respectively; and the number of occurrences of each of x_1, x_2 and x_3 as the bases in different terms among 15 terms is 10.

Evidently $h_r(x_1, \dots, x_k)$ has some enumerating problems. We give the solutions of the problems from some combinatorial enumerations for the compositions of a positive integer. The paper has two main parts: (a) counting for compositions and (b) counting for $h_r(x_1, \dots, x_k)$.

The main results are as shown:

1. The number of occurrences of an integer *t* as the summands in the compositions of *n* into *r* parts

$$= r\binom{n-t-1}{r-2}, \; n \ge r \ge 2, \; n - r + 1 \ge t \ge 1$$

2. The number of occurrences of *t* as the summands in all compositions of *n*

$$= (n - t + 3)\, 2^{n-t-2}, \; n > t \ge 1$$

3. The number of occurrences of *t* as the exponents in different terms among all the terms of $h_r(x_1, \dots, x_k)$

$$= \sum_{i=2}^{r} i \binom{k}{i} \binom{r-t-1}{i-2}, \qquad r, k \ge 2, \qquad r - 1 \ge t \ge 1$$

4. The number of occurrences of each variable as the bases in different terms among all the terms of $h_r(x_1, \dots, x_k)$

$$= 1 + \frac{1}{k}\sum_{t=1}^{r-1}\sum_{i=2}^{r} i\binom{k}{i}\binom{r-t-1}{i-2}$$

2. Counting for Compositions

For counting, we use some simple notations as shown.

1. Compositions of n into r summands = $C(n, r)$.

The notation $C(n, r)$ without any qualification means all compositions of n into r summands. Otherwise we use an adjective to specify the compositions. For example, we write simply ‘some $C(n, r)$’ to mean some compositions of n into r summands.

2. Number of $C(n, r)$ = $NC(n, r)$.

3. Number of occurrences of t in $C(n, r)$ = $N(t)C(n, r)$.

4. Some particular $C(n, r)$ that start with a common summand k = $k + C(n - k, r - 1)$.

We use the symbol of equivalence (≡) between $C(n, r)$ and its implication; and similarly between $k + C(n - k, r - 1)$ and its implication.

Examples: $C(4, 3) \equiv 1 + 1 + 2, \quad 1 + 2 + 1, \quad 2 + 1 + 1.$

$NC(4, 3) = 3$; $N(1)C(4, 3) = 6$; and $N(2)C(4, 3) = 3$.

Some particular $C(6, 4)$, which start with a common summand 2, are:

$2 + C(4, 3) \equiv 2 + 1 + 1 + 2, \quad 2 + 1 + 2 + 1, \quad 2 + 2 + 1 + 1.$

2.1 Number of Occurrences of t in the Compositions of n into r Parts

The number of the compositions of a positive integer n into r parts or summands is $\binom{n-1}{r-1}$. This is a known result. Here we obtain the result in a process of recursive substitution starting with a basic sequential arrangement of n into r summands. The procedure and result lead to count the number of occurrences of t in the compositions of n into r parts.

First we count $NC(n, r)$ for $n \ge r \ge 1$.

(a) By convention, n itself is a composition of n so that r is equal to 1 for the composition. Therefore, for $n \ge 1$ and $r = 1$, we have: $NC(n, r) = NC(n, 1) = 1$.

(b) For $n \ge r \ge 2$, we can write a basic sequential arrangement of $C(n, r)$ in the following way.

$$C(n, r) \equiv 1 + C(n-1, r-1), \quad 2 + C(n-2, r-1), \dots, \quad (n-r+1) + C(r-1, r-1) \qquad (1.1)$$

Consequently for $n \ge r \ge 2$,

$$NC(n, r) = NC(n-1, r-1) + NC(n-2, r-1) + \dots + NC(r-1, r-1) \qquad (1.2)$$

(1.1) and (1.2) yield the successive results as shown.

(i) $C(n, 2) \equiv 1 + (n-1), \quad 2 + (n-2), \dots, \quad (n-1) + 1.$

Hence $NC(n, 2) = n - 1$.

(ii) $C(n, 3) \equiv 1 + C(n-1, 2), \quad 2 + C(n-2, 2), \quad \dots, \quad (n-2) + C(2, 2)$

Hence $NC(n, 3) = NC(n-1, 2) + NC(n-2, 2) + \dots + NC(2, 2)$

$= (n-2) + \dots + 1$

$= \binom{n-1}{2}$

Similarly

$NC(n, 4) = NC(n-1, 3) + NC(n-2, 3) + \dots + NC(3, 3)$

$= \sum_{i=3}^{n-1}\binom{i-1}{2}$

$= \binom{n-1}{3}$

In general for $n \geq r \geq 2$, $NC(n, r) = \binom{n-1}{r-1}$; and including the initial result with this, we get:

$$NC(n,r) = \binom{n-1}{r-1}, \ n \geq r \geq 1 \tag{2}$$

We count below $N(t)C(n, r)$ by the above process and results.

The basic conditions of n, t and r are: $n \geq t \geq 1$, $n \geq r \geq 1$ and $r \gtreqless t$.

(a) When $n \geq 1$ and $r = 1$ then $t = n$ so that $N(t)C(n, r) = N(n)C(n, 1) = 1$. For $n > t \geq 1$, mathematically we can write: $N(t)C(n, 1) = 0$. These are the initial results in the counting of $N(t)C(n, r)$.

(b) When n and r are the fixed integers for $n \geq r \geq 2$ then we find some fixed $C(n, r)$. In one of these $C(n, r)$, each of $r - 1$ summands is smallest or 1 so that the rest is greatest. Consequently the greatest value of a summand t is $n - r + 1$. That is, when $n \geq r \geq 2$ then the condition of t is: $n - r + 1 \geq t \geq 1$. From (1.1) we find:

(i) Some $C(n, r)$ start with a common summand t; and these are: $t + C(n - t, r - 1)$.

(ii) t can occur at other places of different compositions under some or all of $C(n - 1, r - 1)$, $C(n - 2, r - 1)$, …, $C(r - 1, r - 1)$.

The smallest positive integer: 1 can occur as the summands in different compositions under $C(n - 1, r - 1)$, $C(n - 2, r - 1)$, …, $C(r - 1, r - 1)$. Yet occurrences of $t \geq 2$ have some limitations. Since $n - r + 1 \geq t$, it follows that if $n - r \leq t - 2$ then t cannot occur in a $C(n, r)$. More precisely, we cannot find the occurrences of

2 in $C(r - 1, r - 1)$;

3 in $C(r, r - 1)$ and $C(r - 1, r - 1)$;

4 in $C(r + 1, r - 1)$, $C(r, r - 1)$ and $C(r - 1, r - 1)$;

… …

In general t cannot occur in any composition under $C(r + t - 3, r - 1)$, $C(r + t - 4, r - 1)$, …, $C(r - 1, r - 1)$ for $r \geq 2$ and $t \geq 2$. Other compositions under $C(n - 1, r - 1)$, $C(n - 2, r - 1)$, …, $C(r + t - 2, r - 1)$ may contain t. Then from (i) and (ii), we get: for $n \geq r \geq 2$ and $n - r + 1 \geq t \geq 1$,

$$\begin{aligned} N(t)C(n, r) &= NC(n - t, r - 1) + [N(t)C(n - 1, r - 1) + N(t)C(n - 2, r - 1) + \dots \\ &\quad + N(t)C(r + t - 2, r - 1)] \\ \Rightarrow N(t)C(n, r) &= \binom{n-t-1}{r-2} + [N(t)C(n - 1, r - 1) + N(t)C(n - 2, r - 1) + \dots \\ &\quad + N(t)C(r + t - 2, r - 1)] \end{aligned} \tag{3}$$

When $r = 2$, then for $n \geq 2$ and $n - 1 \geq t \geq 1$,

$$\begin{aligned} N(t)C(n, 2) &= 1 + N(t)C(t, 1) = 1 + 1 \\ &= 2 \end{aligned} \tag{3.1}$$

Evidently $N(t)C(n, 2)$ is constant and independent of t and n. (3.1) is the primary case of (3); and then the rest of (3) is: for $n \geq r \geq 3$ and $n - r + 1 \geq t \geq 1$,

$$\begin{aligned} N(t)C(n, r) &= \binom{n-t-1}{r-2} + [N(t)C(n - 1, r - 1) + N(t)C(n - 2, r - 1) + \dots \\ &\quad + N(t)C(r + t - 2, r - 1)] \end{aligned} \tag{3.2}$$

Now our aim is to count $N(t)C(n, r)$ applying (3.1) and (3.2).

1. To count $N(t)C(n, r)$ for $t = 1$

For $n \geq 2$, $N(1)C(n, 2) = 2$

For $n \geq 3$, $N(1)C(n, 3) = n - 2 + [N(1)C(n - 1, 2) + N(1)C(n - 2, 2) + \dots + N(1)C(2, 2)]$

$$= 3(n - 2).$$

For $n \geq 4$, $N(1)C(n, 4) = \binom{n-2}{2} + [N(1)C(n - 1, 3) + N(1)C(n - 2, 3) + \dots + N(1)C(3, 3)]$

$$= \binom{n-2}{2} + 3[(n - 3) + (n - 4) + \dots + 1]$$

$$= 4\binom{n-2}{2}.$$

Similarly for $n \geq 5$, $N(1)C(n, 5) = \binom{n-2}{3} + 4\sum_{i=4}^{n-1}\binom{i-2}{2}$

$$= 5\binom{n-2}{3}.$$

In general for $n \geq r \geq 2$, $N(1)C(n, r) = r\binom{n-2}{r-2}$

2. To count $N(t)C(n, r)$ for $t = 2$

n, r and t have the conditions: $n \geq r \geq 2$ and $n - r + 1 \geq t \geq 1$. Hence when $t = 2$ then the conditions of n and r are: $n - 1 \geq r \geq 2$ and $n \geq 3$.

For $n \geq 3$, $N(2)C(n, 2) = 2$.

For $n \geq 4$, $N(2)C(n, 3) = n - 3 + [N(2)C(n - 1, 2) + N(2)C(n - 2, 2) + \ldots + N(2)C(3, 2)]$
$= 3(n - 3)$.

For $n \geq 5$, $N(2)C(n, 4) = \binom{n-3}{2} + [N(2)C(n - 1, 3) + N(2)C(n - 2, 3) + \ldots + N(2)C(4, 3)]$
$= \binom{n-3}{2} + 3[(n - 4) + (n - 5) + \ldots + 1]$
$= 4\binom{n-3}{2}$.

Thus for $n \geq 6$, $N(2)C(n, 5) = \binom{n-3}{3} + 4\sum_{i=5}^{n-1}\binom{i-3}{2}$
$= 5\binom{n-3}{3}$.

In general for $n - 1 \geq r \geq 2$, $N(2)C(n, r) = r\binom{n-3}{r-2}$.

By the similar operation, we get:

For $n - 2 \geq r \geq 2$, $N(3)C(n, r) = r\binom{n-4}{r-2}$.

For $n - 3 \geq r \geq 2$, $N(4)C(n, r) = r\binom{n-5}{r-2}$.

… …

In this way

$$N(t)C(n,r) = r\binom{n-t-1}{r-2}, \quad n \geq r \geq 2, n - r + 1 \geq t \geq 1 \qquad (4)$$

2.2 Number of Occurrences of t in All Compositions of n

Let the number be denoted by $N(t)C(n)$.

When $n \geq 1$ and $t = n$, then $N(t)C(n) = N(n)C(n) = 1$.

The restrictions in (4) are:

$$n \geq r \geq 2, \quad n - r + 1 \geq t \geq 1 \quad \Leftrightarrow \quad n > t \geq 1, \quad n - t + 1 \geq r \geq 2$$

Hence for $n > t \geq 1$,

$$N(t)C(n) = \sum_{r=2}^{n-t+1} N(t)C(n,r)$$

$$= \sum_{r=2}^{n-t+1} r\binom{n-t-1}{r-2}$$

$$\Rightarrow N(t)C(n) = (n - t + 3)\, 2^{n-t-2}, \quad n > t \geq 1 \qquad (5)$$

2.3 Other Results from (2), (4) and (5)

(a) Number of summands in the compositions of *n*

From (5), we get:

Number of the summands in the compositions of n for $n \geq 2$

= Number of occurrence of t for $t = n$ + Number of occurrences of t for $n > t \geq 1$

$$= 1 + \sum_{t=1}^{n-1}(n - t + 3)\, 2^{n-t-2}$$

$$= 1 + 2^{n-2}\left[(n+3)\sum_{t=1}^{n-1} 2^{-t} - \sum_{t=1}^{n-1} t\, 2^{-t}\right]$$

$$= (n+1)\, 2^{n-2} \qquad (6)$$

Obviously (6) holds for $n = 1$ also.

(b) A proposition from (5)

Proposition 1. *If t_1 and t_2 are the summands in the compositions of n_1 and n_2 respectively such that $n_1 - t_1 = n_2 - t_2$, then the number of occurrences of t_1 in the compositions of n_1 is equal to the number of occurrences of t_2 in the compositions of n_2.*

(c) Number-number relationship

By Pascal's Identity, we get:

$$r\binom{n-2}{r-2} = r\binom{n-1}{r-1} - r\binom{n-2}{r-1}$$

The above relation implies the following number-number relationship from (4) and (2).

Number of occurrences of 1 in $C(n, r)$

= Number of the summands in $C(n, r)$ – Number of the summands in $C(n - 1, r)$ (7)

(d) Number-sum relationship

From (6), we get a number-sum relationship as shown.

Number of the summands in the compositions of all n integers: 1, 2, …, n

$$= \sum_{i=1}^{n} (i+1)2^{i-2}$$

$$= n\,2^{n-1}$$

= Sum of the summands in the compositions of n. (8)

3. Counting for Complete Homogeneous Symmetric Polynomial: $h_r(x_1, \dots, x_k)$

3.1 Number of Terms of the Polynomial

The result is known. Here we count the number applying (2) and Vandermonde's identity. Let some terms of the polynomial contain some fixed m of k variables. The number of these terms = $NC(r, m) = \binom{r-1}{m-1}$.

We have $k \gtreqless r$ in the problem. Hence we find: (i) either $1 \le m \le k < r$ (ii) or $1 \le m \le r \le k$.

Case 1: When $1 \le m \le k < r$ then the number of terms

$$= \sum_{m=1}^{k}\binom{k}{m}\binom{r-1}{m-1}$$

$$= \sum_{m=1}^{k}\binom{k}{k-m}\binom{r-1}{m-1}$$

$$= \binom{k+r-1}{k-1} = \binom{k+r-1}{r}$$

Case 2: When $1 \le m \le r \le k$ then the number of terms

$$= \sum_{m=1}^{r}\binom{k}{m}\binom{r-1}{m-1}$$

$$= \sum_{m=1}^{r}\binom{k}{m}\binom{r-1}{r-m}$$

$$= \sum_{m=0}^{r}\binom{k}{m}\binom{r-1}{r-m}$$

$$= \binom{k+r-1}{r}$$

It follows that the number of terms does not depend on equality or any inequality between k and r, which are all taken into consideration in the process of solution. Thus we find:

The number of terms of $h_r(x_1, \dots, x_k) = \binom{k+r-1}{r}$ (9)

3.2 Number of Occurrences of an Integer t as the Powers

Appling (4), we can count the number of occurrences of an integer t as the powers in different terms among all $\binom{k+r-1}{r}$ terms of $h_r(x_1, \dots, x_k)$.

The condition of t is: $r \ge t \ge 1$.

Case 1. The terms in which the integer r occurs as the powers on the variables are: $x_1^r, \dots, x_k^r$.

Therefore when $t = r$ then the number of occurrences of t is k.

Case 2. When $t < r$, clearly then $r, k \ne 1$. From (4), we get:

The number of occurrences of t

$$= \sum_{i=2}^{r} i\binom{k}{i}\binom{r-t-1}{i-2}, \quad r, k \ge 2, \quad r-1 \ge t \ge 1 \tag{10}$$

(10) has some technical terms for some particular values of k, r and t such that the values of these terms are all 0. The particulars in the context are described below.

(i) If m is an integer in (2, …, r) then the product $m\binom{k}{m}\binom{r-t-1}{m-2}$ or $m\binom{k}{m}\binom{r-t-1}{r-t+1-m}$ is one among r – 1 terms of (10). The value of the term is obviously 0 if $m > r - t + 1$. For example, if the triplet (k, r, t) is (12, 7, 4) then the values of the last three terms of (10) where $m \in$ (5, 6, 7) are all 0. This implies that if the number of bases in a term of $h_7(x_1, \ldots, x_{12})$ is 5, 6 or 7 then the number of occurrences of 4 as the powers on the bases is 0, or in other words 4 cannot occur as the powers on any of these bases.

(ii) When $r > k$ then the last $r - k$ terms have the factors: $\binom{k}{k+1}$, …, $\binom{k}{r}$ in succession such that the values of these $r - k$ terms are all 0. In other words, for $r > k$, the number of occurrences of t is equal to the summation: $\sum_{i=2}^{k} i\binom{k}{i}\binom{r-t-1}{i-2}$.

3.3 Number of Occurrences of a Variable x_m as the Bases

From Case 1 and Case 2 of Topic 3.2, we get:

Total number of bases in all terms of the polynomial

$$= k + \sum_{t=1}^{r-1} \sum_{i=2}^{r} i\binom{k}{i}\binom{r-t-1}{i-2} \tag{11.1}$$

The number of occurrences of every variable $x_m \in (x_1, \ldots, x_k)$ in complete homogeneous symmetric polynomial of degree r in the variables: $x_1, \ldots, x_k$ is same. Hence from (11.1), we get:

The number of occurrences of a variable x_m as the bases

$$= 1 + \frac{1}{k} \sum_{t=1}^{r-1} \sum_{i=2}^{r} i\binom{k}{i}\binom{r-t-1}{i-2} \tag{11.2}$$

References

Andrews, G. E. (2004). The Theory of Partitions, Cambridge University Press, Chapter 4, 54.

Arthur, T. B., & Jennifer, Q. (2003). Proofs That Really Count: The Art of Combinatorial Proof, MAA, 63-68.

Hall, H. S., & Knight, S. R. (1967). Higher Algebra, Macmillan and Co, London, 164 -166.

John, R. (1979). Combinatorial identities, R. E. Krieger Pub. Co.

Miklós, B. (2006). A walk through combinatorics, World Scientific Publishing Co. Pte. Ltd. 65-69, 89.

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