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A Spline Group – Korovkin Approximation Theorem

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Abstract

In this paper, using homogeneous groups, we prove a Korovkin type approximation theorem for a spline group by using the notion of a generalization of positive linear operator.

Keywords: Piecewise polynomial functions, Spline, Lie group, Group homomorphism, Korovkin type theorem, Positive linear operators

1. Introduction and Preliminaries

In this work, we prove a Korovkin approximation theorem by applying the notion of spline with homogeneous groups. Several mathematicians have worked on extending or generalizing the Korovkin's theorems in many ways and to several setting, including function spaces, abstract Banach lattices, Banach algebras and Banach spaces. This theory is very useful in real analysis, functional analysis, harmonic analysis, measure theory, probability theory, summability theory and partial differential equations. But the foremost applications are concerned with constructive approximation theory, which uses it as a valuable tool. Even today, the development of Korovkin type approximation theorem is far from complete.

In this field, Mursaleen work as follows:

Statistical lacunary summability and strongly θ_q-convergence ($0 < q < \infty$) and establish some relations between lacunary statistical convergence, statistical lacunary summability and strongly θ_q-convergence (Mursaleen & Alotaibi, 2011, pp. 373-381). And introduce provided a Korovkin type approximation theorem by using the test functions $1, e^{-x}, e^{-2x}$ here the rate of statistical summability $(C, 1)$ and apply the classical Baskakov operator (Mohiuddine, Alotaibi & Mursaleen, 2012), also he study the rate of functions of two variables through statistical A–summability to Korovkin second theorem via statistical summability $(C, 1)$ (Mursaleen, & Alotaibi, 2012). Recently, (Ergur, & Duman, 2013) they have defined generalize and develop the Korovkin type approximation theory by using an appropriate abstract space. This was done, presented a Korovkin type theorem for an interleave between Riesz's representation theory and Lebesgue-Stieltjes integral-i, for Riesz's functional supremum formula via statistical limit (Al-Muhja, 2014). Malliavin calculus, initiated by (Graczyk, 1991, pp. 183-205), was a stable semigroup of measures on homogeneous Lie groups are a natural generalization of the notion of a strictly stable measure on $\mathcal{R}^n$.

First of all, we recall some will accept the notes and definitions used in this paper. The concept of a Korovkin approximation for positive linear operator was introduced by (Fast, 1951, pp. 241-244) and further studied many others. Let $K \subseteq \mathbb{N}$, the A-density of K denoted by $\delta_A(K)$ is defined to be $\delta_A(K) = \lim_j \sum_{n\in K} a_{jn}$ provided that the limit exists. Using this A-density, we say that a sequence $x = (x_n)$ is A-statistically convergent to L if and only if

$$\delta_A\big(K(\epsilon)\big) = 0 \text{ for every } \epsilon > 0, \text{ where } K(\epsilon) = \{n \in \mathbb{N}: |x_n - L| \geq \epsilon\} \tag{1}$$

or, equivalently

$$\lim_j \sum_{n\,:\,|x_n - L| \geq \epsilon} a_{jn} = 0.$$

This limit is denoted by $st_A - \lim_n x_n = L$.

For a given sequence (x_n), the A-transform x denoted by $\left((Ax)_j\right)$ is given by $(Ax)_j = \sum_{n=1}^{\infty} a_{jn} x_n$ provided the series converges for each $j \in \mathbb{N}$. We say that A is regular if $\lim_j (Ax)_j = L$, whenever $\lim_n x_n = L$.

Theorem 1.1 (Altomare & Campiti, 1994) An infinite summability matrix $A = (a_{jn})$ is regular if and only if it satisfies all of the following properties

$$\sup_j \sum_{n=1}^{\infty} |a_{jn}| < \infty,$$
$$\lim_j a_{jn} = 0 \text{ for each } n \in \mathbb{N},$$
$$\lim_j \sum_{n=1}^{\infty} a_{jn} = 1.$$

The classical Korovkin approximation theorem is stated as follows (Korovkin, 1960):

Theorem 1.2 Let (T_n) be a sequence of positive linear operators from $C[a,b]$ into $C[a,b]$. Then $\lim_n \|T_n(f,x) - f(x)\|_\infty = 0$, for all $f \in C[a,b]$ if and only if $\lim_n \|T_n(f_i,x) - f_i(x)\|_\infty = 0$, for $i = 0,1,2$, where $f_0(x) = 1$, $f_1(x) = x$ and $f_2(x) = x^2$.

2. Spline Group and Operator

The problem of this paper is to study a spline using some properties a Korovkin approximation theorem (Kopotun, 2006, pp. 36-43; Kopotun, 2007, pp. 913-945; Fast, 1951, pp. 241-244), as well as the use of homogeneous groups (Graczyk, 1991, pp. 183-205):

Let s be piecewise polynomial of degree $\leq r$ on t_n such that $s \in C^{v-1}[-1,1]$, where $t_n = \left(\cos\frac{(n-i)\pi}{n}\right)_{i=0}^{n}$, $n \in \mathbb{N}$, $1 \leq v \leq r$ is the Chebyshev partition of $[-1,1]$.

Let $\mathcal{S}_r(\mathrm{z}_n)$ be the space of all piecewise polynomial functions (which we refer to as "splines") of degree r (order $r+1$), with the knots $\mathrm{z}_n = (\mathrm{z}_i)_{i=0}^{n}$,

$$-1 = \mathrm{z}_0 < \mathrm{z}_1 < \cdots < \mathrm{z}_{n-1} < \mathrm{z}_n = 1. \tag{2}$$

In other words, $s \in \mathcal{S}_r(\mathrm{z}_n)$ if, on each interval $(\mathrm{z}_i, \mathrm{z}_{i+1})$, $0 \leq i \leq n-1$, s is in π_r, where π_r denotes the space of algebraic polynomials of degree $\leq r$.

Let G be a family of all functions, and $(G,\circ)$ be a group, such that

$$G := \{s \in \mathcal{S}_r(\mathrm{z}_n) : \exists\, \tau \in \mathcal{S}_r(\mathrm{z}_n) \ni s \circ \tau \textit{ is a continuous function}, \ \mathrm{z}_n = (\mathrm{z}_i)_{i=0}^{n}\}, \tag{3}$$

we also denote $(G,\circ) = (G_s,\circ)$. It is well known that $(G_s,\circ)$, is said to be spline group. We define the norm on $\mathcal{S}_r(\mathrm{z}_n)$, by formula $\|G_s\|_{\mathcal{S}_r(\mathrm{z}_n)} := sup_{s \in \mathcal{S}_r(\mathrm{z}_n)} |G_s(s)|$. Let $(G_s,\circ,\|\cdot\|)$ be a norm group with the norm function $\|\cdot\|: G_s \to [0,\infty)$. Given $\epsilon > 0$, does there exist a $\delta(\epsilon) > 0$, such that if a mapping

$$\Upsilon: G_s \to G_s,$$

satisfies the inequality

$$\|\Upsilon(s \circ \tau) - \Upsilon(s) \circ \Upsilon(\tau)\|_{\mathcal{S}_r(\mathrm{z}_n)} < \delta, \tag{4}$$

for all $s, \tau \in G_s$, then there is a homomorphism

$$\Upsilon: G_s \to G_s,$$

with

$$\|\Upsilon(G_s\,;\cdot) - G_s(\cdot)\|_{\mathcal{S}_r(\mathrm{z}_n)} < \epsilon, \text{ for all } s \in G_s. \tag{5}$$

A spline group G_s, endowed with the Lie product

$$G_s \times G_s \to G_s : m(s,\tau) \mapsto s \circ \tau\ ,$$
$$G_s \to G_s\ : i(\tau) \mapsto \tau^{-1} \tag{6}$$

on G_s, is called a spline Lie group (Cho, Saadati, & Vahidi, 2012; Carter, Segal, & Macdonald, 1995; Kirillov, 2008).

Hence, we introduce:

Let G_s be a spline Lie group. A mapping $\mathcal{H}: G_s \to G_s$ is called a spline Lie group homomorphism if $\mathcal{H}$ is satisfies (6), and group homomorphism for all $s, \tau \in G_s$.

Since $(G_s,\circ)$ is denote a spline group. Hence, in this paper, we define the operator by using (3) with formula:-

Let G_s be a spline Lie group. A family $\mho_\tau$, where $\tau(\cdot) > 0$, of probability measures on G_s is called a continuous semigroup of measures if

$$\mho_s * \mho_\tau = \mho_{s\circ\tau} \ , \tag{7}$$

where s, τ is positive spline,

$$\mho_\tau \Rightarrow I_e \quad \text{as} \quad \tau = 1, \tag{8}$$

where I_e is identity to spline group and a family of dilations $\{\gamma_\tau\}_{\tau>0}$ is said to be a homogeneous group.

A spline s in $G_s/\{I_e\}$ is called homogeneous of degree $\lambda \in \mathcal{R}$ if

$$s \circ \gamma_\tau = \tau^\lambda s \ \text{ for } \ \tau > 0. \tag{9}$$

A distribution σ on G_s is homogeneous of degree λ if

$$\langle \sigma, \left(\tau^{-\sum_{i=1}^{\eta} d_i}\right) \ s \circ \gamma_{\tau-1} \rangle = \tau^\lambda \langle \sigma, s \rangle, \tag{10}$$

for $s \in G_s$, $d_i \in \mathcal{R}$, $\eta \in \mathbb{N}$ and $\tau > 0$.

A linear differential operator Υ on G_s is homogeneous of degree λ if

$$\Upsilon(s \circ \gamma_\tau) = \tau^\lambda (\Upsilon s) \circ \gamma_\tau, \tag{11}$$

for any $s \in G_s$ and $\tau > 0$.

We will a generalization of positive linear operator Υ_j by using our Lemma as follow:

Lemma 2.1 (A-statistically homogeneous group) Let $A = (a_{jn})$ be nonnegative regular summability matrix. For all $s \in G_s/\{I_e\}$, satisfied (3), and Υ_j a sequence of positive linear operators, we have $\Upsilon_j : G_s/\{I_e\} \to G_s/\{I_e\}$, homogeneous group.

Proof. For $s \in G_s/\{I_e\}$, by using (4), (9) and (11), it is clear that,

$$st - \lim_j \sum_{n=1}^{\infty} a_{jn} \ \left\| \Upsilon_j(s \circ \gamma_\tau) - \Upsilon_j(s) \circ \Upsilon_j(\gamma_\tau) \right\|_{\mathcal{S}_r(z_n)}$$

$$= st - \lim_j \sum_{n=1}^{\infty} a_{jn} \ \left\| \tau^\lambda (\Upsilon_j s) \circ \gamma_\tau - \Upsilon_j(s) \circ \gamma_\tau \right\|_{\mathcal{S}_r(z_n)}$$

$$= st - \lim_j \sum_{n=1}^{\infty} a_{jn} \ \left\| \tau^\lambda (\Upsilon_j s) \circ \gamma_\tau - \frac{\tau^\lambda}{\tau^\lambda} (\Upsilon_j s) \circ \gamma_\tau \right\|_{\mathcal{S}_r(z_n)}$$

$$= st - \lim_j \sum_{n=1}^{\infty} a_{jn} \ \left\| \left(1 - \frac{1}{\tau^\lambda}\right) \tau^\lambda (\Upsilon_j s) \circ \gamma_\tau \right\|_{\mathcal{S}_r(z_n)}$$

if $\tau = 1$, then

$$= st - \lim_j \sum_{n=1}^{\infty} a_{jn} \ \left\| \left(1 - \frac{1}{\tau^\lambda}\right) \tau^\lambda (\Upsilon_j s) \circ \gamma_\tau \right\|_{\mathcal{S}_r(z_n)}$$

$$= 0 \tag{12}$$

hence from (1), we have

$$\delta_A(K) = 0.$$

Therefore the Lemma is proved, for $s \in G_s/\{I_e\}$. □

In other words, if a sequence Υ_j is a homomorphism from G_s into G_s, then $\Upsilon_j : G_s/\{I_e\} \to G_s/\{I_e\}$, is homogeneous group and A-statistically convergent.

3. Group and Korovkin Theorem

Now, we present the following results:

Theorem 3.1 Let $A = (a_{jn})$ be nonnegative regular summability matrix, and let Υ_j be a sequence of positive linear operators from $\mathcal{S}_r(z_n)$ into $\mathcal{S}_r(z_n)$. Then for all $s \in \mathcal{S}_r(z_n)$, we have

$$st - \lim_j \sum_{n=1}^{\infty} a_{jn} \left\|\Upsilon_j(G_s; .) - G_s\right\|_{\mathcal{S}_r(z_n)} = 0, \tag{13}$$

if and only if

$$\begin{aligned} &s \circ s_\circ = 1, &&\ni \ s_\circ \in G_{s\circ}\ , \\ &G_s(s \circ \tau) = \tau^\lambda s, &&\ni \ \tau \in G_{s1}\ , \\ &G_s(s \circ \upsilon) = s \circ \gamma_{\tau-1}\,, &&\ni \ \upsilon \in G_{s2}\ ; \end{aligned} \tag{14}$$

such that

$$st - \lim_j \sum_{n=1}^{\infty} a_{jn} \left\|\Upsilon_j(G_{si}; .) - G_{si}\right\|_{\mathcal{S}_r(z_n)} = 0 \quad ; \quad i = 0,1,2 \quad , \tag{15}$$

and G_{si} is a subgroup from G_s ; $i = 0,1,2$.

Proof. Since each of $1,\ \tau^\lambda s,\ s \circ \gamma_{\tau-1}$ belongs to $\mathcal{S}_r(z_n)$, condition (15), follow immediately from (13). Now, assume that (15) holds. For the splines $\tau,\ \upsilon$ given in (14), we get

$$|G_s(t) - G_s(s)| < \epsilon + \vartheta(\mho_\tau),$$

where

$$\begin{aligned} \vartheta(\mho_\tau) &= (\tau^\lambda t) - (t \circ \gamma_{\tau-1}) \\ &= G_s(t \circ \tau) - G_s(t \circ \upsilon). \end{aligned} \tag{16}$$

This is,

$$-\epsilon - \vartheta(\mho_\tau) < G_s(t) - G_s(s) < \epsilon + \vartheta(\mho_\tau).$$

Now, applying the operators $\Upsilon_j(s \circ s_\circ; s)$ for every $j \in \mathbb{N}$, to this inequality,

$$\begin{aligned} -\epsilon\, \Upsilon_j(s \circ s_\circ; s) - \vartheta(\mho_\tau)\, \Upsilon_j(s \circ s_\circ; s) &< \Upsilon_j(G_s(t); s) - G_s(s)\, \Upsilon_j(s \circ s_\circ; s) \\ &< \epsilon\, \Upsilon_j(s \circ s_\circ; s) + \vartheta(\mho_\tau)\, \Upsilon_j(s \circ s_\circ; s) \end{aligned} \tag{17}$$

also,

$$\begin{aligned} \Upsilon_j(G_s(t); s) - G_s(s) &= \Upsilon_j(G_s(t); s) - G_s(s)\Upsilon_j(s \circ s_\circ; s) + G_s(s)\Upsilon_j(s \circ s_\circ; s) - G_s(s) \\ &= \Upsilon_j(G_s(t); s) - G_s(s)\Upsilon_j(s \circ s_\circ; s) + G_s(s)\left[\Upsilon_j(1; s) - 1\right]. \end{aligned} \tag{18}$$

If follows from (17) and (18), that

$$\begin{aligned} \Upsilon_j(G_s(t); s) - G_s(s) &< \epsilon\, \Upsilon_j(s \circ s_\circ; s) + \vartheta(\mho_\tau)\, \Upsilon_j(s \circ s_\circ; s) + G_s(s)\left[\Upsilon_j(1; s) - 1\right] \\ &< \epsilon\, \Upsilon_j(s \circ s_\circ; s) + \Upsilon_j(\vartheta(\mho_\tau); s) + G_s(s)\left[\Upsilon_j(1; s) - 1\right]. \end{aligned} \tag{19}$$

Now, from (16)

$$\Upsilon_j(\vartheta(\mho_\tau); s) = M\left(\Upsilon_j(G_s(t \circ \tau) - G_s(t \circ \upsilon); s)\right)$$

$$= M\left(\Upsilon_j(G_s(t \circ \tau); s) - \Upsilon_j(G_s(t \circ \upsilon); s)\right),$$

where $M = \epsilon + 1$.

Using (19), we obtain

$$\Upsilon_j(G_s(t); s) - G_s(s) < \epsilon\, \Upsilon_j(s \circ s_\circ; s) + M\left(\Upsilon_j(G_s(t \circ \tau); s) - G_s(s) + G_s(s) - \Upsilon_j(G_s(t \circ \upsilon); s)\right) +$$

$$G_s(s)\left[\Upsilon_j(1; s) - 1\right]$$

$$< \epsilon\, \left(\Upsilon_j(s \circ s_\circ; s) - (s \circ s_\circ) + (s \circ s_\circ)\right) + M\left(\Upsilon_j(G_s(t \circ \tau); s) - G_s(s) + G_s(s) - \right.$$

$$\Upsilon_j(G_s(t \circ v); s)\Big) + G_s(s)\big[\Upsilon_j(1; s) - 1\big].$$

Therefore,

$$\left|\Upsilon_j(G_s(t); s) - G_s(s)\right| < \epsilon \left|\Upsilon_j(s \circ s_\circ; s) - (s \circ s_\circ)\right| + \epsilon + M\Big(\left|\Upsilon_j(G_s(t \circ \tau); s) - G_s(s)\right| +$$
$$\left|\Upsilon_j(G_s(t \circ v); s) - G_s(s)\right|\Big) + |G_s(s)| \left|\Upsilon_j(1; s) - 1\right|,$$

since $|G_s(s)| \le 1$ for all $s \in G_s$ and (14). Now, taking supremum over s, we get

$$\left\|\Upsilon_j(G_s(t); s) - G_s(s)\right\|_{\mathcal{S}_r(z_n)} \le \epsilon + \epsilon \left\|\Upsilon_j(1; s) - 1\right\|_{\mathcal{S}_r(z_n)} + M \left\|\Upsilon_j(G_s(t \circ \tau); s) - G_s(s)\right\|_{\mathcal{S}_r(z_n)} +$$

$$M\left\|\Upsilon_j(G_s(t \circ v); s) - G_s(s)\right\|_{\mathcal{S}_r(z_n)} + \left\|\Upsilon_j(1; s) - 1\right\|_{\mathcal{S}_r(z_n)}$$

$$\le \epsilon + (\epsilon + 1) \left\|\Upsilon_j(1; s) - 1\right\|_{\mathcal{S}_r(z_n)} + M \left\|\Upsilon_j(G_s(t \circ \tau); s) - G_s(s)\right\|_{\mathcal{S}_r(z_n)} +$$

$$M\left\|\Upsilon_j(G_s(t \circ v); s) - G_s(s)\right\|_{\mathcal{S}_r(z_n)}$$

$$< \epsilon + M\Big[\left\|\Upsilon_j(1; s) - 1\right\|_{\mathcal{S}_r(z_n)} + \left\|\Upsilon_j(G_s(t \circ \tau); s) - G_s(s)\right\|_{\mathcal{S}_r(z_n)} +$$

$$\left\|\Upsilon_j(G_s(t \circ v); s) - G_s(s)\right\|_{\mathcal{S}_r(z_n)} \Big]. \tag{20}$$

For a given $\xi > 0$, choose $\acute{\epsilon} > 0$ such that $\acute{\epsilon} < \xi$. Define the following sets,

$$D = \Big\{j\colon \left\|\Upsilon_j(G_s(t); s) - G_s(s)\right\|_{\mathcal{S}_r(z_n)} \ge \xi\Big\},$$

$$D_\circ = \Big\{j\colon \left\|\Upsilon_j(1; s) - 1\right\|_{\mathcal{S}_r(z_n)} \ge \tfrac{\xi - \acute{\epsilon}}{3}\Big\},$$

$$D_1 = \Big\{j\colon \left\|\Upsilon_j(G_s(t \circ \tau); s) - \tau^\lambda s\right\|_{\mathcal{S}_r(z_n)} \ge \tfrac{\xi - \acute{\epsilon}}{3}\Big\},$$

$$D_2 = \Big\{j\colon \left\|\Upsilon_j(G_s(t \circ v); s) - s \circ \gamma_{\tau - 1}\right\|_{\mathcal{S}_r(z_n)} \ge \tfrac{\xi - \acute{\epsilon}}{3}\Big\}.$$

Then $D \subset D_\circ \cup D_1 \cup D_2$ and $\delta_A(D) \le \delta_A(D_\circ) + \delta_A(D_1) + \delta_A(D_2)$. Therefore, using conditions (14) and (15), we have

$$st - \lim_j \textstyle\sum_{n=1}^{\infty} a_{jn} \left\|\Upsilon_j(G_s; .) - G_s\right\|_{\mathcal{S}_r(z_n)} = 0.$$

This completes the proof of the theorem. □

Theorem 3.2 If $(z_{ni})_{i=0}^{\infty}$ is defined by (2), then there exists a set $\mathfrak{J} = \{z_{n0} < z_{n1} < \cdots < z_{nm} < \cdots\} \subseteq \mathbb{N}$, such that $\delta_\theta(\mathfrak{J}) = 1$ and Υ_j homogeneous group if and only if a sequence $\Upsilon = (\Upsilon_j)$ is lacunary statistically convergence to L.

4. Auxiliary Results and Proof Theorem 3.2

Recently the concept of lacunary statistically convergence:

Let $K \subseteq \mathbb{N}$. Then

$$\delta_\theta(K) = \lim_{\acute{r}} \frac{1}{h_{\acute{r}}} \left|\{k_{\acute{r}-1} < i < k_{\acute{r}} : i \in K\}\right| \tag{21}$$

is said to be θ-density of K.

A sequence $x = (x_k)$ is said to be lacunary statistically convergent (Fridy, & Orhan, 1993, pp. 43-51) to L, if for every $\epsilon > 0$, the set $K_\epsilon \coloneqq \{k \in \mathbb{N}\colon |x_k - L| \ge \epsilon\}$ has θ-density zero, i.e. $\delta_\theta(K_\epsilon) = 0$. In this case we write

$S_\theta - \lim x = L$. That is,

$$\lim_{\acute{r}} |\{k_{\acute{r}-1} < i < k_{\acute{r}} : |x_k - L| \geq \epsilon\}| = 0. \tag{22}$$

In this case, we write $S_\theta - \lim_i x_i = L$, and we denote the set of all lacunary statistically convergent sequence by S_θ.

The following theorems were proved by (Mursaleen & Alotaibi, 2011, pp. 373-381):

Theorem 4.1 If a sequence $x = (x_k)$ is bounded and lacunary statistically convergent to L then it is statistically lacunary summable to L.

Theorem 4.2 (a) If $0 < q < \infty$ and a sequence $x = (x_k)$ is strongly θ_q-convergent to the limit L, then it is lacunary statistically convergent to L.

(b) If $x = (x_k)$ is bounded and lacunary statistically convergent to L, then $x_k \to L|C_\theta|_q$.

Theorem 4.3 A sequence $x = (x_k)$ is statistically lacunary summable to L if and only if there exists a set $K = \{r_1 < r_2 < \cdots < r_n < \cdots\} \subseteq \mathbb{N}$, such that $\delta(K) = 1$ and $\theta - \lim x_{r_n} = L$.

Theorem 4.4 A sequence $x = (x_k)$ is lacunary statistically convergent to L if and only if there exists a set $K = \{k_1 < k_2 < \cdots < k_n < \cdots\} \subseteq \mathbb{N}$, such that $\delta_\theta(K) = 1$ and $\lim x_{k_n} = L$.

Now, we prove our theorem:

Proof theorem 3.2 Assume that t_n is the Chebyshev partition of $[0,\infty]$. Suppose that $\mathfrak{J} \subseteq \mathbb{N}$, such that $\delta_\theta(\mathfrak{J}) = 1$ and Υ_j homogeneous group holds. Then, from Lemma 2.1 and (10) there is a positive integer η, we have

$$0 \leftarrow \frac{1}{h_{\acute{\eta}}} \sum_{\acute{\eta}=-\sum_{i=1}^{\eta} d_i} \left|\langle \sigma, \tau^{\acute{\eta}} s \circ \gamma_{\tau-1}\rangle\right|^2 > \frac{1}{h_{\acute{\eta}}} \sum_{\acute{\eta}=-\sum_{i=1}^{\eta} d_i} |\sigma - L|.\left|\left(\tau^{\acute{\eta}}\left(\Upsilon_j s\right) \circ \gamma_{\tau-1}\right) - L\right|$$

$$> \frac{\epsilon}{h_{\acute{\eta}}} \sum_{\acute{\eta}=-\sum_{i=1}^{\eta} d_i} \left|\left(\tau^{\acute{\eta}}\left(\Upsilon_j s\right) \circ \gamma_{\tau-1}\right) - L\right|. \tag{23}$$

Put $\mathfrak{J}_\epsilon := \{\eta \in \mathbb{N}: |\overline{\Upsilon}_j - L| \geq \epsilon\}$, where $\overline{\Upsilon}_j = \left(\tau^{\acute{\eta}}\left(\Upsilon_j s\right) \circ \gamma_{\tau-1}\right)$ and $\acute{\mathfrak{J}} = \{z_{\eta+1}, z_{\eta+2}, \ldots\}$. Then $\delta_\theta(\acute{\mathfrak{J}}) = 1$ and $\mathfrak{J}_\epsilon \subseteq \mathbb{N} - \acute{\mathfrak{J}}$, therefore $\lim_\eta \frac{1}{h_{\acute{\eta}}} |\mathfrak{J}_\epsilon| = 0$, from (21), which implies that $\delta_\theta(\mathfrak{J}_\epsilon) = 0$.

Hence $\Upsilon = (\Upsilon_j)$ is lacunary statistically convergence to L.

Conversely, let $\Upsilon = (\Upsilon_j)$ be lacunary statistically convergence to L. Since $(z_{ni})_{i=0}^{\infty}$ is defined by (2), from theorem 4.4, there exists a set $\mathfrak{J} = \{z_{n0} < z_{n1} < \cdots < z_{nm} < \cdots\} \subseteq \mathbb{N}$, such that $\delta_\theta(\mathfrak{J}) = 1$. Now, from (12), we have Υ_j is homogeneous group. This completes the proof of the theorem. □

Now, we can write more of corollary with using theorems 4.1-4, and theorem 3.2.

Corollary 4.5 A sequence $\Upsilon = (\Upsilon_j)$ is lacunary statistically convergent to L, then there exists a set $\mathfrak{J} = \{z_{n0} < z_{n1} < \cdots < z_{nm} < \cdots\} \subseteq \mathbb{N}$, such that $\delta_\theta(\mathfrak{J}) = 1$ and satisfies (1).

Proof. By using theorem 3.2 and Lemma 2.1, it is easily prove that. □

Corollary 4.6 If $0 < q < \infty$ and a sequence $\Upsilon = (\Upsilon_j)$ is strongly θ_q-convergent to the limit L, if and only if there exists a set $\mathfrak{J} = \{z_{n0} < z_{n1} < \cdots < z_{nm} < \cdots\} \subseteq \mathbb{N}$, such that $\delta_\theta(\mathfrak{J}) = 1$ and Υ_j homogeneous group.

Proof. From Theorem 4.2 (a) and theorem 3.2, it is easily prove that. □

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