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On Co-screen Conformality of 1-lightlike Submanifolds in a Lorentzian Manifold

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Abstract

In this paper, the co-screen conformal 1-lightlike submanifolds of a Lorentzian manifold are introduced as a generalization of co-screen locally half-lightlike submanifolds in (Wang, Wang & Liu, 2013; Wang & Liu, 2013) and two examples are given which one is co-screen locally conformal and the other is not. Some results are obtained on these submanifolds which the co-screen distribution is conformal Killing on the ambient manifold. The induced Ricci tensor of co-screen conformal 1-lightlike submanifolds is investigated.

Keywords: Lorentzian manifold, conformal killing, ricci tensor

Subject Clasification: 53C25, 53C50.

1. Introduction

Inspired by Einstein's theory of general relativity, the Kaluza-Klein's theory and the string theory, many physicists consider that the universe, we live in, can be a 4-dimensional submanifold embedded in high dimensional space-time manifold and many mathematicians study not only submanifolds of Riemannian manifolds but also study semi-Riemannian manifolds. One can consider that semi-Riemannian submanifolds are two types which one is non-degenerate submanifolds and the other is lightlike submanifolds. In (Duggal & Bejancu, 1996; Duggal & Jin, 2007; Duggal & Sahin, 2010), Duggal and his colleagues published books related with geometry of lightlike submanifolds and they presented general theory of lightlike submanifolds. Since then large numbers of papers have been published on lightlike submanifolds of semi-Riemannian manifolds.

Unfortunately, due to degenerate metric on lightlike submanifolds and the screen distribution is not canonical, induced nations of the submanifold (e.g sectional curvature, Ricci curvature, shape operator etc.) depend on choosing screen distribution that creates a problem. Therefore, it is necessary to find some classes of lightlike submanifold, whose geometry is essentially the same as that of their chosen screen distribution. Therefore, many mathematicians have been presented variety of methods to overcome this problem and have identified some special submanifolds. For example, the authors are used specific suitable methods for this problem in (Akivis & Goldberg, 1998; Akivis & Goldberg, 1999; Akivis & Goldberg, 2000; Bolós, 2005; Bonnor, 1992; Leistner, 2006). Furthermore, Kupeli (Kupeli, 1996) has shown that any screen distribution of a lightlike submanifold is isometric to the factor bundle on the tangent space the submanifold. In (Atindogbe & Duggal, 2004), Atindogbe and Duggal introduced screen locally conformal lightlike submanifold as a special lightlike submanifold of a semi-Riemannian manifold whose screen distribution is integrable and induced notions of the submanifold are independent of the screen distribution as follows:

A lightlike hypersurface $(M, g, S(TM))$ of a semi-Riemannian manifold is called *screen locally conformal* if there is the following relation between the shape operator A_N and the local shape operator A^*_ξ of the submanifold

$$A_N = \varphi A^*_\xi, \tag{1}$$

where φ is a non-vanishing smooth function on a neighborhood in M.

Screen conformal lightlike submanifolds are studied on lightlike hypersurfaces in (Atindogbe, Ezin & Tossa, 2006; Gülbahar, Kılıç & Keleş, 2013a; Gülbahar, Kılıç & Keleş, 2013b; Jin, 2009a; Jin, 2010; Jin, 2014) on half-lightlike submanifolds in (Duggal & Sahin, 2004; Jin, 2009c) on coisotropic lightlike submanifolds in (Duggal, 2007), on indefinite complex space forms in (Jin, 2009b; Jin, 2010b), on indefinite contact space form in (Massamba, 2008; Massamba, 2012), on warped product manifold in (Sahin, 2005). Furthermore, screen conformal submersions are studied in (Sahin, 2007).

Recently, another special lightlike submanifolds whose screen distribution is integrable and the induced notions of the submanifold are independent of the screen distribution is defined in (Wang, Wang & Liu, 2013; Wang & Liu, 2013) and given by

$$A_u = \varphi^c A^*_\xi, \tag{2}$$

where φ^c is a non-vanishing smooth function on a neighborhood in M, u is is a unit vector field of screen transversal bundle of the submanifold, A_u, A^*_ξ are the shape operator and the local shape operator on M, respectively.

2. Preliminaries

Let $\widetilde{M}$ be a semi-Riemannian manifold equipped with semi-Riemannian metric $\widetilde{g}$ of index $\widetilde{q}$. The manifold $(\widetilde{M}, \widetilde{g})$ is called a *Lorentzian manifold* if $\widetilde{q} = 1$.

Let (M, g) be an $(n+1)$-dimensional lightlike submanifold of an $(n+m+2)$-dimensional Lorentzian manifold. The radical space Rad T_pM on the tangent space at a point $p \in M$ is one-dimensional subspace, defined by

$$\text{Rad } T_pM = \{\xi \in T_pM : \ g_p(\xi, X) = 0, \ \forall X \in T_pM\}. \tag{3}$$

The submanifold (M, g) is called a *lightlike hypersurface* if m is equal to zero. The complementary vector bundle $S(TM)$ of Rad TM in TM is called the *screen bundle* of M (Duggal & Bejancu, 2007). For these submanifolds, any screen bundle is non-degenerate and

$$TM = \text{Rad } TM \oplus_{orth} S(TM), \tag{4}$$

where $\oplus_{orth}$ denotes the orthogonal direct sum. Consider a complementary vector bundle $S(TM^\perp)$ of Rad TM in $TM^\perp$. Then, we have the following orthogonal direct sum

$$TM^\perp = \text{Rad } TM \oplus_{orth} S(TM^\perp). \tag{5}$$

Here, $S(TM^\perp)$ is the non-degenerate distribution with respect to $\widetilde{g}$. It is said to be *co-screen distribution* on M (Jin, 2009c; Jin, 2010b).

Let tr(TM) and ltr(TM) be complementary but not orthogonal vector bundles to TM in $T\widetilde{M}|_M$ and Rad TM in tr(TM), respectively. In this situation, we have

$$T\widetilde{M}|_M = (\text{Rad } TM \oplus \text{ltr } (TM)) \oplus_{orth} S(TM) \oplus_{orth} S(TM^\perp). \tag{6}$$

Let (M, g) be an $(n+1)$-dimensional lightlike submanifold of an $(n+m+2)$ dimensional $(\widetilde{M}, \widetilde{g})$ and $\mathcal{U}$ be a local coordinate neighborhood of M. Then, there exists a quasi-orthonormal frame of $\widetilde{M}$ along M, on $\mathcal{U}$:

$$\{\xi, e_1, e_2, \ldots, e_n, N, u_1, u_2, \ldots, u_m\}, \tag{7}$$

where $\Gamma(S(TM)) = \text{Span}\{e_1, e_2, \ldots, e_n\}$, $\Gamma(\text{Rad } TM|_{\mathcal{U}}) = \text{Span}\{\xi\}$, $\Gamma(\text{ltr}(TM)|_{\mathcal{U}}) = \text{Span}\{N\}$, $\Gamma(\text{tr}(M)|_{\mathcal{U}}) = \text{Span}\{u_1, u_2, \ldots, u_m\}$. We note that the quasi-orthonormal basis of $\widetilde{M}$ satisfies that

$$\widetilde{g}(\xi, N) = 1, \ \widetilde{g}(N, N) = \widetilde{g}(N, u) = \widetilde{g}(\xi, u) = 0, \tag{8}$$

for all $u \in \Gamma(\text{tr}(TM)|_{\mathcal{U}})$.

Let $\widetilde{\nabla}$ be the Levi-Civita connection of $\widetilde{M}$ and P be the projection morphism of $\Gamma(TM)$ to $\Gamma(S(TM))$. The Gauss and Weingarten formulas are given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + B(X,Y)N + \sum_{\alpha=1}^{m} D^{\alpha}(X,Y)u_{\alpha}, \tag{9}$$

$$\widetilde{\nabla}_X N = -A_N X + \rho N + \sum_{\alpha=1}^{m} \rho_{\alpha} u_{\alpha}, \tag{10}$$

$$\widetilde{\nabla}_X u_{\beta} = -A_{u_{\beta}} X + \varepsilon N + \sum_{\alpha=1}^{m} \varepsilon_{\alpha} u_{\alpha}, \tag{11}$$

$$\nabla_X PY = \nabla^*_X PY + C(X,PY)\xi, \tag{12}$$

$$\nabla_X \xi = -A_{\xi} X - \rho\xi, \tag{13}$$

for any $X, Y \in \Gamma(TM)$, where ∇ and ∇^* are the induced linear connections on TM and $S(TM)$, respectively; B and D^{α} are coefficients of the lightlike second fundamental form and coefficients of the screen second fundamental form of TM, respectively, C is coefficients of the local second fundamental form on $S(TM)$, A_N , $A_{u_{\beta}}$ are the shape operators on M, A^*_{ξ} is the shape operator on $S(TM)$ and ε, ε_{α}, ρ, ρ_{α} are 1-forms on M.

Let us define a local 1-form η by

$$\eta(X) = \widetilde{g}(X,N), \ \forall X \in \Gamma(TM|_{\mathcal{U}}). \tag{14}$$

Consider (9), (14) and $\widetilde{\nabla}$ is a metric connection on $\widetilde{M}$ it is known that the induced connection ∇ is not a metric connection (Duggal & Bejancu, 2007).

The second fundamental form h and the local second fundamental form h^* are given by, respectively,

$$h(X,Y) = h^{\ell}(X,Y) + h^{s}(X,Y) \text{ and } h^*(X,PY) = C(X,PY)\xi, \tag{15}$$

where

$$h^{\ell}(X,Y) = B(X,Y)N \text{ and } h^{s}(X,Y) = \sum_{\alpha=1}^{m} D^{\alpha}(X,Y)u_{\alpha},$$

for any $X, Y \in TM$. Here, we note that h^{ℓ} is the light part of the second fundamental form and h^s is the non-degenerate part of the second fundamental form.

It is known that $B = 0$ on Rad TM and it is independent of the screen distribution $S(TM)$ for any $\alpha \in \{1, \dots m\}$ and the following relations satisfy that

$$B(X,PY) = g(A^*_{\xi}X,Y), \ \ \widetilde{g}(A^*_{\xi}X,N) = 0, \tag{16}$$

$$D^{\alpha}(X,PY) = g(A_{u_{\alpha}}X,PY), \ \ \widetilde{g}(\nabla_X N, u_{\alpha}) = \rho_{\alpha}, \tag{17}$$

$$C(X,PY) = g(A_N X,PY), \tag{18}$$

for any $X, Y \in TM$ and $\alpha \in \{1, \dots, m\}$.

The submanifold $(M, g, S(TM))$ is called *totally umbilical* if there exist smooth functions $\lambda \in F(\mathrm{tr}(TM))$ and $\lambda_{\alpha} \in F(S(TM)^{\perp})$ for any $\alpha \in \{1, \dots, m\}$ such that

$$B(X,Y) = \lambda g(X,Y) \text{ and } D^{\alpha}(X,Y) = \lambda_{\alpha} g(X,Y), \tag{19}$$

for all $X, Y \in \Gamma(TM)$.

A 1-lightlike submanifold is said to be *irrotational* if $\widetilde{\nabla}_X \xi \in \Gamma(TM)$ for any $X \in \Gamma(TM)$, where $\xi \in \Gamma(\mathrm{Rad}\,TM)$ (Kupeli, 1996).

Let $\{e_1, \dots, e_n\}$ be an orthonormal basis of $\Gamma(S(TM))$ and define

$$\mu^* = \frac{1}{n}\sum_{i=1}^{n} C(e_i,e_i), \ \ \mu_1 = \frac{1}{n}\sum_{i=1}^{n} B(e_i,e_i), \ \ \mu_2 = \frac{1}{n}\sum_{i=1}^{n}\sum_{\alpha=1}^{m} D^{\alpha}(e_i,e_i). \tag{20}$$

The manifold $(M, g, S(TM))$ is called *minimal* if $\mu_1 = \mu_2 = 0$ and $D^\alpha = 0$ on Rad TM (Bejan & Duggal, 2005; Duggal & Sahin, 2010).

The mean curvature vector on TM and on $\Gamma(S(TM))$ are given by

$$H(p) = \frac{1}{n} trace|_{S(TM)} h = \mu_1 N + \sum_{\alpha=1}^{m} \mu_2 u_\alpha, \tag{21}$$

$$H^*(p) = \mu^* \xi + \mu_1 N + \sum_{\alpha=1}^{m} \mu_2 u_\alpha, \tag{22}$$

respectively.

Let $\Pi = sp\{e_i, e_j\}$ be 2-dimensional non-degenerate plane of the tangent space T_pM at $p \in M$. Then, the number

$$K_{ij} = \frac{g(R_p(e_j, e_i)e_i, e_j)}{g_p(e_i, e_i)g_p(e_j, e_j) - g_p(e_i, e_j)^2} \tag{23}$$

is called *the sectional curvature* of the section Π at $p \in M$. Since the operator C isn't symmetric the sectional curvature function doesn't need to be symmetric on any lightlike submanifold of a semi-Riemannian manifold (Duggal & Sahin, 2010).

Let ξ be a null vector of T_pM. A plane Π of T_pM is called a *null plane* if it contains ξ and e_i such that $\widetilde{g}(\xi, e_i) = 0$ and $\widetilde{g}(e_i, e_i) \neq 0$. *The null sectional curvature* of Π be given in (Beem, Ehrlich, & Easley, 1996) as follows:

$$K_\xi(\Pi) = \frac{g(R_p(e_i, \xi)\xi, e_i)}{g_p(e_i, e_i)}. \tag{24}$$

We note that the null sectional curvature measures differences in length of two spacelike geodesic constructed from the degenerate plane section Π and it is independent of the choice of the spacelike vector e_i but it depends quadratically on the null vector ξ (Albujer & Haesen, 2010).

Let $(M, g, S(TM))$ be an $(n+1)$ dimensional 1-lightlike submanifold of an $\widetilde{m}$-dimensional Lorentzian manifold $(\widetilde{M}, \widetilde{g})$ and $\{e_1, \ldots\ldots, e_n\}$ be an orthonormal basis of $\Gamma(S(TM))$. The induced Ricci type tensor $R^{(0,2)}$ of M is defined by

$$R^{(0,2)}(X, Y) = \sum_{i=1}^{n} g(R(e_i, X)Y, e_i) + \widetilde{g}(R(\xi, X)Y, N). \tag{25}$$

It is known that $R^{(0,2)}$ is not symmetric and has no geometric meaning. The tensor $R^{(0,2)}$ is called the *Ricci curvature* if it is symmetric (Duggal & Sahin, 2010).

3. Co-screen Conformal 1-lightlike Submanifolds

We begin this section with the canonical theorems for 1-lightlike submanifolds of a Lorentzian manifold.

Let (M, g) be an $(n+1)$-dimensional 1-lightlike submanifold of an $(m+n+2)$-dimensional Lorentzian manifold $(\widetilde{M}, \widetilde{g})$. Let us consider two quasi-orthonormal frame $\{\xi, N, e_i, u_\alpha\}$ and $\{\xi, N', e'_i, u'_\alpha\}$ induced on $\mathcal{U}$. In this case, the followings can be written

$$e'_a = \sum_{b=1}^{n} e_a^b (e_b - f_b \xi), \quad u'_\alpha = \sum_{\beta=1}^{m} u_\alpha^\beta (u_\beta - Q_\beta \xi) \tag{26}$$

and

$$N' = N + N_1 \xi + \sum_{a=1}^{n} f_a e_a + \sum_{\alpha=1}^{m} Q_\alpha u_\alpha, \tag{27}$$

where e_a^b, u_α^β, N_1, f_a and Q_α are differentiable functions on $\mathcal{U}$ for $a \in \{1, \ldots, n\}$ and $\alpha \in \{1, \ldots, m\}$. Since $\widetilde{g}(N, N) = 0$, we have

$$N_1 = -\frac{1}{2} \sum_{a=1}^{n} (f_a)^2 - \frac{1}{2} \sum_{\alpha=1}^{m} (Q_\alpha)^2. \tag{28}$$

Let h^ℓ and h'^ℓ are light parts of the second fundamental forms and B and B' are coefficients of the light parts of the second fundamental forms on screen distributions $S(TM)$ and $S(TM)'$, respectively. Taking $\widetilde{\xi} = \theta\,\xi$ and thus $\widetilde{N} = \frac{1}{\theta}N$, where θ is some function, we obtain

$$h'^\ell(X,Y) = \widetilde{g}(\nabla_X Y, \widetilde{\xi})\widetilde{N} = \widetilde{g}(\nabla_X Y, \xi)N = h^\ell(X,Y), \tag{29}$$

which implies that h^ℓ is independent of the screen distribution $S(TM)$.

Let h^s and h'^s are non-degenerate parts of the second fundamental forms and D and D' are coefficients of the non-degenerate parts of the second fundamental forms on screen distributions $S(TM)$ and $S(TM)'$, respectively. Using (26), we have

$$\begin{aligned} D'^\alpha(X,Y) &= \widetilde{g}(\widetilde{\nabla}_X Y, u_\alpha) \\ &= \sum_{\beta=1}^{m} u_\alpha^\beta (D^\beta(X,Y) - Q_\beta B(X,Y)), \end{aligned} \tag{30}$$

which implies that non-degenerate part of the second fundamental form h^s depends on the screen distribution.

Using similar method, one can easily get

$$C'(X,PY) = C(X,PY) - \frac{1}{2}\|e\|^2 B(X,Y) + g(\nabla_X PY, e) \tag{31}$$

where $e = \sum_{a=1}^{n} f_a e_a$ is called the *characteristic vector field*.

Let us consider the first derivative of a screen distribution $S(TM)$ given by

$$S(p) = \mathrm{Span}\{[X,Y]_p : \; X_p, Y_p \in S(TM), \; p \in M\}, \tag{32}$$

where [,] denotes the Lie-bracket. Then, we have the following:

Theorem 1 *Let $(M, g, S(TM))$ be an $(n+1)$-dimensional 1-lightlike submanifold of a Lorentzian manifold. If the first derivative S defined by* (32) *coincides with $S(TM)$, then $S(TM)$ is a canonical screen of M, up to an orthogonal transformation with a canonical lightlike transversal vector bundle and the screen second fundamental form h^* is independent of a screen distribution.*

The proof is same as that of Theorem 2.1 in (Duggal, 2007), so we omit it here.

Now, we recall a class of 1-lightlike submanifolds of a Lorentzian manifold which admits an integrable canonical screen distribution as follows.

Definition 2 (Atindogbe & Duggal, 2004) *A* 1*-lightlike submanifold $(M, g, S(TM))$ of a Lorentzian $(\widetilde{M}, \widetilde{g})$ is called a screen locally conformal if*

$$B(X,Y) = \varphi C(X,Y), \quad \forall X, Y \in \Gamma(TM|_{\mathcal{U}}), \tag{33}$$

where $\mathcal{U}$ is a local coordinate neighborhood of M and φ is a smooth function on a neighborhood $\mathcal{U}$ in M. If φ is non-zero constant then the submanifold is called screen homothetic.

Now, we state the following theorem:

Theorem 3 *Let $(M, g, S(TM))$ be an $(n+1)$-dimensional 1-lightlike submanifold of an $(n+m+2)$-dimensional Lorentzian manifold $(\widetilde{M}, \widetilde{g})$. Suppose that $S(TM)$ is integrable and $S(TM)$ is totally umbilical immersed in $\widetilde{M}$ and it is parallel along integral curves of the radical distribution. Then, M is screen locally conformal if and only if $\mu^*\mu_1 \neq 0$.*

Proof. Let us denote M' as a leaf of $S(TM)$. Then, we have

$$\nabla_X Y = \nabla^*_X Y + C(X,Y)\xi + B(X,Y)N + \sum_{\alpha=1}^{m} D^\alpha(X,Y)u_\alpha, \tag{34}$$

for all $X, Y \in TM'$. The mean curvature vector of M' is a vector field of the rank $(m+2)$. From (22) and since M' is totally umbilical, it is clear that

$$C(X,Y)\xi + B(X,Y)N + \sum_{\alpha=1}^{m} D^\alpha(X,Y)u_\alpha = g(X,Y)(\mu^*\xi + \mu_1 N + \sum_{\alpha=1}^{m} \mu_2 u_\alpha). \tag{35}$$

In other words,

$$B(X,Y) = \mu_1 g(X,Y), \quad C(X,Y) = \mu^* g(X,Y) \text{ and } D^\alpha(X,Y) = \mu_2 g(X,Y), \tag{36}$$

for any $\alpha \in \{1,\dots,m\}$. Also, it is known that the mean curvature vector H^* on $\Gamma(S(TM))$ satisfies that

$$\widetilde{g}(H^*, H^*) = 2\mu_1\mu^* + \mu_2^2 \geq 0. \tag{37}$$

If $\mu_1\mu^* \neq 0$ then $A_N X = \frac{\mu_1}{\mu^*} A^*_\xi X$ for all $X \in TM'$. Since $S(TM)$ is parallel along integral curves of the radical distribution, we have $A_N\xi = 0$ so that

$$A_N X = \frac{\mu_1}{\mu^*} A^*_\xi X, \tag{38}$$

for all $X \in TM$ which implies that M is screen locally conformal. The proof of the converse part is straightforward.

Using same proof way of Theorem 2.3 in (Duggal & Bejancu, 1996), we immediately have the following theorem:

Theorem 4 *Let $(M, g, S(TM))$ be a 1-lightlike submanifold of a Lorentzian manifold. Then, the following assertions are equivalent:*

1) $S(TM)$ is integrable ;

2) $h^*(X,Y) = h^*(Y,X)$ for all $X, Y \in \Gamma(S(TM))$;

3) the shape operator A_N on M is symmetric.

From the above theorem, it is clear that screen conformal 1-lightlike submanifolds have the important features that their screen distributions are always integrable and the sectional curvature function is always symmetric and it has significant geometric meanings as in Riemannian manifolds.

We give now the following definition that shows that there is another type lightlike submanifold which its screen distributions are always integrable and the sectional curvature function defined on it is always symmetric.

Definition 5 *Let M be an $(n+1)$-dimensional lightlike submanifold of a Lorentzian manifold. The submanifold M is called co-screen locally conformal on a coordinate neighborhood $\mathcal{U}$ if there exists a non-zero smooth function φ^c such that*

$$A_N X = \varphi^c A_{u_\alpha} X, \quad \forall X \in \Gamma(TM), \tag{39}$$

for any null transversal vector field $N \in \Gamma(ltr(TM))$ and $\alpha \in \{1,\dots m\}$.

We now state the following theorem to characterize the co-screen conformal 1-lightlike submanifolds.

Theorem 6 *Let $(M, g, S(TM))$ be a 1-lightlike submanifold of a Lorentzian manifold, then M is co-screen conformal if and only if*

$$C(X, PY) = \varphi^c D^\alpha(X, PY), \textit{ for all } X, Y \in \Gamma(TM), \tag{40}$$

where φ^c is a non-zero smooth function on M.

Proof. Let $(M, g, S(TM))$ be a 1-lightlike submanifold of a Lorentzian manifold. Then, it follows (17) and (18), we have

$$C(X, PY) = g(A_N X, PY) = \varphi^c g(A_{u_\alpha} X, PY) = \varphi^c D^\alpha(X, PY), \tag{41}$$

for any $X, Y \in \Gamma(TM)$.

Conversely, since the shape operator A_N is $S(TM)$-valued, $\rho_\alpha = 0$ for all $\alpha \in \{1,\dots m\}$. Thus, we obtain $C(X, PY) = \varphi^c D^\alpha(X, PY)$ which implies that M is co-screen locally conformal.

Theorem 7 *The conditions given in Theorem 4 are always satisfied for any co-screen locally conformal 1-lightlike submanifold of a Lorentzian manifold.*

Example 8 Consider in R^7_1 with signature $(-,+,+,+,+,+,+)$ a submanifold M given by the equations

$$x_4 = (x_1^2 - x_2^2)^{\frac{1}{2}}, \quad x_3 = (1 - x_5^2)^{\frac{1}{2}}, \quad x_6 = (1 - x_7^2)^{\frac{1}{2}}, \quad x_2, x_5, x_7 > 0. \tag{42}$$

Then, we have

$$\begin{aligned} TM &= \text{Span}\{\xi = x_1\partial x_1 + x_2\partial x_2 + x_4\partial x_4, \\ &\quad U_1 = x_4\partial x_1 + x_1\partial x_4, U_2 = -x_5\partial x_3 + x_3\partial x_5, \\ &\quad U_3 = x_6\partial x_7 + x_7\partial x_6\}, \end{aligned}$$

and

$$TM^{\perp} = \text{Span}\{\xi, u_1 = x_3\partial x_3 + x_5\partial x_5,\ u_2 = -x_6\partial x_6 + x_7\partial x_7\}.$$

Thus, Rad $TM = \text{Span}\{\xi\}$ is a distribution on M and $S(TM^{\perp}) = \text{Span}\{u_1, u_2\}$. Hence, M is a 1-lightlike submanifold of R^7_1 with $S(TM) = \text{Span}\{U_1, U_2, U_3\}$. Also, the lightlike transversal bundle ltr(TM) is spanned by

$$N = \frac{1}{2x_1}\{-x_1\partial x_1 + x_2\partial x_2 + x_4\partial x_4\}.$$

By direct calculations, we get the manifold $(M, g, S(TM))$ isn't co-screen conformal since φ^c can't be vanishing function.

Example 9 Let $\widetilde{M} = R^6_1$ be be a semi-Euclidean space of signature $(-, +, +, +, +, +, +, +)$ with respect to the canonical basis

$$\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial x_6\}.$$

Let M be a submanifold of R^8_1 given by

$$\begin{aligned} &x_1 = \sinh u^1,\ \ x_2 = \cosh u^1,\ \ x_3 = u^1,\ \ x_4 = u^2, \\ &x_5 = \frac{1}{\sqrt{2}}u^3,\ \ x_6 = -\frac{1}{\sqrt{2}}u^3, \end{aligned} \tag{43}$$

where all of u^1, u^2, u^3, u^4 are non-vanishing coordinate functions. Then, we have

$$\begin{aligned} &\text{Rad}\, TM = \text{Span}\{\xi = \cosh u^1\partial x_1 + \sinh u^1\partial x_2 + \partial x_3\}, \\ &S(TM) = \text{Span}\{e_1 = \partial x_4,\ \ e_2 = \frac{1}{\sqrt{2}}\partial x_5 - \frac{1}{\sqrt{2}}\partial x_6\}, \\ &\text{ltr}(TM) = \text{Span}\{N = -\frac{1}{2}\cosh u^1\partial x_1 - \frac{1}{2}\sinh u^1\partial x_2 + \frac{1}{2}\partial x_3\}, \\ &\text{tr}(TM) = \text{Span}\{u_1 = \frac{1}{\sqrt{2}}\partial x_5 + \frac{1}{\sqrt{2}}\partial x_6,\ \ \ u_2 = \sinh u^1\partial x_1 + \cosh u^1\partial x_2\}. \end{aligned} \tag{44}$$

By straightforward computations, it can be obtained that the submanifold is co-screen conformal with φ^c is arbitrary.

Now, we give the following:

Theorem 10 *Let $(M, g, S(TM))$ be a totally geodesic, totally umbilical or minimal screen locally conformal 1-lightlike submanifold of a Lorentzian manifold $(\widetilde{M}, \widetilde{g})$. Any leaf M' of $S(TM)$ immersed in $\widetilde{M}$ as a $(m+2)$-dimensional non-degenerate submanifold if and only if the following assertions must be occurred:*

a) M is irrotational.

b) M is co-screen locally conformal.

Proof. Let M is a co-screen locally conformal irrotational 1-lightlike submanifold. Suppose that X, Y be tangent vector fields of the leaf M' of a screen distribution and h' is its second fundamental form in $\widetilde{M}$. If we put (16), (17), (18) and (39) in (34) we get

$$\widetilde{\nabla}_X Y = \nabla^*_X Y + g(A^*_{\xi}X, Y)(\varphi\xi + N) + \sum_{\alpha=1}^{n}\widetilde{g}(A_{u_\alpha}X, Y)u_\alpha, \tag{45}$$

which implies that

$$h'(X, Y) = B(X, Y)(\varphi\xi + N + u_\alpha), \text{ for any } \alpha = \{1, \ldots, m\}, \tag{46}$$

and so

$$h'(X, Y) = \sqrt{\varphi}B(X, Y)(\sqrt{\varphi}\xi + \frac{1}{\sqrt{\varphi}}N + u_\alpha) \tag{47}$$

where $(\sqrt{\varphi}\xi + \frac{1}{\sqrt{\varphi}}N + u_\alpha)$ is a unit spacelike vector field on M'. Since M is irrotational $D^\alpha(X, \xi) = 0$ for all $\alpha \in \{1, \ldots, m\}$ and $B(X, \xi) = 0$ for all $X \in \Gamma(TM')$. Therefore, the leaf M' of $S(TM)$ immersed in $\widetilde{M}$ as a $(m+2)$-dimensional non-degenerate submanifold. The proof of the converse part is straightforward.

Corollary 11 *If $(M, g, S(TM))$ is a lightlike hypersurface of a Lorentzian manifold $(\widetilde{M}, \widetilde{g})$. Then, any leaf M' of $S(TM)$ immersed in $\widetilde{M}$ as a 2-dimensional non-degenerate submanifold.*

Remark 12 *The above corollary is also valid for lightlike hypersurfaces of a semi-Riemannian manifold that proved in (Atindogbe, Ezin & Tossa, 2006).*

Theorem 13 *Let $(M, g, S(TM))$ be a co-screen conformal 1-lightlike submanifold of a Lorentzian manifold. The co-screen distribution $S(TM^\perp)$ is a conformal Killing on $\widetilde{M}$ if and only if $S(TM)$ is totally umbilical.*

Proof. Let $\widetilde{L}$ denote the Lie derivative on $\widetilde{M}$. If $S(TM^\perp)$ is conformal Killing on $\widetilde{M}$, then

$$(\widetilde{L}_{u_\alpha}\widetilde{g})(X, Y) = \widetilde{g}(\widetilde{\nabla}_X u_\alpha, Y) + \widetilde{g}(X, \widetilde{\nabla}_Y u_\alpha) = \delta\, g(X, Y), \tag{48}$$

for any $u_\alpha \in \Gamma(S(TM^\perp))$ and $X, Y \in \Gamma(TM)$. Here, δ is a smooth function. Putting (17) in (48), we get

$$(\widetilde{L}_{u_\alpha}\widetilde{g})(X, Y) = -2D^\alpha(X, Y) = \delta g(X, Y), \tag{49}$$

for any $X, Y \in \Gamma(TM)$. Since M is co-screen conformal, we obtain

$$C(X, Y) = -\frac{\delta}{2\varphi}g(X, Y), \tag{50}$$

which shows that $S(TM)$ is totally umbilical.

Now, we assume that $S(TM)$ is totally umbilical. Then,

$$C(X, Y) = \lambda' g(X, Y), \quad \forall X, Y \in \Gamma(S(TM)), \tag{51}$$

where λ' is a smooth function. Putting $\lambda' = -\frac{\delta}{2\varphi}$, it is clear that $S(TM^\perp)$ is a conformal Killing on $\widetilde{M}$.

Corollary 14 *Let $(M, g, S(TM))$ be co-screen conformal 1-lightlike submanifold of a Lorentzian manifold. The co-screen distribution $S(TM^\perp)$ is a Killing distribution on $\widetilde{M}$ if and only if $S(TM)$ is totally geodesic and M is minimal.*

Using similar proof method of Theorem 3.10 in (Wang & Liu, 2013) we have also the following:

Theorem 15 *Let $(M, g, S(TM))$ be a co-screen conformal 1-lightlike submanifold of a Lorentzian manifold. Then,*

1) Any leaf of (STM) is totally geodesic on M.

2) The submanifold M is a lightlike product manifold of M' and F where M' is a leaf of $S(TM)$ and F is a null curve of M.

3) $D^\alpha = 0$ on $S(TM)$ for all $\alpha \in \{1, \ldots, m\}$.

4. Ricci Curvature on Co-screen Conformal 1-lightlike Submanifolds

In this section, we study on the sectional curvature, the null sectional curvature and the induced Ricci curvature on co-screen conformal 1-lightlike submanifolds of a Lorentzian manifold. We begin this section herewith the following lemma.

Lemma 16 *Let $(M, g, S(TM))$ be a co-screen conformal 1-lightlike submanifold of a Lorentzian manifold. Let us denote the Riemannian curvature tensors R and $\widetilde{R}$ of the submanifold M and the ambient manifold $\widetilde{M}$, respectively.*

Then, we have the following relations:

$$\widetilde{g}(\widetilde{R}(X,Y)PZ,PW) = g(R(X,Y)PZ,PW) + B(X,PZ)C(Y,PW) - B(Y,PZ)C(X,PW) + \frac{1}{(\varphi^c)^2}[C(X,PZ)C(Y,PW) - C(Y,PZ)C(X,PW)], \quad (52)$$

$$\widetilde{g}(\widetilde{R}(X,Y)Z,N) = \widetilde{g}(R(X,Y)Z,N) + \frac{1}{\varphi^c}[C(X,Z)\rho_\alpha(Y) - C(Y,Z)\rho_\alpha(X)] + B(X,Z)\eta(A_N Y) - B(Y,Z)\eta(A_N X), \quad (53)$$

$$\widetilde{g}(\widetilde{R}(X,Y)\xi,N) = \widetilde{g}(R(X,Y)\xi,N) + \sum_{\alpha=1}^{m}\rho_\alpha(X)\varepsilon_\alpha(Y) - \rho_\alpha(Y)\varepsilon_\alpha(X), \quad (54)$$

$$\widetilde{g}(\widetilde{R}(X,Y)Z,\xi) = (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) + B(Y,Z)\rho(X) - B(X,Z)\rho(Y) + \frac{1}{\varphi^c}[\sum_{\alpha=1}^{m} C(Y,Z)\varepsilon_\alpha(X) - C(X,Z)\varepsilon_\alpha(Y)], \quad (55)$$

where $X, Y, Z, W \in \Gamma(TM)$, $N \in \Gamma(ltr(TM))$ *and* $\xi \in Rad\,TM$.

Let $(M, g, S(TM))$ be $(n+1)$-dimensional co-screen conformal 1-lightlike submanifold, $\{e_1, \ldots, e_n, \xi, N, u_1, \ldots, u_m\}$ be a quasi-orthonormal basis on $(\widetilde{M}, \widetilde{g})$. From (52) and (53), we have

$$\widetilde{g}(\widetilde{R}(e_i,X)Y,e_i) = g(R(e_i,X)Y,e_i) + B(e_i,Y)C(X,e_i) - B(X,Y)C(e_i,e_i) + \frac{1}{(\varphi^c)^2}[C(e_i,Y)C(X,e_i) - C(X,Y)C(e_i,e_i)] \quad (56)$$

and

$$\widetilde{g}(\widetilde{R}(e_i,\xi)Y,N) = \widetilde{g}(R(X,\xi)Y,N). \quad (57)$$

Theorem 17 *Let* $(M, g, S(TM))$ *be an* $(n+1)$*-dimensional co-screen conformal 1-lightlike submanifold of a Loretzian manifold* $(\widetilde{M}, \widetilde{g})$*. Then, the induced Ricci curvature tensor of* M *is symmetric.*

Proof. From (20), (56) and (57) we get

$$\begin{aligned} \mathrm{Ric}(X,Y) &= \sum_{i=1}^{n} g(R(e_i,X)Y,e_i) + \widetilde{g}(R(X,\xi)Y,N) \\ &= \widetilde{\mathrm{Ric}}_{S(TM)}(X,Y) + \widetilde{g}(\widetilde{R}(X,\xi)Y,N) + \mu^*[B(X,Y) + \frac{1}{(\varphi^c)^2}C(X,Y)] \\ &\quad - \sum_{i=1}^{n}[B(Y,e_i)C(X,e_i) + \frac{1}{(\varphi^c)^2}C(Y,e_i)C(X,e_i)], \end{aligned} \quad (58)$$

where $\widetilde{\mathrm{Ric}}_{S(TM)}(X,Y)$ is n-plane section Ricci curvature of $T_p\widetilde{M}$ for any $X, Y \in \Gamma(TM)$. Since the operator C is symmetric on these submanifolds, it is clear that the Ricci tensor is symmetric.

Using (56) we have the following theorem:

Theorem 18 *If* $(M, g, S(TM))$ *is co-screen conformal 1-lightlike submanifold of a Lorentzian space form* $\widetilde{M}(c)$ *with constant curvature c. Then, the null sectional curvature of* M *is given by*

$$K_\xi(\Pi) = D^\alpha(\xi,\xi)D^\alpha(e_i,e_i) - [D^\alpha(\xi,e_i)]^2, \;\; for \;\; any \;\; \alpha \in \{1, \ldots, m\}, \quad (59)$$

where $\Pi = Span\{\xi, e_i\}$ *is a degenerate plane section of* T_pM.

We note that if the submanifold $(M, g, S(TM))$ is irrotational then the null sectional curvature given by (59) vanishes identically.

Definition 19 *We define the screen Ricci curvature at a unit vector* $e_i \in \Gamma(S(TM))$ *as*

$$Ric_{S(TM)}(e_i) = \sum_{j=1}^{n} K_{ij}, \;\; 1 \le i \ne j \le n, \quad (60)$$

Since K_{ij} are symmetric for all $i, j \in \{1, \dots, n\}$ in co-screen conformal 1-lightlike submanifolds of a Lorentzian manifold, it is clear that the screen Ricci curvature is well defined.

We note that the screen Ricci curvature vanishes identically if $n = 1$, it is equal to the sectional curvature if $n = 2$.

Theorem 20 *Let $(M, g, S(TM))$ be a 4-dimensional co-screen conformal 1-lightlike submanifold of a Lorentzian manifold. The screen Ricci curvature $Ric_{S(TM)}$ is constant at every unit vector on $\Gamma(S(TM))$ if and only if the following conditions are occurred.*

a) The sectional curvature function on M is constant.

b) Any leaf M' of $S(TM)$ immersed in $\widetilde{M}$ is $(m+2)$-dimensional non-degenerate submanifold with constant curvature.

Proof. Let $\{e_1, e_2, e_3\}$ be an orthonormal basis of $\Gamma(S(TM))$. If $\mathrm{Ric}_{S(TM)}$ is constant, then

$$\begin{aligned}\mathrm{Ric}_{S(TM)}(e_1) &= K_{12} + K_{13} = \lambda,\\ \mathrm{Ric}_{S(TM)}(e_2) &= K_{21} + K_{23} = \lambda,\\ \mathrm{Ric}_{S(TM)}(e_3) &= K_{31} + K_{32} = \lambda,\end{aligned}$$

where λ is a constant. Thus, we have

$$K_{12} = \frac{1}{2}[\mathrm{Ric}_{S(TM)}(e_1) + \mathrm{Ric}_{S(TM)}(e_2) - \mathrm{Ric}_{S(TM)}(e_3)] = \frac{1}{2}\lambda,$$

which shows that the sectional curvature is constant and any leaf M' of $S(TM)$ immersed in $\widetilde{M}$ is $(m+2)$-dimensional non-degenerate submanifold with constant curvature. The proof of the converse part is straightforward.

Theorem 21 *Let $(M, g, S(TM))$ be an $(n+1)$-dimensional co-screen conformal 1-lightlike submanifold of a semi-Euclidean space with the screen Ricci curvature on the submanifold vanishes identically. Then, at least one of the following situations are occurred:*

a) $\mu_1 C(X, X) = \sum_{i=1}^{n} B(e_i, X)C(e_i, X)$ for all $X \in \Gamma(S(TM))$.

b) $\varphi^c = \mp 1$.

Proof. If we take trace in (56), we have

$$\left(\frac{(\varphi^c)^2 - 1}{(\varphi^c)^2}\right)\sum_{i=1}^{n}[B(e_i, X)C(e_i, X)] - \mu_1 C(X, X) = 0,$$

which implies that

$$\mu_1 C(X, X) = \sum_{i=1}^{n} B(e_i, X)C(e_i, X),$$

or

$$\varphi^c = \mp 1. \tag{61}$$

This is proof of the theorem.

Theorem 22 *Let $(M, g, S(TM))$ be an $(n+1)$-dimensional flat co-screen conformal 1-lightlike submanifold with $B = 0$ of a semi-Euclidean space. Then, the local shape operator of M takes the form as follows:*

$$A^*_\xi = \begin{pmatrix} a & -a & \cdots & -a \\ -a & a & \cdots & -a \\ \vdots & \vdots & \ddots & \vdots \\ -a & -a & \cdots & a \end{pmatrix}, \tag{62}$$

for a is a real number.

Proof. Since M is flat, using (53) and (54) one can write

$$D(Y,Z)\varepsilon_\alpha(X) = D(X,Z)\varepsilon_\alpha(Y),$$
$$D(X,Z)\rho_\alpha(Y) = D(Y,Z)\varepsilon_\alpha(X),$$

i.e

$$D(Y,Z)[\varepsilon_\alpha(X) - \rho_\alpha(Y)] = D(X,Z)[\varepsilon_\alpha(Y) - \rho_\alpha(X)]. \tag{63}$$

for all $X, Y, Z \in TM$. If we put $Y = X + Z$ then we have

$$D(Z,Z)[\varepsilon_\alpha(X) - \rho_\alpha(X)] = D(X,Z)[\varepsilon_\alpha(Z) - \rho_\alpha(Z)], \tag{64}$$

i.e

$$\frac{\varepsilon_\alpha(X) - \rho_\alpha(X)}{\varepsilon_\alpha(Z) - \rho_\alpha(Z)} = \frac{D(X,Z)}{D(Z,Z)}. \tag{65}$$

By using similar method, one has

$$\frac{\varepsilon_\alpha(X) - \rho_\alpha(X)}{\varepsilon_\alpha(Y) - \rho_\alpha(Y)} = \frac{D(X,Z)}{D(Z,Z)}, \tag{66}$$

which implies that $D(Y,Y) = D(Z,Z)$. Taking into consideration (52) it can be obtained $D(Y,Y) = -D(X,Y)$. Thus the shape operator A^*_ξ takes the form as (62).

Definition 23 *Let $(M, g, S(TM))$ be an $(n+1)$-dimensional co-screen conformal 1-lightlike submanifold of a Loretzian manifold $(\widetilde{M}, \widetilde{g})$. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $\Gamma(S(TM))$. We define the screen scalar curvature at a point $p \in M$ as*

$$r_{S(TM)}(p) = \frac{1}{2}\sum_{i,j=1}^{n} K_{ij}. \tag{67}$$

Using (52) and (67), we state the following lemma:

Lemma 24 *Let $(M, g, S(TM))$ be an $(n+1)$-dimensional co-screen conformal 1-lightlike submanifold of a Loretzian manifold $(\widetilde{M}, \widetilde{g})$. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $\Gamma(S(TM))$. Then we have*

$$2r_{S(TM)}(p) = 2\widetilde{r}_{S(TM)}(p) + \mu^*(\mu^* + \mu_1) - \sum_{i,j=1}^{n}[B_{ij}C_{ij} + (\frac{C_{ij}}{\varphi^c})^2], \tag{68}$$

where $B_{ij} = B(e_i, e_j)$ and $C_{ij} = C(e_i, e_j)$ for all $i, j \in \{1, \ldots, n\}$.

Let us denote

$$|B|^2 = \sum_{i,j}^{n}(B_{ij})^2 \text{ and } |C|^2 = \sum_{i,j}^{n}(C_{ij})^2. \tag{69}$$

We note that both the second fundamental forms B and C are independent of the screen distribution in co-screen conformal 1-lightlike submanifolds of a Lorentzian manifold it is clear that the norms of these operators are well-defined.

Theorem 25 *Let $(M, g, S(TM))$ be an $(n+1)$-dimensional co-screen conformal 1-lightlike submanifold of a Loretzian manifold $(\widetilde{M}, \widetilde{g})$. Then, we have*

$$2r_{S(TM)}(p) \leq 2\widetilde{r}_{S(TM)}(p) + \mu^*(\mu^* + \mu_1) + \frac{1}{2}|B|^2 + \frac{1 - 2(\varphi^c)^2}{(\varphi^c)^2}|C|^2. \tag{70}$$

The equality case of (70) holds for the point $p \in M$ if and only if either the submanifold M is totally geodesic or is also screen locally conformal with $\varphi = -1$.

Proof. If we put

$$\sum_{i,j=1}^{n} B_{ij}C_{ij} = \frac{1}{2}\{\sum_{i,j=1}^{n}(B_{ij}+C_{ij})^2 - (B_{ij})^2 - (C_{ij})^2\} \tag{71}$$

in Lemma 24, we get

$$\begin{aligned} 2r_{S(TM)}(p) &= 2\widetilde{r}_{S(TM)}(p) + \mu^*(\mu^*+\mu_1) - \frac{1}{2}\sum_{i,j=1}^{n}(B_{ij}+C_{ij})^2 \\ &\quad + \frac{1}{2}\sum_{i,j=1}^{n}(B_{ij})^2 + \frac{1}{2(\varphi^c)^2}\sum_{i,j=1}^{n}(C_{ij})^2, \end{aligned} \tag{72}$$

which implies (70) inequality. The equality case of (70) holds for the point $p \in M$ if and only if

$$\sum_{i,j=1}^{n}(B_{ij}+C_{ij})^2 = 0, \tag{73}$$

which shows that the submanifold M is totally geodesic or is also screen locally conformal with $\varphi = -1$.

Taking into consideration (70), we have also the following theorem:

Theorem 26 *Let $(M, g, S(TM))$ be an $(n+1)$-dimensional co-screen conformal 1-lightlike submanifold of a Loretzian manifold $(\widetilde{M}, \widetilde{g})$. Then, we have*

$$2r_{S(TM)}(p) \le 2\widetilde{r}_{S(TM)}(p) + \mu^*(\mu^*+\mu_1) - \frac{1}{2}(trace\overline{A})^2 + \frac{1}{2}|B|^2 + \frac{1-2(\varphi^c)^2}{(\varphi^c)^2}|C|^2, \tag{74}$$

where

$$\overline{A} = \begin{pmatrix} B_{11}+C_{11} & B_{12}+C_{12} & \cdots & B_{1n}+C_{1n} \\ B_{21}+C_{21} & B_{22}+C_{22} & \cdots & B_{2n}+C_{2n} \\ \vdots & & & \\ B_{n1}+C_{n1} & B_{n2}+C_{n2} & \cdots & B_{nn}+C_{nn} \end{pmatrix}. \tag{75}$$

The equality case of (74) *holds for the point $p \in M$ if and only if $B(X,Y) = -C(X,Y)$ for all mutually orthogonal vectors $X, Y \in \Gamma(S(TM))$.*

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