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Quantum Moment Equations for a One-Band and a Two-Band kp Pauli-type Hamiltonian

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Abstract

The hydrodynamic moment equations for a quantum system described by a One-Band Pauli type Hamiltonian and a Two-Band kp Pauli type Hamiltonian are derived.

Keywords: quantum moment equations, Pauli-type Hamiltonian

1. Introduction

In (Wigner, 1932) E. Wigner introduced the quasi-distribution

$$w(r,p) = \frac{1}{h^d} \int_{\mathbb{R}^d} \rho\left(r + \frac{\xi}{2}, r - \frac{\xi}{2}\right) e^{-i\frac{p\cdot\xi}{\hbar}}\, d\xi \tag{1}$$

for a quantum system with d degrees of freedom in the mixed state described by the density matrix

$$\rho(x,y) = \langle x| S |y\rangle \tag{2}$$

corresponding to the statistical operator S. As usual h denotes the Planck constant and $\hbar = \frac{h}{2\pi}$.

In (Barletti, 2005), L. Barletti, studied the Quantum Moment Equations for the Two-Band Kp Hamiltonian of this type

$$H_{Kp} = \begin{pmatrix} -\frac{\hbar^2}{2m}\triangle + V_1 & -\frac{\hbar^2}{m}K\nabla \\ \frac{\hbar^2}{m}K\nabla & -\frac{\hbar^2}{2m}\triangle + V_2 \end{pmatrix} \tag{3}$$

where m is the electron mass, $K = \langle u_1| \nabla |u_2\rangle$ is the matrix element of the gradient operator between the real Blooch fuction u_1 and u_2 which is assumed to be constant and the functions V_1, V_2 are the potentials of the electron in the conduction and in the valence band rispectively. The Hamiltonian (3) describes an electron that "sees" two energy bands avaible and a Zener tunneling between the two-band is possible. This is the case of Interband Resonant Tunneling Diode where a conductio electron may becomes a valence electron after tunneling through a double barrier (Borgioli, Frosali, Zweifel, 2003). If we work with the (3) Hamiltonian we don't consider the electro as a 1/2 spin particle.

We will consider the following One-Band Pauli Hamiltonian

$$H_P = \left[\frac{1}{2m}\left(\frac{\hbar}{i}\nabla - \frac{e}{c}A\right)^2 + e\phi\right]\sigma_0 - \frac{e\hbar}{2mc}B^i\sigma_i \tag{4}$$

where

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

with

$$A = \left(A^1(x), A^2(x), A^3(x)\right), B = \left(B^1(x), B^2(x), B^3(x)\right)$$

and $B = curl(A)$; we well derive the equations of quantum hydrodynamic moments for this type of Hamiltonian. Moreover we study the Two-band case, too. These types of Hamiltonian operators consider the electron as a 1/2 spin particle.

We well derive the equations of quantum hydrodynamic moments for this type of Hamiltonians and we prove for a pure state the system of order-0 and order-1 equations is closed.

2. Moment Equations for the Pauli Hamiltonian

We start to consider the Pauli Hamiltonian

$$H_P = \left[\frac{1}{2m}\left(\frac{\hbar}{i}\nabla - \frac{e}{c}A\right)^2 + e\phi\right]\sigma_0 - \frac{e\hbar}{2mc}B^i\sigma_i$$

let us take

$$H_P = H^i\sigma_i = \sum_{i=0}^{4} H^i\sigma_i \tag{5}$$

with $H^0 = \left[\frac{1}{2m}\left(\frac{\hbar}{i}\nabla - \frac{e}{c}A\right)^2 + e\phi\right]$ and $H^i = -\frac{e\hbar}{2mc}B^i$, for $i = 1, 2, 3$; then for the density matrix we get

$$\begin{aligned} i\hbar\partial_t\varrho &= H^i_x\sigma_i\varrho - H^i_y\varrho\sigma_i \\ &= \tfrac{1}{2}H^i_x\sigma_i\varrho + \tfrac{1}{2}H^i_x\sigma_i\varrho \\ &+ \tfrac{1}{2}H^i_y\sigma_i\varrho - \tfrac{1}{2}H^i_y\sigma_i\varrho - \\ &- \tfrac{1}{2}H^i_y\varrho\sigma_i - \tfrac{1}{2}H^i_y\varrho\sigma_i + \\ &+ \tfrac{1}{2}H^i_x\varrho\sigma_i - \tfrac{1}{2}H^i_x\varrho\sigma_i \\ &= \tfrac{1}{2}\left(H^i_x + H^i_y\right)\sigma_i\varrho + \\ &+ \tfrac{1}{2}\left(H^i_x - H^i_y\right)\sigma_i\varrho - \\ &- \tfrac{1}{2}\left(H^i_x + H^i_y\right)\varrho\sigma_i + \\ &+ \tfrac{1}{2}\left(H^i_x - H^i_y\right)\varrho\sigma_i \\ &= \tfrac{1}{2}\left(H^i_x + H^i_y\right)[\sigma_i, \varrho]_- + \\ &+ \tfrac{1}{2}\left(H^i_x - H^i_y\right)[\sigma_i, \varrho]_+ \end{aligned} \tag{6}$$

where $[a, b]_\pm = ab \pm ba$.

From (6) we obtain

$$\begin{aligned} i\hbar\partial_t\varrho &= \tfrac{1}{2m}\left[\left(\tfrac{\hbar}{i}\nabla_x - \tfrac{e}{c}A(x)\right)^2 - \left(\tfrac{\hbar}{i}\nabla_y - \tfrac{e}{c}A(y)\right)^2\right]\varrho + \\ &+ e(\phi(x) - \phi(y))\varrho - \\ &- \tfrac{e\hbar}{4mc}\left(B^i(x) - B^i(y)\right)[\sigma_i, \varrho]_+ - \\ &- \tfrac{e\hbar}{4mc}\left(B^i(x) + B^i(y)\right)[\sigma_i, \varrho]_- \end{aligned} \tag{7}$$

with $\varrho = \begin{pmatrix} \varrho_{1,1} & \varrho_{1,2} \\ \varrho_{2,1} & \varrho_{2,2} \end{pmatrix}$ and $\varrho_{1,2}(x, y; t) = \varrho^*_{2,1}(y, x; t)$.

Let us take $P = \frac{\hbar}{i}\nabla - \frac{e}{c}A$, then by (7) we get

$$\begin{aligned} i\hbar\partial_t\varrho &= \tfrac{1}{2m}\left(P_x^2 - P_y^2\right)\varrho + e(\phi(x) - \phi(y))\varrho - \\ &- \tfrac{e\hbar}{4mc}\left[\left(B^i(x) - B^i(y)\right)[\sigma_i, \varrho]_+ + \left(B^i(x) + B^i(y)\right)[\sigma_i, \varrho]_-\right] \end{aligned} \tag{8}$$

and

$$\begin{aligned} \partial_t\varrho &= \tfrac{1}{m}\left[\left(\tfrac{P_x - P_y}{2}\right)\left(\tfrac{P_x + P_y}{i\hbar}\right)\right]\varrho - \tfrac{ei}{\hbar}(\phi(x) - \phi(y))\varrho + \\ &+ \tfrac{ei}{4mc}\left[\left(B^i(x) - B^i(y)\right)[\sigma_i, \varrho]_+ + \left(B^i(x) + B^i(y)\right)[\sigma_i, \varrho]_-\right]. \end{aligned} \tag{9}$$

By Wigner transform we have

$$\begin{aligned} \partial_t w &= \tfrac{1}{m}\widetilde{P}_r\widetilde{P}_p w - \tfrac{ei}{\hbar}\Theta_-(\phi)w + \\ &+ \tfrac{ei}{4mc}\left[\Theta_-\left(B^i\right)[\sigma_i, w]_+ + \Theta_+\left(B^i\right)[\sigma_i, w]_-\right] \end{aligned} \tag{10}$$

where $w = \mathcal{W}\varrho$ and

$$\begin{aligned}
\widetilde{\mathcal{P}}_r &= \mathcal{W}\left(\frac{P_x+P_y}{i\hbar}\right)\mathcal{W}^{-1} \\
&= \mathcal{W}\left(\frac{\hbar}{i}\frac{\nabla_x+\nabla_y}{i\hbar}\right)\mathcal{W}^{-1}+ \\
&+\frac{e}{i\hbar c}\mathcal{W}\left(A\left(x\right)+A\left(y\right)\right)\mathcal{W}^{-1} \\
&= P_r + \frac{e}{i\hbar c}\Theta_+\left(A\right) \\
\widetilde{\mathcal{P}}_p &= \mathcal{W}\left(\frac{P_x-P_y}{2}\right)\mathcal{W}^{-1} \\
&= \frac{\hbar}{i}\mathcal{W}\left(\frac{\nabla_x-\nabla_y}{2}\right)\mathcal{W}^{-1}+ \\
&+\frac{e}{2c}\mathcal{W}\left(A\left(x\right)-A\left(y\right)\right)\mathcal{W}^{-1} \\
&= P_p + \frac{e}{2c}\Theta_-\left(A\right)
\end{aligned}$$

with

$$\begin{aligned}
\widetilde{\nabla}_x &= \mathcal{W}\left(\nabla_x\right)\mathcal{W}^{-1} \\
&= \frac{\nabla_r}{2} + \frac{i}{\hbar}p \\
\widetilde{\nabla}_y &= \mathcal{W}\left(\nabla_y\right)\mathcal{W}^{-1} \\
&= \frac{\nabla_r}{2} - \frac{i}{\hbar}p \\
\widetilde{P}_r &= \frac{\hbar}{i}\mathcal{W}\left(\frac{\nabla_x+\nabla_y}{i\hbar}\right)\mathcal{W}^{-1} \\
&= \frac{\hbar}{i}\frac{\widetilde{\nabla}_x+\widetilde{\nabla}_y}{i\hbar} = -\nabla_r \\
\widetilde{P}_p &= \frac{\hbar}{i}\mathcal{W}\left(\frac{\nabla_x-\nabla_y}{2}\right)\mathcal{W}^{-1} \\
&= \frac{\hbar}{i}\frac{\widetilde{\nabla}_x-\widetilde{\nabla}_y}{2} = p
\end{aligned}$$

and

$$\begin{aligned}
\Theta_-\left(\phi\right)\left(\cdot\right) &= \left(2\pi\right)^{-3}\int_{\mathbb{R}^6}\left[\phi\left(r+\tfrac{\hbar}{2}\xi\right)-\phi\left(r-\tfrac{\hbar}{2}\xi\right)\right]\left(\cdot\right)e^{-i(p-p_1)\xi}\,dp_1d\xi \\
\Theta_\pm\left(A^j\right)\left(\cdot\right) &= \left(2\pi\right)^{-3}\int_{\mathbb{R}^6}\left[A^j\left(r+\tfrac{\hbar}{2}\xi\right)\pm A^j\left(r-\tfrac{\hbar}{2}\xi\right)\right]\left(\cdot\right)e^{-i(p-p_1)\xi}\,dp_1d\xi \\
\Theta_\pm\left(B^j\right)\left(\cdot\right) &= \left(2\pi\right)^{-3}\int_{\mathbb{R}^6}\left[B^j\left(r+\tfrac{\hbar}{2}\xi\right)\pm B^j\left(r-\tfrac{\hbar}{2}\xi\right)\right]\left(\cdot\right)e^{-i(p-p_1)\xi}\,dp_1d\xi
\end{aligned} \tag{11}$$

for $j = 1, 2, 3$; moreover

$$\begin{aligned}
\widetilde{P}_x &= \mathcal{W}\left(\frac{\hbar}{i}\nabla_x - \frac{e}{c}A\left(x\right)\right)\mathcal{W}^{-1} \\
&= \frac{\hbar}{i}\widetilde{\nabla}_x - \frac{e}{c}\widetilde{A}_x \\
&= \frac{\hbar}{i}\frac{\nabla_r}{2} + p - \frac{e}{c}\widetilde{A}_x; \\
\widetilde{P}_y &= \frac{\hbar}{i}\mathcal{W}\left(\frac{\hbar}{i}\nabla_y - \frac{e}{c}A\left(y\right)\right)\mathcal{W}^{-1} \\
&= \frac{\hbar}{i}\widetilde{\nabla}_y - \frac{e}{c}\widetilde{A}_y \\
&= \frac{\hbar}{i}\frac{\nabla_r}{2} - p - \frac{e}{c}\widetilde{A}_y.
\end{aligned}$$

Let us take

$$\widehat{w} = \begin{pmatrix} w_{1,1} \\ w_{1,2} \\ w_{2,1} \\ w_{2,2} \end{pmatrix} \tag{12}$$

by equality (2.6), if

$$-\frac{ei}{\hbar}\Theta_-\left(\phi\right)w + \frac{ei}{4mc}\left[\Theta_-\left(B^i\right)\left[\sigma_i, w\right]_+ + \Theta_+\left(B^i\right)\left[\sigma_i, w\right]_-\right] = 0 \tag{13}$$

we get

$$\partial_t\widehat{w} \quad = -\frac{1}{m}\mathcal{D}\mathcal{P}\widehat{w} \tag{14}$$

where

$$\mathcal{D} = -\widetilde{\mathcal{P}}_r\widehat{\mathbf{1}} \tag{15}$$

and

$$\mathcal{P} = \widetilde{\mathcal{P}}_p\widehat{\mathbf{1}} \tag{16}$$

with $\widehat{\mathbf{1}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

Moreover, by (10), (15) and (16), we obtain

$$\partial_t \widehat{w} \quad = -\tfrac{1}{m} \mathcal{D}\mathcal{P}\widehat{w} + \Theta\widehat{w} \tag{17}$$

where $\Theta\widehat{w}$ is a potential pseudo-differential operator that we well see. We will consider the "local average" of any phase-space quantity f defin by

$$\langle f \rangle (r) = \int_{\mathbb{R}^d} f(r,p)\, dp;$$

then for us $\langle \widehat{w} \rangle = \langle \widehat{w} \rangle (r) = \int_{\mathbb{R}^d} \widehat{w}(r,p)\, dp$.

For for the Pauli Hamiltonian with $\Theta\widehat{w} = 0$ we have the Wigner equation (17) and since the operator $\mathcal{D}$ does not involve the momentum variable p, then $\langle \mathcal{D}\cdot \rangle = \mathcal{D} \langle \cdot \rangle$ and we get the order-0 moment equation

$$\partial_t \langle \widehat{w} \rangle + \frac{1}{m} \mathcal{D} \langle \mathcal{P}\widehat{w} \rangle = 0 \tag{18}$$

moreover, since $\mathcal{D}$ and $\mathcal{P}$ commute, we have the order-1 moment equation

$$\partial_t \langle \mathcal{P}\widehat{w} \rangle + \frac{1}{m} \mathcal{D} \langle \mathcal{P} \otimes \mathcal{P}\widehat{w} \rangle = 0 \tag{19}$$

and more in general the order-m moment equation

$$\partial_t \left\langle \mathcal{P}^{\otimes m}\widehat{w} \right\rangle + \frac{1}{m} \mathcal{D} \left\langle \mathcal{P}^{\otimes (m+1)}\widehat{w} \right\rangle = 0 \tag{20}$$

where $\mathcal{P}^{\otimes m} = \overbrace{\mathcal{P} \otimes \cdots \otimes \mathcal{P}}^{m\text{-time}}$.

Remark 1. *For the free Pauli Hamiltonian, from (17), with $\phi = 0$ and $A = B = 0$, we have*

$$\partial_t \widehat{w} = -\frac{1}{m} \nabla_r \cdot p\widehat{w} \tag{21}$$

that it is the usual Wigner equation for free 1/2 spin particle.

Let us introduce the following notations:

$$n = \langle \widehat{w} \rangle \tag{22}$$

and

$$J = \langle P\widehat{w} \rangle \tag{23}$$

then by (18), (22) and (23) we get

$$\partial_t n + \frac{1}{m} \mathcal{D} J = 0. \tag{24}$$

As in (L. Barletti, 2003) we use the following convetion: every operation between colum-vectors has to be understood component-wise. For example we have

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = \begin{pmatrix} a_1 b_1 \\ a_2 b_2 \\ a_3 b_3 \\ a_4 b_4 \end{pmatrix} \tag{25}$$

and

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} / \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = \begin{pmatrix} a_1/b_1 \\ a_2/b_2 \\ a_3/b_3 \\ a_4/b_4 \end{pmatrix}; \tag{26}$$

moreover, if a_i, b_i are vectors (such as the for components of $\langle \mathcal{P}\widehat{w} \rangle$)

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = \begin{pmatrix} a_1 \cdot b_1 \\ a_2 \cdot b_2 \\ a_3 \cdot b_3 \\ a_4 \cdot b_4 \end{pmatrix} \tag{27}$$

and

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} \otimes \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = \begin{pmatrix} a_1 \otimes b_1 \\ a_2 \otimes b_2 \\ a_3 \otimes b_3 \\ a_4 \otimes b_4 \end{pmatrix}. \tag{28}$$

Since $\widetilde{P}_x = \frac{\hbar}{i}\widetilde{\nabla}_x - \frac{e}{c}\widetilde{A}_x$, $\widetilde{P}_y = \frac{\hbar}{i}\widetilde{\nabla}_y - \frac{e}{c}\widetilde{A}_y$ and

$$\begin{aligned} \mathcal{D} = \widetilde{\mathcal{P}}_r 1 = \mathcal{W}\left(\tfrac{P_x+P_y}{i\hbar}\right)\mathcal{W}^{-1} = \tfrac{\widetilde{P}_x+\widetilde{P}_y}{i\hbar} \\ \mathcal{P} = \widetilde{\mathcal{P}}_p 1 = \mathcal{W}\left(\tfrac{P_x-P_y}{2}\right)\mathcal{W}^{-1} = \tfrac{\widetilde{P}_x-\widetilde{P}_y}{2} \end{aligned} \tag{29}$$

we obtain

$$\begin{aligned} \mathcal{P}\otimes\mathcal{P} = \mathcal{P}^{\otimes 2} &= \left(\tfrac{\widetilde{P}_x-\widetilde{P}_y}{2}\right)^{\otimes 2} \\ &= \tfrac{1}{4}\left(\widetilde{P}_x^{\otimes 2} + \widetilde{P}_y^{\otimes 2} - 2\widetilde{P}_x\otimes\widetilde{P}_y\right) \\ &= \tfrac{1}{4}\left(\widetilde{P}_x^{\otimes 2} + \widetilde{P}_y^{\otimes 2} + 2\widetilde{P}_x\otimes\widetilde{P}_y\right) - \widetilde{P}_x\otimes\widetilde{P}_y \\ &= \tfrac{1}{4}\left(\widetilde{P}_x + \widetilde{P}_y\right)^{\otimes 2} - \widetilde{P}_x\otimes\widetilde{P}_y \\ &= -\tfrac{\hbar^2}{4}\left(\tfrac{\widetilde{P}_x+\widetilde{P}_y}{i\hbar}\right)^{\otimes 2} - \widetilde{P}_x\otimes\widetilde{P}_y \\ &= -\tfrac{\hbar^2}{4}\mathcal{D}^{\otimes 2} - \widetilde{P}_x\otimes\widetilde{P}_y \end{aligned} \tag{30}$$

where the component-wise commutativity between $\widetilde{P}_x$ and $\widetilde{P}_y$ was used. Moreover we get

$$\left\langle \mathcal{P}^{\otimes 2}\widehat{w}\right\rangle = -\frac{\hbar^2}{4}\mathcal{D}^{\otimes 2}\langle\widehat{w}\rangle - \left\langle \widetilde{P}_x\otimes\widetilde{P}_y\widehat{w}\right\rangle \tag{31}$$

and

$$\begin{aligned} \langle\mathcal{P}\widehat{w}\rangle^{\otimes 2} &= \left\langle \tfrac{\widetilde{P}_x-\widetilde{P}_y}{2}\widehat{w}\right\rangle^{\otimes 2} \\ &= \tfrac{1}{4}\left(\left\langle\widetilde{P}_x\widehat{w}\right\rangle^{\otimes 2} + \left\langle\widetilde{P}_y\widehat{w}\right\rangle^{\otimes 2} - 2\left\langle\widetilde{P}_x\widehat{w}\right\rangle\otimes\left\langle\widetilde{P}_y\widehat{w}\right\rangle\right) \\ &= \tfrac{1}{4}\left\langle\widetilde{P}_x + \widetilde{P}_y\widehat{w}\right\rangle^{\otimes 2} - \left\langle\widetilde{P}_x\widehat{w}\right\rangle\otimes\left\langle\widetilde{P}_y\widehat{w}\right\rangle \\ &= -\tfrac{\hbar^2}{4}\langle\mathcal{D}\widehat{w}\rangle^{\otimes 2} - \left\langle\widetilde{P}_x\widehat{w}\right\rangle\otimes\left\langle\widetilde{P}_y\widehat{w}\right\rangle \\ &= -\tfrac{\hbar^2}{4}\left(\mathcal{D}\langle\widehat{w}\rangle\right)^{\otimes 2} - \left\langle\widetilde{P}_x\widehat{w}\right\rangle\otimes\left\langle\widetilde{P}_y\widehat{w}\right\rangle \\ &= -\tfrac{\hbar^2}{4}\left(\mathcal{D}\langle\widehat{w}\rangle\right)\otimes\left(\mathcal{D}\langle\widehat{w}\rangle\right) - \left\langle\widetilde{P}_x\widehat{w}\right\rangle\otimes\left\langle\widetilde{P}_y\widehat{w}\right\rangle \end{aligned} \tag{32}$$

then

$$\langle\mathcal{P}\otimes P\widehat{w}\rangle = \frac{J\otimes J}{n} + Q(n) - nT \tag{33}$$

where

$$Q(n) = -\frac{\hbar^2}{4}\left((\mathcal{D}\otimes\mathcal{D})n - \frac{(\mathcal{D}n)\otimes(\mathcal{D}n)}{n}\right) \tag{34}$$

and

$$nT = \left\langle\widetilde{\mathcal{P}}_x\otimes\widetilde{\mathcal{P}}_y\widehat{w}\right\rangle - \frac{\left\langle\widetilde{\mathcal{P}}_x\widehat{w}\right\rangle\otimes\left\langle\widetilde{\mathcal{P}}_y\widehat{w}\right\rangle}{n}. \tag{35}$$

Proposition 1. *Let $w = \mathcal{W}\varrho$ be the Wigner transform of the mixed state $\varrho = \sum_{s=1}^{+\infty} \lambda_s \varrho^s$, where $\lambda_s \geq 0$, $\sum_{s=1}^{+} \lambda_s = 1$ and each ϱ^s is a pure-state; then*

$$T = \sum_{s=1}^{+\infty} \frac{\lambda_s n^s}{n} \left(\frac{\left\langle \widetilde{\mathcal{P}}_x \widehat{w}^s \right\rangle}{n^s} - \frac{\left\langle \widetilde{\mathcal{P}}_x \widehat{w} \right\rangle}{n} \right) \otimes \left(\frac{\left\langle \widetilde{\mathcal{P}}_x \widehat{w}^s \right\rangle}{(n^s)^*} - \frac{\left\langle \widetilde{\mathcal{P}}_x \widehat{w} \right\rangle}{(n)^*} \right)^* \tag{36}$$

where $\widehat{w}^s = \mathcal{W}\widehat{\varrho}^s$, $n^s = \langle w^s \rangle$ and $*$ denotes adjunction:

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}^* = \begin{pmatrix} \overline{a} \\ \overline{b} \\ \overline{c} \\ \overline{d} \end{pmatrix}.$$

Proof. Let us define

$$q = \left\langle \widetilde{\mathcal{P}}_x \widehat{w} \right\rangle = \langle P_x \varrho \rangle \tag{37}$$

and

$$q^s = \left\langle \widetilde{\mathcal{P}}_x \widehat{w}^s \right\rangle \tag{38}$$

then

$$q = \langle P_x \varrho \rangle = \sum_{s=1}^{+\infty} \lambda_s q_s. \tag{39}$$

Since $\langle P_x \varrho \rangle = \left\langle P_y \varrho \right\rangle^*$, for a pure state, we have

$$\left\langle \widetilde{\mathcal{P}}_x \otimes \widetilde{\mathcal{P}}_y \widehat{w}^s \right\rangle = \frac{q^s \otimes (q^s)^*}{n^s} \tag{40}$$

From (35) and (40) we get

$$\begin{aligned} T &= \tfrac{1}{n} \left\langle \widetilde{\mathcal{P}}_x \otimes \widetilde{\mathcal{P}}_y \widehat{w} \right\rangle - \tfrac{\left\langle \widetilde{\mathcal{P}}_x \widehat{w} \right\rangle \otimes \left\langle \widetilde{\mathcal{P}}_y \widehat{w} \right\rangle}{n^2} \\ &= \sum_{s=1}^{+\infty} \lambda_s \tfrac{\left\langle \widetilde{\mathcal{P}}_x \otimes \widetilde{\mathcal{P}}_y \widehat{w}^s \right\rangle}{n} - \tfrac{\left\langle \widetilde{\mathcal{P}}_x \widehat{w} \right\rangle \otimes \left\langle \widetilde{\mathcal{P}}_y \widehat{w} \right\rangle}{n^2} \\ &= \sum_{s=1}^{+\infty} b^s \tfrac{q^s}{n^s} \otimes \tfrac{(q^s)^*}{n^s} - \tfrac{\left\langle \widetilde{\mathcal{P}}_x \widehat{w} \right\rangle \otimes \left\langle \widetilde{\mathcal{P}}_y \widehat{w} \right\rangle}{n^2} \end{aligned} \tag{41}$$

where

$$b^s = \frac{\lambda_s n^s}{n}. \tag{42}$$

Since

$$\begin{aligned} \sum_{s=1}^{+\infty} b^s \left(\tfrac{q^s}{n^s} - \tfrac{q}{n} \right) \otimes \left(\tfrac{q^s}{(n^s)^*} - \tfrac{q}{(n)^*} \right)^* &= \sum_{s=1}^{+\infty} b^s \left(\tfrac{q^s}{n^s} - \tfrac{q}{n} \right) \otimes \left(\left(\tfrac{q^s}{(n^s)^*} \right)^* - \left(\tfrac{q}{(n)^*} \right)^* \right) \\ &= \sum_{s=1}^{+\infty} b^s \left(\tfrac{q^s}{n^s} \otimes \left(\tfrac{q^s}{(n^s)^*} \right)^* - \tfrac{q^s}{n^s} \otimes \left(\tfrac{q}{(n)^*} \right)^* \right) + \\ &+ \sum_{s=1}^{+\infty} b^s \left(-\tfrac{q}{n} \otimes \left(\tfrac{q^s}{(n^s)^*} \right)^* + \tfrac{q}{n} \otimes \left(\tfrac{q}{(n)^*} \right)^* \right) \\ &= \sum_{s=1}^{+\infty} b^s \tfrac{q^s}{n^s} \otimes \left(\tfrac{q^s}{(n^s)^*} \right)^* - \left(\sum_{s=1}^{+\infty} b^s \tfrac{q^s}{n^s} \right) \otimes \left(\tfrac{q}{(n)^*} \right)^* - \\ &- \tfrac{q}{n} \otimes \left(\sum_{s=1}^{+\infty} b^s \left(\tfrac{q^s}{(n^s)^*} \right)^* \right) + \tfrac{q}{n} \otimes \left(\tfrac{q}{(n)^*} \right)^* \end{aligned} \tag{43}$$

and

$$\begin{aligned} &\sum_{s=1}^{+\infty} b^s \tfrac{q^s}{n^s} = \tfrac{q}{n} \\ &\sum_{s=1}^{+\infty} b^s \left(\tfrac{q^s}{(n^s)^*} \right)^* = \sum_{s=1}^{+\infty} \tfrac{\lambda_s}{n} (q^s)^* = \tfrac{1}{n} (q)^* \end{aligned} \tag{44}$$

we get

$$\begin{array}{ll}\sum_{s=1}^{+\infty} b^s\left(\frac{q^s}{n^s}-\frac{q}{n}\right)\otimes\left(\frac{q^s}{(n^s)^*}-\frac{q}{(n)^*}\right)^* & =\sum_{s=1}^{+\infty} b^s\frac{q^s}{n^s}\otimes\left(\frac{q^s}{n^s}\right)^*-\frac{q}{n}\otimes\left(\frac{q}{n}\right)^*-\frac{q}{n}\otimes\frac{(q)^*}{n}+\frac{q}{n}\otimes\left(\frac{q}{n}\right)^* \\ & =\sum_{s=1}^{+\infty} b^s\frac{q^s}{n^s}\otimes\left(\frac{q^s}{(n^s)^*}\right)^*-\frac{q}{n}\otimes\frac{(q)^*}{n} \\ & =\sum_{s=1}^{+\infty} b^s\frac{q^s}{n^s}\otimes\frac{(q^s)^*}{n^s}-\frac{\langle\widetilde{\mathcal{P}}_x\widehat{w}\rangle\otimes\langle\widetilde{\mathcal{P}}_y\widehat{w}\rangle}{n^2} \\ & =T.\end{array} \tag{45}$$

□

Corollary 1. *If $w=\mathcal{W}\varrho$ is the Wigner transform of a pure-state density matrix, then $T=0$.*

From (37) we get

$$\begin{array}{ll} q & =\left\langle\widetilde{\mathcal{P}}_x\widehat{w}\right\rangle \\ & =\left\langle\left[\left(\frac{\widetilde{\mathcal{P}}_x-\widetilde{\mathcal{P}}_y}{2}\right)+\left(\frac{\widetilde{\mathcal{P}}_x+\widetilde{\mathcal{P}}_y}{2}\right)\right]\widehat{w}\right\rangle \\ & =\left\langle\left(\frac{\widetilde{\mathcal{P}}_x-\widetilde{\mathcal{P}}_y}{2}\right)\widehat{w}\right\rangle+\frac{i\hbar}{2}\left\langle\left(\frac{\widetilde{\mathcal{P}}_x+\widetilde{\mathcal{P}}_y}{i\hbar}\right)\widehat{w}\right\rangle \\ & =\langle\mathcal{P}\widehat{w}\rangle+\frac{i\hbar}{2}\langle\mathcal{D}\widehat{w}\rangle \\ & =J+\frac{i\hbar}{2}\mathcal{D}n\end{array} \tag{46}$$

and $q^s=J^s+\frac{i\hbar}{2}\mathcal{D}n^s$, with $J^s=\langle\mathcal{P}\widehat{w}^s\rangle$, then

$$\begin{array}{ll} T & =\sum_{s=1}^{+\infty} b^s\left(\frac{q^s}{n^s}-\frac{q}{n}\right)\otimes\left(\frac{q^s}{(n^s)^*}-\frac{q}{(n)^*}\right)^* \\ & =\sum_{s=1}^{+\infty} b^s\left(\frac{J^s+\frac{i\hbar}{2}\mathcal{D}n^s}{n^s}-\frac{J+\frac{i\hbar}{2}\mathcal{D}n}{n}\right)\otimes\left(\frac{J^s+\frac{i\hbar}{2}\mathcal{D}n^s}{(n^s)^*}-\frac{J+\frac{i\hbar}{2}\mathcal{D}n}{(n)^*}\right)^* \\ & =\sum_{s=1}^{+\infty} b^s\left(\frac{J^s}{n^s}-\frac{J}{n}\right)\otimes\left(\frac{J^s}{(n^s)^*}-\frac{J}{(n)^*}\right)^*+ \\ & -\frac{i\hbar}{2}\sum_{s=1}^{+\infty} b^s\left(\frac{J^s}{n^s}-\frac{J}{n}\right)\otimes\left(\frac{\mathcal{D}n^s}{(n^s)^*}-\frac{\mathcal{D}n}{(n)^*}\right)^*+ \\ & +\frac{i\hbar}{2}\sum_{s=1}^{+\infty} b^s\left(\frac{\mathcal{D}n^s}{n^s}-\frac{\mathcal{D}n}{n}\right)\otimes\left(\frac{J^s}{(n^s)^*}-\frac{J}{(n)^*}\right)^*+ \\ & +\frac{\hbar^2}{4}\sum_{s=1}^{+\infty} b^s\left(\frac{\mathcal{D}n^s}{n^s}-\frac{\mathcal{D}n}{n}\right)\otimes\left(\frac{\mathcal{D}n^s}{(n^s)^*}-\frac{\mathcal{D}n}{(n)^*}\right)^*\end{array} \tag{47}$$

By decomposition (46) and (47) we have

$$T=T_c+T_{os} \tag{48}$$

where

$$T_c=\sum_{s=1}^{+\infty} b^s\left(\frac{J^s}{n^s}-\frac{J}{n}\right)\otimes\left(\frac{J^s}{(n^s)^*}-\frac{J}{(n)^*}\right)^* \tag{49}$$

is a "gauge-current temperature" and

$$T_{os}=\frac{\hbar^2}{4}\sum_{s=1}^{+\infty} b^s\left(\frac{\mathcal{D}n^s}{n^s}-\frac{\mathcal{D}n}{n}\right)\otimes\left(\frac{\mathcal{D}n^s}{(n^s)^*}-\frac{\mathcal{D}n}{(n)^*}\right)^* \tag{50}$$

is an "gauge-osmotic temperature".

From (20), (24) and (33) we obtain

$$\partial_t J+\frac{1}{m}\mathcal{D}\left(\frac{J\otimes J}{n}+Q(n)+nT\right)=0. \tag{51}$$

Moreover, for $T=0$, the system

$$\left\{\begin{array}{c}\partial_t n+\frac{1}{m}\mathcal{D}J=0 \\ \partial_t J+\frac{1}{m}\mathcal{D}\left(\frac{J\otimes J}{n}+Q(n)\right)=0\end{array}\right. \tag{52}$$

it is a closed system of Madelung-like QHD equations for a free Pauli Hamiltonian.

2.1 Moments of the Potential Terms

In the following we will using a multi-index notation: a multi-index $\alpha = (\alpha_1, ..., \alpha_N)$ is a -uple of non negative integer and $|\alpha| = \alpha_1 + \cdots + \alpha_N$, $\alpha! = \alpha_1! \cdots \alpha_N!$; moreover $\alpha \le \beta$ if and only if $\alpha_i \le \beta_i$ for $i = 1, ..., N$.

Let us consider

$$\partial_t w = \frac{1}{m}\widetilde{P}_r\widetilde{P}_p w + Vw \tag{53}$$

where

$$\begin{aligned} Vw &= -\frac{ei}{\hbar}\Theta_-(\phi)\, w + \\ &+\frac{ei}{4mc}\left[\Theta_-\left(B^i\right)[\sigma_i, w]_+ + \Theta_+\left(B^i\right)[\sigma_i, w]_-\right] \end{aligned} \tag{54}$$

since

$$[\sigma_1, w]_\pm = \begin{pmatrix} \pm\sigma_1 & 1 \\ 1 & \pm\sigma_1 \end{pmatrix}\widehat{w} = W_1^\pm \widehat{w}, \tag{55}$$

$$[\sigma_2, w]_\pm = \begin{pmatrix} \mp\sigma_2 & -i1 \\ i1 & \mp\sigma_2 \end{pmatrix}\widehat{w} = W_2^\pm \widehat{w} \tag{56}$$

and

$$[\sigma_3, w]_\pm = W_3^\pm \widehat{w} \tag{57}$$

with

$$W_3^+ = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} \tag{58}$$

and

$$W_3^- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \tag{59}$$

from (55), (56), (57), (58) and (59) we get

$$\partial_t \widehat{w} = -\frac{1}{m}\mathcal{D}\mathcal{P}\widehat{w} + \Theta\widehat{w} \tag{60}$$

where

$$\begin{aligned} \Theta\widehat{w} &= -\frac{ei}{\hbar}\Theta_-(\phi)\,\widehat{w} + \\ &+\frac{ei}{4mc}\left[\Theta_-\left(B^1\right)W_1^+ + \Theta_+\left(B^1\right)W_1^-\right]\widehat{w} + \\ &+\frac{ei}{4mc}\left[\Theta_-\left(B^2\right)W_2^+ + \Theta_+\left(B^2\right)W_2^-\right]\widehat{w} + \\ &+\frac{ei}{4mc}\left[\Theta_-\left(B^3\right)W_3^+ + \Theta_+\left(B^3\right)W_3^-\right]\widehat{w}. \end{aligned} \tag{61}$$

We define

$$\begin{aligned} I_1 = \; & -\frac{ei}{\hbar}\begin{pmatrix} \Theta_-(\phi) & & & \\ & \Theta_-(\phi) & & \\ & & \Theta_-(\phi) & \\ & & & \Theta_-(\phi) \end{pmatrix} + \\ & +\frac{ei}{2mc}\begin{pmatrix} \Theta_-\left(B^3\right) & 0 & 0 & 0 \\ 0 & \Theta_+\left(B^3\right) & 0 & 0 \\ 0 & 0 & -\Theta_+\left(B^3\right) & 0 \\ 0 & 0 & 0 & -\Theta_-\left(B^3\right) \end{pmatrix} \end{aligned} \tag{62}$$

and

$$\begin{aligned} I_2 &= \tfrac{ei}{4mc}\left\{\left(\Theta_-\left(B^1\right)-\Theta_+\left(B^1\right)\right)\begin{pmatrix}\sigma_1 & 0\\ 0 & \sigma_1\end{pmatrix}-\right.\\ &-\left(\Theta_-\left(B^2\right)-\Theta_+\left(B^2\right)\right)\begin{pmatrix}\sigma_2 & 0\\ 0 & \sigma_2\end{pmatrix}+\\ &+\left(\Theta_-\left(B^1\right)+\Theta_+\left(B^1\right)\right)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}+\\ &\left.+\left(\Theta_-\left(B^2\right)+\Theta_+\left(B^2\right)\right)\begin{pmatrix}0 & -i1\\ i1 & 0\end{pmatrix}\right\}. \end{aligned} \tag{63}$$

Let us consider the expansion of the pseudo-differential operator in a formal Taylor series with respec to $i\hbar\nabla_p$:

$$\widetilde{\phi}\left(r \pm \frac{i\hbar}{2}\nabla_p\right) = \sum_{k=0}^{+\infty}\sum_{|\alpha|=k}\left(\pm\frac{i\hbar}{2}\right)^k \frac{\nabla^\alpha\phi(r)}{\alpha!}\nabla_p^\alpha, \tag{64}$$

let us consider

$$\begin{aligned} \mathbb{I}_1^1 &= \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix} \quad \mathbb{I}_1^2 = \begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix}\\ \mathbb{I}_2^1 &= \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} \quad \mathbb{I}_2^2 = \begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&1\end{pmatrix} \end{aligned} \tag{65}$$

then we get

$$-\frac{ei}{\hbar}\Theta_-(\phi)\,\mathbf{1} = -\frac{ei}{\hbar}\sum_{k=0}^{+\infty}\sum_{|\alpha|=k}\sum_{j=1,2}\left(\frac{i\hbar}{2}\right)^k\frac{\nabla^\alpha\phi(r)}{\alpha!}\mathbb{H}_j^k\nabla_p^\alpha \tag{66}$$

where

$$\mathbb{H}_j^k = \mathbb{I}_j^1 - (-1)^k\,\mathbb{I}_j^2 \tag{67}$$

for $j = 1, 2$ and $k = 0, 1, \ldots$.

Let us consider

$$\begin{aligned} \mathbb{J}_1^1 &= \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix} \quad \mathbb{J}_1^2 = \begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&-1&0\\0&0&0&0\end{pmatrix}\\ \mathbb{J}_2^1 &= \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix} \quad \mathbb{J}_2^2 = \begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&-1\end{pmatrix} \end{aligned} \tag{68}$$

then

$$\begin{aligned} &\tfrac{ei}{2mc}\begin{pmatrix}\Theta_-\left(B^3\right)&0&0&0\\0&\Theta_+\left(B^3\right)&0&0\\0&0&-\Theta_+\left(B^3\right)&0\\0&0&0&-\Theta_-\left(B^3\right)\end{pmatrix}=\\ &=\tfrac{ei}{2mc}\sum_{k=0}^{+\infty}\sum_{|\alpha|=k}\sum_{j=1,2}\left(\tfrac{i\hbar}{2}\right)^k\tfrac{\nabla^\alpha B^3(r)}{\alpha!}\mathbb{Y}_j^k\nabla_p^\alpha \end{aligned} \tag{69}$$

where

$$\mathbb{Y}_j^k = \mathbb{J}_j^1 - (-1)^k\,\mathbb{J}_j^2 \tag{70}$$

for $j = 1, 2$ and $k = 0, 1, \ldots$.

From (66) and (69) we get

$$\begin{aligned}\langle \mathcal{P}^{\otimes n} I_1 \widehat{w}\rangle &= -\frac{ei}{\hbar}\sum_{k=0}^{+\infty}\sum_{|\alpha|=k}\sum_{j=1,2}\left(\frac{i\hbar}{2}\right)^k \frac{\nabla^\alpha \phi(r)}{\alpha!}\left\langle \left(\widehat{p\mathbf{1}}\right)^{\otimes n}\mathbb{H}_j^k \nabla_p^\alpha \widehat{w}\right\rangle + \\ &+\frac{ei}{2mc}\sum_{k=0}^{+\infty}\sum_{|\alpha|=k}\sum_{j=1,2}\left(\frac{i\hbar}{2}\right)^k \frac{\nabla^\alpha B^3(r)}{\alpha!}\left\langle \left(\widehat{p\mathbf{1}}\right)^{\otimes n}\mathbb{Y}_j^k \nabla_p^\alpha \widehat{w}\right\rangle.\end{aligned} \tag{71}$$

Let us consider

$$\begin{aligned}\mathbb{N}_1 &= \begin{pmatrix} 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{pmatrix} \quad \mathbb{N}_2 = \begin{pmatrix} 0 & -i & 0 & 0\\ i & 0 & 0 & 0\\ 0 & 0 & 0 & -i\\ 0 & 0 & i & 0\end{pmatrix}\\ \mathbb{N}_3 &= \begin{pmatrix} 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\end{pmatrix} \quad \mathbb{N}_4 = \begin{pmatrix} 0 & 0 & 0 & -i\\ 0 & 0 & -i & 0\\ 0 & i & 0 & 0\\ i & 0 & 0 & 0\end{pmatrix}\end{aligned} \tag{72}$$

then

$$\begin{aligned}\langle \mathcal{P}^{\otimes n} I_2 \widehat{w}\rangle &= \frac{ei}{2mc}\left\{-\sum_{k=0}^{+\infty}\sum_{|\alpha|=k}\left(-\frac{i\hbar}{2}\right)^k \frac{\nabla^\alpha B^1(r)}{\alpha!}\left\langle \left(\widehat{p\mathbf{1}}\right)^{\otimes n}\mathbb{N}_1^k \nabla_p^\alpha \widehat{w}\right\rangle + \right.\\ &+\sum_{k=0}^{+\infty}\sum_{|\alpha|=k}\left(-\frac{i\hbar}{2}\right)^k \frac{\nabla^\alpha B^2(r)}{\alpha!}\left\langle \left(\widehat{p\mathbf{1}}\right)^{\otimes n}\mathbb{N}_2^k \nabla_p^\alpha \widehat{w}\right\rangle + \\ &+\sum_{k=0}^{+\infty}\sum_{|\alpha|=k}\left(\frac{i\hbar}{2}\right)^k \frac{\nabla^\alpha B^1(r)}{\alpha!}\left\langle \left(\widehat{p\mathbf{1}}\right)^{\otimes n}\mathbb{N}_3^k \nabla_p^\alpha \widehat{w}\right\rangle + \\ &\left.+\sum_{k=0}^{+\infty}\sum_{|\alpha|=k}\left(\frac{i\hbar}{2}\right)^k \frac{\nabla^\alpha B^2(r)}{\alpha!}\left\langle \left(\widehat{p\mathbf{1}}\right)^{\otimes n}\mathbb{N}_4^k \nabla_p^\alpha \widehat{w}\right\rangle\right\}\end{aligned} \tag{73}$$

where

$$\mathbb{N}_j^k = \mathbb{N}_j \tag{74}$$

for $j = 1, 2, 3, 4$ and $k = 0, 1, \ldots$.

If we consider a single component $\left(\widehat{p\mathbf{1}}\right)^\beta$ of the tensor product $\left(\widehat{p\mathbf{1}}\right)^{\otimes n}$, where β is a multi-index with $|\beta| = n$, integration by parts yields

$$\begin{aligned}\left\langle \left(\widehat{p\mathbf{1}}\right)^\beta \mathbb{H}_j^k \nabla_p^\alpha \widehat{w}\right\rangle &= \begin{cases} (-1)^k \left\langle \left(\widehat{p\mathbf{1}}\right)^{\beta-\alpha}\mathbb{H}_j^k \widehat{w}\right\rangle & \text{if } \alpha \le \beta\\ 0 & \text{if } \alpha \nleq \beta\end{cases}\\ \left\langle \left(\widehat{p\mathbf{1}}\right)^\beta \mathbb{Y}_j^k \nabla_p^\alpha \widehat{w}\right\rangle &= \begin{cases} (-1)^k \left\langle \left(\widehat{p\mathbf{1}}\right)^{\beta-\alpha}\mathbb{Y}_j^k \widehat{w}\right\rangle & \text{if } \alpha \le \beta\\ 0 & \text{if } \alpha \nleq \beta\end{cases}\\ \left\langle \left(\widehat{p\mathbf{1}}\right)^\beta \mathbb{N}_s^k \nabla_p^\alpha \widehat{w}\right\rangle &= \begin{cases} (-1)^k \left\langle \left(\widehat{p\mathbf{1}}\right)^{\beta-\alpha}\mathbb{N}_s^k \widehat{w}\right\rangle & \text{if } \alpha \le \beta\\ 0 & \text{if } \alpha \nleq \beta\end{cases}\end{aligned} \tag{75}$$

for $j = 1, 2$, $s = 1, 2, 3, 4$ and $k = 0, 1, \ldots$.

From (71), (73) and (75) we get

$$\begin{aligned}\langle \mathcal{P}^{\otimes n} I_1 \widehat{w}\rangle &= -\frac{ei}{\hbar}\sum_{k=0}^{n}\sum_{|\alpha|=k}\sum_{j=1,2}\left(-\frac{i\hbar}{2}\right)^k \frac{\nabla^\alpha \phi(r)}{\alpha!}\left\langle \mathcal{P}^{\otimes n-\alpha}\mathbb{H}_j^k \widehat{w}\right\rangle + \\ &+\frac{ei}{2mc}\sum_{k=0}^{n}\sum_{|\alpha|=k}\sum_{j=1,2}\left(-\frac{i\hbar}{2}\right)^k \frac{\nabla^\alpha B^3(r)}{\alpha!}\left\langle \mathcal{P}^{\otimes n-\alpha}\mathbb{Y}_j^k \widehat{w}\right\rangle\end{aligned} \tag{76}$$

and

$$\begin{aligned}\langle \mathcal{P}^{\otimes n} I_2 \widehat{w}\rangle &= -\frac{ei}{2mc}\sum_{k=0}^{n}\sum_{|\alpha|=k}\left(\frac{i\hbar}{2}\right)^k \frac{\nabla^\alpha B^1(r)}{\alpha!}\left\langle P^{\otimes n-\alpha}\mathbb{N}_1^k \widehat{w}\right\rangle + \\ &+\frac{ei}{2mc}\sum_{k=0}^{n}\sum_{|\alpha|=k}\left(\frac{i\hbar}{2}\right)^k \frac{\nabla^\alpha B^2(r)}{\alpha!}\left\langle P^{\otimes n-\alpha}\mathbb{N}_2^k \widehat{w}\right\rangle + \\ &+\frac{ei}{2mc}\sum_{k=0}^{n}\sum_{|\alpha|=k}\left(-\frac{i\hbar}{2}\right)^k \frac{\nabla^\alpha B^1(r)}{\alpha!}\left\langle P^{\otimes n-\alpha}\mathbb{N}_3^k \widehat{w}\right\rangle \\ &+\frac{ei}{2mc}\sum_{k=0}^{n}\sum_{|\alpha|=k}\left(-\frac{i\hbar}{2}\right)^k \frac{\nabla^\alpha B^2(r)}{\alpha!}\left\langle P^{\otimes n-\alpha}\mathbb{N}_4^k \widehat{w}\right\rangle\end{aligned} \tag{77}$$

3. One-band Madelung Equations

Proposition (1) and corollary (1) imply that the $n = 0$ and $n = 1$ moment equations for a pure state are closed and yield a analogue of QHD Madelung equations.

For $n = 0$ we get:

$$\begin{aligned}\langle\Theta\widehat{w}\rangle &= -\tfrac{ei}{\hbar}\phi(r)\sum_{j=1,2}\left\langle\mathbb{H}_j^0\widehat{w}\right\rangle + \tfrac{ei}{2mc}B^3(r)\sum_{j=1,2}\left\langle\mathbb{Y}_j^0\widehat{w}\right\rangle - \\ &-\tfrac{ei}{2mc}B^1(r)\left\langle\mathbb{N}_1^0\widehat{w}\right\rangle + \tfrac{ei}{2mc}B^2(r)\left\langle\mathbb{N}_2^0\widehat{w}\right\rangle + \\ &+\tfrac{ei}{2mc}B^1(r)\left\langle\mathbb{N}_3^0\widehat{w}\right\rangle + \tfrac{ei}{2mc}B^2(r)\left\langle\mathbb{N}_4^0\widehat{w}\right\rangle \\ &= \tfrac{ei}{2mc}B^1(r)\mathbb{M}_1\langle\widehat{w}\rangle + \tfrac{ei}{2mc}B^2(r)\mathbb{M}_2\langle\widehat{w}\rangle\end{aligned} \tag{78}$$

where

$$\mathbb{M}_1 = \begin{pmatrix} 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix} \quad \mathbb{M}_2 = \begin{pmatrix} 0 & -i & 0 & -i \\ i & 0 & -i & 0 \\ 0 & i & 0 & -i \\ i & 0 & i & 0 \end{pmatrix} \tag{79}$$

For $n = 1$ we get:

if $k = 0$

$$\begin{aligned}-\tfrac{ei}{\hbar}\sum_{j=1,2}\phi(r)\left\langle\mathcal{P}\mathbb{H}_j^0\widehat{w}\right\rangle &= -\tfrac{ei}{\hbar}\sum_{j=1,2}\phi(r)\mathbb{H}_j^0\langle\mathcal{P}\widehat{w}\rangle \\ \tfrac{ei}{2mc}\sum_{j=1,2}B^3(r)\left\langle\mathcal{P}\mathbb{Y}_j^0\widehat{w}\right\rangle &= \tfrac{ei}{2mc}\sum_{j=1,2}B^3(r)\mathbb{Y}_j^0\langle\mathcal{P}\widehat{w}\rangle \\ \tfrac{-ei}{2mc}B^1(r)\langle\mathcal{P}\mathbb{N}_1\widehat{w}\rangle &= \tfrac{-ei}{2mc}B^1(r)\mathbb{N}_1\langle\mathcal{P}\widehat{w}\rangle \\ \tfrac{ei}{2mc}B^2(r)\langle\mathcal{P}\mathbb{N}_2\widehat{w}\rangle &= \tfrac{ei}{2mc}B^2(r)\mathbb{N}_2\langle\mathcal{P}\widehat{w}\rangle \\ \tfrac{ei}{2mc}B^1(r)\langle\mathcal{P}\mathbb{N}_3\widehat{w}\rangle &= \tfrac{ei}{2mc}B^1(r)\mathbb{N}_3\langle\mathcal{P}\widehat{w}\rangle \\ \tfrac{ei}{2mc}B^2(r)\langle\mathcal{P}\mathbb{N}_4\widehat{w}\rangle &= \tfrac{ei}{2mc}B^2(r)\mathbb{N}_4\langle\mathcal{P}\widehat{w}\rangle\end{aligned} \tag{80}$$

and

$$\begin{aligned}\mathbb{G}J &= \Big[-\tfrac{ei}{\hbar}\sum_{j=1,2}\phi(r)\mathbb{H}_j^0 + \tfrac{ei}{2mc}\sum_{j=1,2}B^3(r)\mathbb{Y}_j^0 - \\ &-\tfrac{ei}{2mc}B^1(r)\mathbb{N}_1 + \tfrac{ei}{2mc}B^2(r)\mathbb{N}_2 + \tfrac{ei}{2mc}B^1(r)\mathbb{N}_3 + \tfrac{ei}{2mc}B^2(r)\mathbb{N}_4\Big]\langle\mathcal{P}\widehat{w}\rangle;\end{aligned} \tag{81}$$

if $k = 1$ we get

$$\begin{aligned}-\tfrac{ei}{\hbar}\sum_{|\alpha|=1}\sum_{j=1,2}\left(-\tfrac{i\hbar}{2}\right)\nabla^\alpha\phi(r)\left\langle\mathbb{H}_j^1\widehat{w}\right\rangle &= -\tfrac{e}{2}\sum_{|\alpha|=1}\sum_{j=1,2}\nabla^\alpha\phi(r)\mathbb{H}_j^1\langle\widehat{w}\rangle \\ \tfrac{ei}{2mc}\sum_{|\alpha|=1}\sum_{j=1,2}\left(-\tfrac{i\hbar}{2}\right)\tfrac{\nabla^\alpha B^3(r)}{\alpha!}\left\langle\mathbb{Y}_j^1\widehat{w}\right\rangle &= \tfrac{e\hbar}{4mc}\sum_{|\alpha|=1}\sum_{j=1,2}\nabla^\alpha B^3(r)\mathbb{Y}_j^1\langle\widehat{w}\rangle \\ -\tfrac{ei}{2mc}\sum_{|\alpha|=1}\left(\tfrac{i\hbar}{2}\right)\tfrac{\nabla^\alpha B^1(r)}{\alpha!}\left\langle\mathbb{N}_1^1\widehat{w}\right\rangle &= \tfrac{e\hbar}{4mc}\sum_{|\alpha|=1}\nabla^\alpha B^1(r)\mathbb{N}_1\langle\widehat{w}\rangle \\ \tfrac{ei}{2mc}\sum_{|\alpha|=1}\left(\tfrac{i\hbar}{2}\right)\tfrac{\nabla^\alpha B^2(r)}{\alpha!}\left\langle\mathbb{N}_2^1\widehat{w}\right\rangle &= -\tfrac{e\hbar}{4mc}\sum_{|\alpha|=1}\nabla^\alpha B^2(r)\mathbb{N}_2\langle\widehat{w}\rangle \\ \tfrac{ei}{2mc}\sum_{|\alpha|=k}\left(\tfrac{i\hbar}{2}\right)\tfrac{\nabla^\alpha B^1(r)}{\alpha!}\left\langle\mathbb{N}_3^1\widehat{w}\right\rangle &= -\tfrac{e\hbar}{4mc}\sum_{|\alpha|=1}\nabla^\alpha B^1(r)\mathbb{N}_3\langle\widehat{w}\rangle \\ \tfrac{ei}{2mc}\sum_{|\alpha|=k}\left(\tfrac{i\hbar}{2}\right)\tfrac{\nabla^\alpha B^2(r)}{\alpha!}\left\langle\mathbb{N}_4^1\widehat{w}\right\rangle &= -\tfrac{e\hbar}{4mc}\sum_{|\alpha|=1}\nabla^\alpha B^2(r)\mathbb{N}_4\langle\widehat{w}\rangle\end{aligned} \tag{82}$$

and

$$\begin{aligned}\mathbb{E}n &= \Big[-\tfrac{e}{2}\sum_{|\alpha|=1}\sum_{j=1,2}\nabla^\alpha\phi(r)\mathbb{H}_j^1 + \tfrac{e\hbar}{4mc}\sum_{|\alpha|=1}\sum_{j=1,2}\nabla^\alpha B^3(r)\mathbb{Y}_j^1 - \\ &-\tfrac{e\hbar}{4mc}\sum_{|\alpha|=1}\nabla^\alpha B^1(r)\mathbb{N}_1 + \tfrac{e\hbar}{4mc}\sum_{|\alpha|=1}\nabla^\alpha B^2(r)\mathbb{N}_2 + \\ &+\tfrac{e\hbar}{4mc}\sum_{|\alpha|=1}\nabla^\alpha B^1(r)\mathbb{N}_3 + \tfrac{e\hbar}{4mc}\sum_{|\alpha|=1}\nabla^\alpha B^2(r)\mathbb{N}_4\Big]\langle\widehat{w}\rangle.\end{aligned} \tag{83}$$

Then we can write the moment equations for $n = 0$ and $n = 1$ inthe following form:

$$\begin{cases}\partial_t n + \frac{1}{m}\mathcal{D}J = \frac{ei}{2mc}\left[B^1(r)\mathbb{M}_1 + B^2(r)\mathbb{M}_2\right]n \\ \partial_t J + \frac{1}{m}\mathcal{D}\left(\frac{J\otimes J}{n} + Q(n) + nT\right) = \mathbb{G}J + \mathbb{E}n\end{cases}. \tag{84}$$

By Corollary (1) for a pure state we have $T = 0$, then

$$\begin{cases} \partial_t n + \frac{1}{m}\mathcal{D}J = \frac{ei}{2mc}\left[B^1(r)\mathbb{M}_1 + B^2(r)\mathbb{M}_2\right]n \\ \partial_t J + \frac{1}{m}\mathcal{D}\left(\frac{J\otimes J}{n} + Q(n)\right) = \mathbb{G}J + \mathbb{E}n \end{cases} \tag{85}$$

is a closed Madelung-like system for a pure state.

4. A Two-band Kp Pauli Hamiltonian

Now we consider the following Hamiltonian:

$$H_{kp}^{\text{Pauli}} = \begin{pmatrix} \frac{1}{2m}\left(\frac{\hbar}{i}\nabla - \frac{e}{c}A\right)^2\sigma_0 + V_1^j\sigma_j & \frac{K}{m}\left(\frac{\hbar}{i}\nabla - \frac{e}{c}A\right)\sigma_0 \\ -\frac{K}{m}\left(\frac{\hbar}{i}\nabla - \frac{e}{c}A\right)\sigma_0 & \frac{1}{2m}\left(\frac{\hbar}{i}\nabla - \frac{e}{c}A\right)^2\sigma_0 + V_2^j\sigma_j \end{pmatrix} \tag{86}$$

where $K = \langle u_1|\left(\frac{\hbar}{i}\nabla - \frac{e}{c}A\right)|u_2\rangle$. The Hamiltonoan (86) it is the equivalent of the Hamiltonian (3) considering the electron as a particle. The Hamiltonian (86) describes an 1/2 spin electron that "sees" two energy bands avaible and a Zener tunneling between the two-band is possible. Let us take

$$P = \begin{pmatrix} \left(\frac{\hbar}{i}\nabla - \frac{e}{c}A\right)\sigma_0 & K\sigma_0 \\ -K\sigma_0 & \left(\frac{\hbar}{i}\nabla - \frac{e}{c}A\right)\sigma_0 \end{pmatrix} \tag{87}$$

then

$$P^2 = \begin{pmatrix} \left(\frac{\hbar}{i}\nabla - \frac{e}{c}A\right)^2\sigma_0 & 2K\left(\frac{\hbar}{i}\nabla - \frac{e}{c}A\right)\sigma_0 \\ -2K\left(\frac{\hbar}{i}\nabla - \frac{e}{c}A\right)\sigma_0 & \left(\frac{\hbar}{i}\nabla - \frac{e}{c}A\right)^2\sigma_0 \end{pmatrix} - K^2\begin{pmatrix} \sigma_0 & 0 \\ 0 & \sigma_0 \end{pmatrix} \tag{88}$$

and

$$H_{kp}^{\text{Pauli}} = \frac{1}{2m}P^2 + \frac{K^2}{2m}\begin{pmatrix} \sigma_0 & 0 \\ 0 & \sigma_0 \end{pmatrix} + \begin{pmatrix} V_1^j\sigma_j & 0 \\ 0 & V_2^j\sigma_j \end{pmatrix}. \tag{89}$$

We put $p = \left(\frac{\hbar}{i}\nabla - \frac{e}{c}A\right)$ and we consider

$$H_{kp}^{\text{Pauli}} = \frac{1}{2m}p^2\mathbf{1} - \frac{Kp}{m}W + V_1^jZ_j^1 + V_2^jZ_j^2 \tag{90}$$

where

$$W = \begin{pmatrix} 0 & -\sigma_0 \\ \sigma_0 & 0 \end{pmatrix}, Z_j^1 = \begin{pmatrix} \sigma_j & 0 \\ 0 & 0 \end{pmatrix}, Z_j^2 = \begin{pmatrix} 0 & 0 \\ 0 & \sigma_j \end{pmatrix} \tag{91}$$

then

$$\begin{aligned} i\hbar\partial_t\varrho &= \left[\tfrac{1}{2m}p_x^2 - \tfrac{1}{2m}p_y^2\right]\varrho - \tfrac{k\left(p_x - p_y\right)}{2m}\left[W;\varrho\right]_+ - \tfrac{k\left(p_x + p_y\right)}{2m}\left[W;\varrho\right]_- + \\ &+ \tfrac{\left(V_1^j(x) - V_1^j(y)\right)}{2}\left[Z_j^1;\varrho\right]_+ + \tfrac{\left(V_2^j(x) - V_2^j(y)\right)}{2}\left[Z_j^2;\varrho\right]_+ + \\ &+ \tfrac{\left(V_1^j(x) + V_1^j(y)\right)}{2}\left[Z_j^1;\varrho\right]_- + \tfrac{\left(V_2^j(x) + V_2^j(y)\right)}{2}\left[Z_j^2;\varrho\right]_-. \end{aligned} \tag{92}$$

If $V_1^j = V_2^j = 0$, we get

$$i\hbar\partial_t\varrho = \left[\tfrac{1}{2m}p_x^2 - \tfrac{1}{2m}p_y^2\right]\varrho - \tfrac{k\left(p_x - p_y\right)}{2m}\left[W;\varrho\right]_+ - \tfrac{k\left(p_x + p_y\right)}{2m}\left[W;\varrho\right]_- \ , \tag{93}$$

since

$$[W;\varrho]_\pm = \widehat{W}_\pm\widehat{\varrho} \tag{94}$$

where $\widehat{\varrho} = (\varrho_{1,1}, \ldots \varrho_{1,4}, \ldots, \varrho_{4,1}, \ldots, \varrho_{4,4})^t$ and

$$\widehat{W}_\pm = \begin{pmatrix} \mp W & 0 & -1 & 0 \\ 0 & \mp W & 0 & -1 \\ 1 & 0 & \mp W & 0 \\ 0 & 1 & 0 & \mp W \end{pmatrix}, \tag{95}$$

then equation (93) is equivalent to

$$i\hbar\partial_t\widehat{\varrho} = \frac{\left(p_x^2 - p_y^2\right)}{2m}\widehat{\varrho} - \frac{k\left(p_x - p_y\right)}{2m}\widehat{W}_+\widehat{\varrho} - \frac{k\left(p_x + p_y\right)}{2m}\widehat{W}_-\widehat{\varrho} \tag{96}$$

and

$$i\hbar\partial_t\widehat{\varrho} = \frac{1}{2m}\left[\left(p_x - p_y\right)\widehat{1} - k\widehat{W}_-\right]\left[\left(p_x + p_y\right)\widehat{1} - k\widehat{W}_+\right]\widehat{\varrho} - \frac{k^2}{2m}\widehat{W}_-\widehat{W}_+\widehat{\varrho} \tag{97}$$

with $\widehat{1} = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}$; since $\widehat{W}_-\widehat{W}_+ = \widehat{1}$, we get

$$\partial_t\widehat{\varrho} = \frac{1}{m}\left[\left(\frac{p_x - p_y}{2}\right)\widehat{1} - \frac{k}{2}\widehat{W}_-\right]\left[\left(\frac{p_x + p_y}{i\hbar}\right)\widehat{1} + \frac{ik}{\hbar}\widehat{W}_+\right]\widehat{\varrho} + \frac{ik^2}{2m\hbar}\widehat{\varrho}. \tag{98}$$

Now we write the equation of evolution of the time-dipendent Wigner matrix

$$\partial_t\widehat{w} = -\frac{1}{m}\mathcal{P}\mathcal{D}\widehat{w} + \frac{ik^2}{2m\hbar}\widehat{w} \tag{99}$$

where

$$\mathcal{P} = \left[\left(\frac{\widetilde{p}_x - \widetilde{p}_y}{2}\right)\widehat{1} - \frac{k}{2}\widehat{W}_-\right] \tag{100}$$

and

$$\mathcal{D} = -\left[\left(\frac{\widetilde{p}_x + \widetilde{p}_y}{i\hbar}\right)\widehat{1} + \frac{ik}{\hbar}\widehat{W}_+\right], \tag{101}$$

with

$$\begin{cases} \widetilde{p}_x = \frac{\hbar}{i}\widetilde{\nabla}_x - \frac{e}{c}\widetilde{A}_x = \frac{\hbar}{i}\frac{\nabla_r}{2} + p - \frac{e}{c}\widetilde{A}_x \\ \widetilde{p}_y = \frac{\hbar}{i}\widetilde{\nabla}_y - \frac{e}{c}\widetilde{A}_y = \frac{\hbar}{i}\frac{\nabla_r}{2} - p - \frac{e}{c}\widetilde{A}_y \end{cases} \tag{102}$$

and

$$\begin{cases} \frac{\widetilde{p}_x - \widetilde{p}_y}{2} = p - \frac{e}{2c}\Theta_-(A) \\ \frac{\widetilde{p}_x + \widetilde{p}_y}{i\hbar} = -\nabla_r - \frac{e}{2c}\Theta_+(A) \end{cases} ; \tag{103}$$

moreover

$$\mathcal{D} = \left[\nabla_r\widehat{1} + \frac{e}{2c}\Theta_+(A)\,\widehat{1} + \frac{ik}{\hbar}\widehat{W}_+\right] \tag{104}$$

and

$$\mathcal{P} = \left[p\widehat{1} - \frac{e}{2c}\Theta_-(A)\,\widehat{1} - \frac{k}{2}\widehat{W}_-\right]. \tag{105}$$

From (100) and (101) we get

$$\mathcal{P}^{\otimes 2} = \frac{1}{4}\left[\left(\widetilde{p}_x - \widetilde{p}_y\right)^{\otimes 2} - 2k\left(\widetilde{p}_x - \widetilde{p}_y\right)\widehat{1}\otimes\widehat{W}_- + k^2\widehat{W}_-\otimes\widehat{W}_-\right] \tag{106}$$

and

$$\mathcal{D}^{\otimes 2} = \left[\left(\frac{\widetilde{p}_x + \widetilde{p}_y}{i\hbar}\right)^{\otimes 2} + 2\frac{ik^2}{\hbar}\left(\frac{\widetilde{p}_x + \widetilde{p}_y}{i\hbar}\right)\widehat{1}\otimes\widehat{W}_+ - \frac{k^2}{\hbar^2}\widehat{W}_+\otimes\widehat{W}_+\right] \tag{107}$$

then

$$\begin{aligned} \mathcal{P}^{\otimes 2} \quad &= -\frac{\hbar^2}{4}\left(\frac{\widetilde{p}_x + \widetilde{p}_y}{i\hbar}\right)^{\otimes 2} - \widetilde{p}_x\otimes\widetilde{p}_y - \frac{k}{2}\left(\widetilde{p}_x - \widetilde{p}_y\right)\widehat{1}\otimes\widehat{W}_- + \frac{k^2}{4}\widehat{W}_-\otimes\widehat{W}_- \\ &= -\frac{\hbar^2}{4}\mathcal{D}^{\otimes 2} - \frac{i\hbar k}{2}\mathcal{D}\otimes\widehat{W}_+ - k\mathcal{P}\otimes\widehat{W}_- - \widetilde{p}_x\otimes\widetilde{p}_y + \frac{k^2}{4}\left[\widehat{W}_+^{\otimes 2} - \widehat{W}_-^{\otimes 2}\right] \end{aligned} \tag{108}$$

and, since $\left[\widehat{W}_+^2 - \widehat{W}_-^2\right] = 0$,

$$\left\langle\mathcal{P}^{\otimes 2}\widehat{w}\right\rangle = -\frac{\hbar^2}{4}\mathcal{D}^{\otimes 2}n - \frac{i\hbar k}{2}\widehat{W}_+\mathcal{D}n - k\widehat{W}_-J - \left\langle\widetilde{p}_x\otimes\widetilde{p}_y\widehat{w}\right\rangle. \tag{109}$$

By (106), (107) and (108) we get

$$\begin{aligned}\langle \mathcal{P}\widehat{w}\rangle^{\otimes 2} &= \left\langle\left(\tfrac{\widetilde{p}_x-\widetilde{p}_y}{2}\right)\widehat{w}\right\rangle^{\otimes 2} - k\left\langle\left(\tfrac{\widetilde{p}_x-\widetilde{p}_y}{2}\right)\widehat{w}\right\rangle \otimes \left\langle \widehat{W}_-\widehat{w}\right\rangle + \tfrac{k^2}{4}\left\langle \widehat{W}_-\widehat{w}\right\rangle^{\otimes 2}\\ &= -\tfrac{\hbar^2}{4}\left\langle\left(\tfrac{\widetilde{p}_x+\widetilde{p}_y}{i\hbar}\right)\widehat{w}\right\rangle^{\otimes 2} - k\left\langle\left(\tfrac{\widetilde{p}_x-\widetilde{p}_y}{2}\right)\widehat{w}\right\rangle \otimes \left\langle \widehat{W}_-\widehat{w}\right\rangle + \tfrac{k^2}{4}\left\langle \widehat{W}_-\widehat{w}\right\rangle^{\otimes 2} +\\ &+ \langle \widetilde{p}_x\widehat{w}\rangle \otimes \left\langle \widetilde{p}_y\widehat{w}\right\rangle\end{aligned} \tag{110}$$

and

$$\langle \mathcal{D}\widehat{w}\rangle^{\otimes 2} = \left\langle\left(\frac{\widetilde{p}_x+\widetilde{p}_y}{i\hbar}\right)\widehat{w}\right\rangle^{\otimes 2} + \frac{2ik}{\hbar}\left\langle\left(\frac{\widetilde{p}_x+\widetilde{p}_y}{i\hbar}\right)\widehat{w}\right\rangle \otimes \left\langle \widehat{W}_+\widehat{w}\right\rangle - \frac{k^2}{\hbar^2}\left\langle \widehat{W}_+\widehat{w}\right\rangle^{\otimes 2}; \tag{111}$$

moreover by (108), (109) and (110) it follows that

$$\begin{aligned}\langle \mathcal{P}\widehat{w}\rangle^{\otimes 2} &= -\tfrac{\hbar^2}{4}(\mathcal{D}n)^{\otimes 2} + \langle \widetilde{p}_x\widehat{w}\rangle \otimes \left\langle \widetilde{p}_y\widehat{w}\right\rangle +\\ &+ \tfrac{i\hbar k}{2}(\mathcal{D}n)\otimes\left(\widehat{W}_+n\right) - kJ\otimes\left(\widehat{W}_-n\right) + \tfrac{k^2}{4}\left[\left(\widehat{W}_+n\right)^{\otimes 2} - \left(\widehat{W}_-n\right)^{\otimes 2}\right]\end{aligned} \tag{112}$$

then

$$\langle \mathcal{P}\otimes P\widehat{w}\rangle = \frac{J\otimes J}{n} + Q(n) + V(n) - nT \tag{113}$$

where

$$\begin{aligned}&Q(n) = -\tfrac{\hbar^2}{4}\left(\mathcal{D}^{2\otimes}n - \tfrac{(\mathcal{D}n)^{2\otimes}}{n}\right);\\ &nT = \left\langle \widetilde{\mathcal{P}}_x\otimes\widetilde{\mathcal{P}}_y\widehat{w}\right\rangle - \tfrac{\langle\widetilde{\mathcal{P}}_x\widehat{w}\rangle\otimes\langle\widetilde{\mathcal{P}}_y\widehat{w}\rangle}{n};\\ &V(n) = k\left[\tfrac{J\otimes\left(\widehat{W}_-n\right)}{n} - \widehat{W}_-J\right] - \tfrac{i\hbar k}{2}\left[\tfrac{(\mathcal{D}n)\otimes\left(\widehat{W}_+n\right)}{n} + \widehat{W}_+\mathcal{D}n\right] - \tfrac{k^2}{4}\left[\tfrac{\left(\widehat{W}_+n\right)^{\otimes 2}}{n} - \tfrac{\left(\widehat{W}_-n\right)^{\otimes 2}}{n}\right].\end{aligned} \tag{114}$$

Proposition 2. *Let $w = \mathcal{W}\varrho$ be the Wigner transform of the mixed state $\varrho = \sum_{s=1}^{+\infty}\lambda_s\varrho^s$, where $\lambda_s \geq 0$, $\sum_{s=1}^{+}\lambda_s = 1$ and each ϱ^s is a pure-state; then*

$$T = \sum_{s=1}^{+\infty}\frac{\lambda_s n^s}{n}\left(\frac{\langle\widetilde{p}_x\widehat{w}^s\rangle}{n^s} - \frac{\langle\widetilde{p}_x\widehat{w}\rangle}{n}\right)\otimes\left(\frac{\langle\widetilde{p}_x\widehat{w}^s\rangle}{(n^s)^*} - \frac{\langle\widetilde{p}_x\widehat{w}\rangle}{(n)^*}\right)^* \tag{115}$$

where $\widehat{w}^s = \mathcal{W}\widehat{\varrho}^s$, $n^s = \langle w^s\rangle$ and $*$ denotes adjunction:

$$\begin{pmatrix} a\\ b\\ c\\ d\end{pmatrix}^* = \begin{pmatrix} \overline{a}\\ \overline{b}\\ \overline{c}\\ \overline{d}\end{pmatrix}.$$

Corollary 2. *If $w = \mathcal{W}\varrho$ is the Wigner transform of a pure-state density matrix, then $T = 0$.*

From (99) and (113) we obtain

$$\partial_t J + \frac{1}{m}\mathcal{D}\left(\frac{J\otimes J}{n} + Q(n) + nT - V(n)\right) - \frac{ik^2}{2m\hbar}J = 0 \tag{116}$$

and for $T = 0$

$$\begin{cases} \partial_t n + \frac{1}{m}\mathcal{D}J - \frac{ik^2}{2m\hbar}n = 0\\ \partial_t J + \frac{1}{m}\mathcal{D}\left(\frac{J\otimes J}{n} + Q(n) - V(n)\right) - \frac{ik^2}{2m\hbar}J = 0\end{cases} \tag{117}$$

is a closed system of Madelung-like QHD equations for a ”free” two-band-gauge Kp Pauli Hamiltonian.

4.1 Moments of the Potential Terms

In the following we will using a multi-index notation: a multi-index $\alpha = (\alpha_1, ..., \alpha_N)$ is a -uple of non negative integer and $|\alpha| = \alpha_1 + \cdots + \alpha_N$, $\alpha! = \alpha_1!\cdots\alpha_N!$; moreover $\alpha \leq \beta$ if and only if $\alpha_i \leq \beta_i$ for $i = 1, ..., N$. For the potential we get

$$\left[Z_j^h;\varrho\right]_\pm = \mathbb{H}_{j,\pm}^h\widehat{\varrho} \tag{118}$$

where

$$\mathbb{H}^h_{j,\pm} = \pm\mathbb{Z}^h_{j,1} + \mathbb{Z}^h_{j,2} \tag{119}$$

for $h = 1,2$ e $j = 1,2,3$.

By (118) and (119) we have

$$\begin{array}{l} \frac{\left(V^j_h(x)-V^j_h(y)\right)}{2}\left[Z^h_j;\varrho\right]_+ = \frac{\left(V^j_h(x)-V^j_h(y)\right)}{2}\mathbb{H}^h_{j,+}\widehat{\varrho} \\ \frac{\left(V^j_h(x)+V^j_h(y)\right)}{2}\left[Z^h_j;\varrho\right]_- = \frac{\left(V^j_h(x)+V^j_h(y)\right)}{2}\mathbb{H}^h_{j,-}\widehat{\varrho} \end{array} \tag{120}$$

and by Wigner transform

$$\begin{array}{l} \frac{1}{2}\Theta_-\left(V^j_h\right)\mathbb{H}^h_{j,+}\widehat{w} \\ \frac{1}{2}\Theta_+\left(V^j_h\right)\mathbb{H}^h_{j,-}\widehat{w} \end{array} \tag{121}$$

for $h = 1,2$ and $j = 1,2,3$.

From (120) and (121) it follows

$$\begin{aligned} \frac{1}{2}\left[\Theta_-\left(V^j_h\right)\mathbb{H}^h_{j,+}\widehat{w} + \Theta_+\left(V^j_h\right)\mathbb{H}^h_{j,-}\widehat{w}\right] &= \frac{1}{2}\left[\left(\Theta_-\left(V^j_h\right) - \Theta_+\left(V^j_h\right)\right)\mathbb{Z}^h_{j,1}\right]\widehat{w}+ \\ &+\frac{1}{2}\left[\left(\Theta_-\left(V^j_h\right) + \Theta_+\left(V^j_h\right)\right)\mathbb{Z}^h_{j,2}\right]\widehat{w} \\ &= -\sum_{k=0}^{+\infty}\sum_{|\alpha|=k}\left(-\frac{i\hbar}{2}\right)^k\frac{\nabla^\alpha V^j_h(r)}{\alpha!}\mathbb{Z}^h_{j,1}\nabla^\alpha_p\widehat{w}+ \\ &+\sum_{k=0}^{+\infty}\sum_{|\alpha|=k}\left(\frac{i\hbar}{2}\right)^k\frac{\nabla^\alpha V^j_h(r)}{\alpha!}\mathbb{Z}^h_{j,2}\nabla^\alpha_p\widehat{w} \end{aligned} \tag{122}$$

By (122) we can write

$$\begin{aligned} \left\langle\mathcal{P}^{\otimes n}\mathcal{V}\widehat{w}\right\rangle &= \sum_{k=0}^{+\infty}\sum_{|\alpha|=k}\left(-\frac{i\hbar}{2}\right)^k\frac{\nabla^\alpha V^j_h}{\alpha!}\left\langle\mathcal{P}^{\otimes n}\mathbb{Z}^h_{j,1}\nabla^\alpha_p\widehat{w}\right\rangle + \\ &+\sum_{k=0}^{+\infty}\sum_{|\alpha|=k}\left(\frac{i\hbar}{2}\right)^k\frac{\nabla^\alpha V^j_h}{\alpha!}\left\langle\mathcal{P}^{\otimes n}\mathbb{Z}^h_{j,2}\nabla^\alpha_p\widehat{w}\right\rangle \end{aligned} \tag{123}$$

where $\mathcal{P}^{\otimes n} = \left(p\widehat{1} - \frac{K}{2}\widehat{W}_-\right)^{\otimes n}$.

If we consider a single component $\left(p\widehat{1} - \frac{K}{2}\widehat{W}_-\right)^\beta$ of the tensor product $\mathcal{P}^{\otimes n}$, integration by parts yelds

$$\left\langle\left(p\widehat{1} - \frac{K}{2}\widehat{W}_-\right)^\beta\mathbb{Z}^h_{j,\nu}\nabla^\alpha_p\widehat{w}\right\rangle = \begin{cases} (-1)^{|\alpha|}\left\langle\left(p\widehat{1} - \frac{K}{2}\widehat{W}_-\right)^{\beta-\alpha}\mathbb{Z}^h_{j,\nu}\widehat{w}\right\rangle & \text{if } \alpha\leq\beta \\ 0 & \text{if } \alpha\nleq\beta \end{cases} \tag{124}$$

then

$$\left\langle\mathcal{P}^{\otimes n}\mathbb{Z}^h_{j,\nu}\nabla^\alpha_p\widehat{w}\right\rangle = (-1)^{|\alpha|}\left\langle\mathcal{P}^{\otimes(n-\alpha)}\mathbb{Z}^h_{j,\nu}\widehat{w}\right\rangle. \tag{125}$$

Let $\mathcal{P}^\gamma = \left(p\widehat{1} - \frac{K}{2}\widehat{W}_-\right)^\gamma$, with $|\gamma| = n-k$, be any component of $\mathcal{P}^{\otimes(n-\alpha)}$; then we can write

$$\begin{aligned} \left(p\widehat{1} - \frac{k}{2}\widehat{W}_-\right)^\gamma &= \sum_{\delta\leq\gamma}\binom{\gamma}{\delta}\left(p\widehat{1}\right)^\delta\left(-\frac{k}{2}\widehat{W}_-\right)^{\gamma-\delta} \\ &= \sum_{\delta\leq\gamma}\binom{\gamma}{\delta}p^\delta\left(-\frac{k}{2}\right)^{\gamma-\delta}\left(\widehat{W}_-\right)^{|\gamma-\delta|} \end{aligned} \tag{126}$$

and

$$\begin{aligned} \left(p\widehat{1} - \frac{k}{2}\widehat{W}_-\right)^\gamma\mathbb{Z}^h_{j,\nu} &= \sum_{\delta\leq\gamma}\binom{\gamma}{\delta}p^\delta\left(-\frac{k}{2}\right)^{\gamma-\delta}\left(\widehat{W}_-\right)^{|\gamma-\delta|}\mathbb{Z}^h_{j,\nu} \\ &= \sum_{\delta\leq\gamma}\binom{\gamma}{\delta}\left(-\frac{k}{2}\right)^{\gamma-\delta}\left(\widehat{W}_-\right)^{|\gamma-\delta|}\mathbb{Z}^h_{j,\nu}p^\delta \\ &= \sum_{\delta\leq\gamma}\binom{\gamma}{\delta}\left(-\frac{k}{2}\right)^{\gamma-\delta}\left(\widehat{W}_-\right)^{|\gamma-\delta|}\mathbb{Z}^h_{j,\nu}\left(\mathcal{P} + \frac{k}{2}\widehat{W}_-\right)^\delta, \end{aligned} \tag{127}$$

since $\mathcal{P}\widehat{W}_- = \widehat{W}_-\mathcal{P}$, we obtain

$$\begin{aligned}\left(p\widehat{1} - \tfrac{k}{2}\widehat{W}_-\right)^{\gamma}\mathbb{Z}^h_{j,v} &= \sum_{\eta\le\delta\le\gamma}\binom{\gamma}{\delta}\binom{\delta}{\eta}\left(-\tfrac{k}{2}\right)^{\gamma-\delta}\left(\widehat{W}_-\right)^{|\gamma-\delta|}\mathbb{Z}^h_{j,v}\left(\tfrac{k}{2}\widehat{W}_-\right)^{\delta-\eta}\mathcal{P}^{\eta} \\ &= \sum_{\eta\le\delta\le\gamma}\binom{\gamma}{\delta}\binom{\delta}{\eta}\left(-\tfrac{k}{2}\right)^{\gamma-\delta}\left(\widehat{W}_-\right)^{|\gamma-\delta|}\mathbb{Z}^h_{j,v}\left(\widehat{W}_-\right)^{|\delta-\eta|}\left(\tfrac{k}{2}\right)^{\delta-\eta}\mathcal{P}^{\eta}\end{aligned} \tag{128}$$

This shows that each component of $\left\langle\mathcal{P}^{\otimes(n-\alpha)}\mathbb{Z}^h_{j,v}\widehat{w}\right\rangle$ is a linear combination of terms $\langle\mathcal{P}^{\gamma}\widehat{w}\rangle$, with $|\gamma| \le n-k$. In conclusion we get

$$\left\langle\mathcal{P}^{\otimes n}\mathcal{V}\widehat{w}\right\rangle = \sum_{k=0}^{n}\sum_{|\alpha|=k}\left(\frac{i\hbar}{2}\right)^{k}\frac{\nabla^{\alpha}V^j_h}{\alpha!}\left\langle\mathcal{P}^{\otimes(n-\alpha)}\mathbb{H}^h_{j,k}\widehat{w}\right\rangle \tag{129}$$

where $\mathbb{H}^h_{j,k} = \mathbb{Z}^h_{j,1} + (-1)^k\,\mathbb{Z}^h_{j,2}$ and $\left\langle\mathcal{P}^{\otimes(n-\alpha)}\mathbb{H}^h_{j,k}\widehat{w}\right\rangle$ is a linear combination of components of $\langle\mathcal{P}^{\gamma}\widehat{w}\rangle$, with $|\gamma| \le n-|\alpha|$.

5. Two-band Madelung Equations

Proposition (2) and corollary (2) imply that the $n = 0$ and $n = 1$ moment equations for a pure state are closed and yield a analogue of QHD Madelung equations.

For $n = 0$, we get

$$\langle\mathcal{V}\widehat{w}\rangle = V^j_h(r)\,\mathbb{H}^h_{j,0}\langle\widehat{w}\rangle \tag{130}$$

where $\mathbb{H}^h_{j,0} = \mathbb{Z}^h_{j,1} + \mathbb{Z}^h_{j,2}$.

For $n = 1$ and $k = 0$, we have

$$V^j_h(r)\left\langle\mathcal{P}\mathbb{H}^h_{j,0}\widehat{w}\right\rangle = V^j_h(r)\,\mathbb{H}^h_{j,0}\langle\mathcal{P}\widehat{w}\rangle + V^j_h(r)\left[\widehat{W}_-;\mathbb{H}^h_{j,0}\right]\langle\widehat{w}\rangle. \tag{131}$$

For $n = 1$ and $k = 1$, we get

$$\left(\frac{i\hbar}{2}\right)\nabla V^j_h(r)\left\langle\mathbb{H}^h_{j,1}\widehat{w}\right\rangle = \left(\frac{i\hbar}{2}\right)\nabla V^j_h(r)\,\mathbb{H}^h_{j,1}\langle\widehat{w}\rangle \tag{132}$$

where $\mathbb{H}^h_{j,1} = \mathbb{Z}^h_{j,1} - \mathbb{Z}^h_{j,2}$; then

$$\langle\mathcal{P}\mathcal{V}\widehat{w}\rangle = V^j_h(r)\,\mathbb{H}^h_{j,0}J + V^j_h(r)\left[\widehat{W}_-;\mathbb{H}^h_{j,0}\right]n + \left(\frac{i\hbar}{2}\right)\nabla V^j_h(r)\,\mathbb{H}^h_{j,1}n. \tag{133}$$

Using (92), (130) and (5.4) we can write the moment equations for $n = 0$ and $n = 1$ in the following form

$$\left\{\begin{array}{r}\partial_t n + \frac{1}{m}\mathcal{D}J - \frac{ik^2}{2m\hbar}n = V^j_h(r)\,\mathbb{H}^h_{j,0}n \\ \partial_t J + \frac{1}{m}\mathcal{D}\left(\frac{J\otimes J}{n} + Q(n) - V(n)\right) - \frac{ik^2}{2m\hbar}J = V^j_h(r)\,\mathbb{H}^h_{j,0}J + \left[V^j_h(r)\left[\widehat{W}_-;\mathbb{H}^h_{j,0}\right] + \frac{i\hbar}{2}\nabla V^j_h(r)\,\mathbb{H}^h_{j,1}\right]n\end{array}\right. . \tag{134}$$

5.1 The $\mathbb{Z}^h_{j,v}$ Matrices

$$\mathbb{Z}^1_{1,1} = \begin{pmatrix}\sigma_1&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&\sigma_1&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&\sigma_1&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&\sigma_1&0\\0&0&0&0&0&0&0&0\end{pmatrix} \quad \mathbb{Z}^1_{1,2} = \begin{pmatrix}0&0&\mathbf{1}&0&0&0&0&0\\0&0&0&\mathbf{1}&0&0&0&0\\\mathbf{1}&0&0&0&0&0&0&0\\0&\mathbf{1}&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}$$

$$\mathbb{Z}^2_{1,1} = \begin{pmatrix}0&0&0&0&0&0&0&0\\0&\sigma_1&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&\sigma_1&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&\sigma_1&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&\sigma_1\end{pmatrix} \quad \mathbb{Z}^2_{1,2} = \begin{pmatrix}0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&\mathbf{1}&0\\0&0&0&0&0&0&0&\mathbf{1}\\0&0&0&0&\mathbf{1}&0&0&0\\0&0&0&0&0&\mathbf{1}&0&0\end{pmatrix}$$

$$\mathbb{Z}^1_{2,1} = \begin{pmatrix} \sigma_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sigma_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \mathbb{Z}^1_{2,2} = \begin{pmatrix} 0 & 0 & -i\mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i\mathbf{1} & 0 & 0 & 0 & 0 \\ i\mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & i\mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbb{Z}^2_{2,1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_2 \end{pmatrix} \quad \mathbb{Z}^2_{2,2} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -i\mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i\mathbf{1} \\ 0 & 0 & 0 & 0 & i\mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i\mathbf{1} & 0 & 0 \end{pmatrix}$$

$$\mathbb{Z}^1_{3,1} = \begin{pmatrix} a_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sigma_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \mathbb{Z}^1_{3,2} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbb{Z}^2_{1,1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_3 \end{pmatrix} \quad \mathbb{Z}^2_{3,2} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

with

$$a_3 = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \quad b_3 = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}.$$

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