Prime Labeling for Some Cycle Related Graphs

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Abstract

Here we investigate prime labeling for some cycle related graphs. We also discuss prime labeling in the context of some graph operations namely fusion, duplication and vertex switching in cycle C_n.

Keywords: Prime labeling, Duplication, Vertex switching

1. Introduction

We begin with finite, connected and undirected graph $G = (V(G), E(G))$ without loops and multiple edges. Here elements of sets $V(G)$ and $E(G)$ are known as vertices and edges respectively. In the present work C_n denotes cycle with n vertices and $N(v)$ denotes the set of all neighboring vertices of v. For all other terminology and notations in graph theory we follow (Gross, J., 1999) where as for number theory we follow (Niven, I., 1972). We will give brief summary of definitions which are useful for the present investigations.

Definition 1.1 If the vertices of the graph are assigned values subject to certain conditions is known as *graph labeling*. Enough literature is available in printed as well as in electronic form on different types of graph labeling and more than 1000 research papers have been published so far in past four decades. A current survey of various graph labeling problems can be found in (Gallian, J., 2009).

Following three are the common features of any graph labeling problem.

(1) a set of numbers from which vertex labels are assigned;

(2) a rule that assigns a value to each edge;

(3) a condition that these values must satisfy.

The present work is targeted to discuss one such labeling known as prime labeling defined as follows.

Definition 1.2

Let $G = (V(G), E(G))$ be a graph with p vertices. A bijection $f : V(G) \to \{1, 2, \ldots, p\}$ is called a *prime labeling* if for each edge $e = uv$, $gcd(f(u), f(v)) = 1$. A graph which admits prime labeling is called a *prime graph*.

The notion of a prime labeling was introduced by Roger Entringer and was discussed in a paper by (Tout, A,1982, p.365-368). Many researchers have studied the prime graphs. For e.g. in (Fu, H., 1994, p.181-186) have proved that path P_n on n vertices is a prime graph. In (Deretsky, T., 1991, p.359-369) have proved that the cycle C_n on n vertices is a prime graph. In (Lee, S., 1988, p.59-67) have proved that wheel W_n is a prime graph if and only if n is even. Around 1980 Roger Etringer conjectured that *All trees have prime labeling* which is not settled till today. The prime labeling for planar grid is investigated by (Sundaram, M., 2006, p.205-209).

Definition 1.3 An *independent set* of vertices in a graph G is a set of mutually non-adjacent vertices.

Definition 1.4 The *independence number* of a graph G is the maximum cardinality of an independent set of vertices. It is denoted by $ind(G)$ or $\alpha(G)$.

Definition 1.5 Let u and v be two distinct vertices of a graph G. A new graph G_1 is constructed by *identifying(fusing)* two vertices u and v by a single vertex x is such that every edge which was incident with either u or v in G is now incident with x in G_1.

Definition 1.6 *Duplication* of a vertex v_k of graph G produces a new graph G_1 by adding a vertex v'_k with $N(v'_k) = N(v_k)$.

In other words a vertex v'_k is said to be duplication of v_k if all the vertices which are adjacent to v_k are now adjacent to v'_k

also.

Definition 1.7 A *vertex switching* G_v of a graph G is obtained by taking a vertex v of G, removing all the edges incident with v and adding edges joining v to every vertex which are not adjacent to v in G.

Definition 1.8 (Shee, S.,& Ho, Y., 1996, p.221-229) Let $G_1, G_2, \ldots, G_n$, $n \geq 2$ be n copies of a fixed graph G. The graph obtained by adding an edge between G_i and G_{i+1} for $i = 1, 2, \ldots n - 1$ is called the *path union* of G.

In the present work we prove that the graphs obtained by identifying any two vertices, duplicating arbitrary vertex and switching of any vertex in cycle C_n admit prime labeling. We also prove that the graph obtained by path union of cycle C_n is a prime graph except for odd n. In addition to this we show that the graph obtained by joining two copies of cycle C_n by a path of arbitrary length is a prime graph except n and k both are odd.

2. Main Results

Theorem 2.1 The graph obtained by identifying any two vertices v_i and v_j (where $d(v_i, v_j) \geq 3$) of cycle C_n is a prime graph.

Proof: Let C_n be the cycle with vertices $v_1, v_2, \ldots, v_n$ and the vertex v_1 be fused with v_m where $m \leq \lceil n/2 \rceil$. Denote the resultant graph as G. Here we note that $|V(G)| = n - 1$. It is obvious that identifying two vertices of cycle C_n produces connected graph which includes two edge disjoint cycles C_{m-1} and C_{n-m+1}.

To define labeling $f : V(G) \to \{1, 2, \ldots, n - 1\}$ following cases are to be considered.

Case 1: If $n \equiv 0(mod 2)$

$$f(v_i) = i; 1 \leq i < m$$
$$f(v_i) = i - 1; m + 1 \leq i \leq n$$

Case 2: If $n \equiv 1(mod 2)$

$$f(v_i) = i; 1 \leq i < m$$
$$f(v_i) = i - 1; m + 1 \leq i \leq n$$

In each case described above the graph under consideration admits prime labeling i.e. G is a prime graph.

Remark 2.2 (i)when $m > \lceil \frac{n}{2} \rceil$ identifying two vertices will repeat all the graphs which are already considered earlier for $m \leq \lceil \frac{n}{2} \rceil$.

(ii)when $d(v_i, v_j) < 3$ then fusion yields a graph which is not simple and it is not desirable for prime labeling.

Illustration 2.3 Consider a graph G obtained by identifying the vertex v_1 with v_6 of cycle C_{11}. It is the case related to $n \equiv 1(mod 2)$. The labeling is as shown in *Fig 1*.

Theorem 2.4 The graph obtained by duplicating arbitrary vertex of cycle C_n is a prime graph.

Proof: Let v_k be any arbitrary vertex of cycle C_n, v'_k be its duplicated vertex and G be the graph resulted due to duplication of vertex v_k. Then $|V(G)| = n + 1$. Define labeling $f : V(G) \to \{1, 2, \ldots, n + 1\}$ as follows.

$$f(v_k) = 4$$
$$f(v'_k) = 2$$
$$f(v_{k+1}) = 3$$
$$f(v_{k+i}) = i + 3; 1 < i < n - 1$$
$$f(v_{k+n-1}) = 1$$

Then for any edge $e = v_i v_j \in G$, $gcd(f(v_i), f(v_j)) = 1$.

Thus function defined above provides prime labeling for graph G i.e. graph G under consideration is a prime graph.

Illustration 2.5 Consider a graph G obtained by duplicating a vertex v_1 in cycle C_6. The labeling is as shown in *Fig 2*.

Theorem 2.6 The switching of any vertex in cycle C_n produces a prime graph.

Proof: Let $G = C_n$ and $v_1, v_2, \ldots, v_n$ be the successive vertices of C_n and G_v denotes the vertex switching of G with respect to the vertex v. It is obvious that $|V(G_v)| = 2n - 5$. Without loss of generality we initiate the labeling from v_1 and proceed in the clockwise direction.

Define labeling $f : V(G_v) \to \{1, 2, \ldots, n\}$ as follows.

$$f(v_i) = i; 1 \leq i \leq n$$

Then for any edge $e = v_i v_j \in G_v$, $gcd(f(v_i), f(v_j)) = 1$.

Thus f admits a prime labeling and consequently G_v is a prime graph.

Illustration 2.7 Consider a graph G obtained by switching the vertex v_1 of cycle C_9. The corresponding prime labeling is shown in *Fig 3*.

Theorem 2.8 The graph obtained by the path union of finite number of copies of cycle C_n is a prime graph except for odd n.

Proof: Let G be the path union of cycle C_n and $G_1, G_2, \ldots, G_k$ be k copies of the cycle C_n. We note that $|V(G)| = nk$. Let us denote the successive vertices of the graph G_i by $u_{i1}, u_{i2}, \ldots, u_{in}$. Let $e_i = u_{i1}u_{(i+1)1}$ be the edge joining G_i and G_{i+1} for $i = 1, 2, \ldots, k - 1$.

To define labeling $f : V(G) \to \{1, 2, \ldots, nk\}$ following cases are to be considered.

Case 1: If $n \equiv 0(mod2)$

Subcase I: $k \equiv 0(mod2)$

$$f(u_{ij}) = (i - 1)n + j; 1 \le i \le k, 1 \le j \le n$$

Subcase II: $k \equiv 1(mod2)$

$$f(u_{ij}) = (i - 1)n + j; 1 \le i \le k, 1 \le j \le n$$

Here we note that $gcd(u_{i1}, u_{(i+1)1}) = 1$ for each $1 \le i \le k$ because

$$
\begin{aligned}
gcd(u_{i1}, u_{(i+1)1}) &= gcd(2mi - (2m - 1), 2m(i + 1) - (2m - 1)), \text{ where } 2m = n \\
&= gcd(2mi - (2m - 1), 2m(i + 1) - (2m - 1) - (2mi - (2m - 1))) \\
&= gcd(2mi - (2m - 1), 2m) \\
&= gcd(2mi - (2m - 1) - 2mi, 2m) \\
&= gcd(-(2m - 1), 2m) \\
&= gcd((2m - 1), 2m) \\
&= gcd((2m - 1), 2m - (2m - 1)) \\
&= gcd(2m - 1, 1) \\
&= 1
\end{aligned}
$$

Case 2: If $n \equiv 1(mod2)$

In this case $|\alpha(G)| < \lfloor n/2 \rfloor$ and as proved in(Fu, H., 1994, p.181-186) the graph is not a prime graph.

Thus we proved that the graph obtained by the path union of finite number of copies of cycle C_n is a prime graph except for odd n.

Illustration 2.9 Consider a graph G obtained by path union of three copies of the cycle C_{10}. It is the case related to $n \equiv 0(mod2)$ and $k \equiv 1(mod2)$. The prime labeling is as shown in *Fig 4*.

Theorem 2.10 The graph obtained by joining two copies of cycle C_n by a path P_k is a prime graph except n and k both are odd.

Proof: Let G be the graph obtained by joining two copies of cycle C_n by a path P_k. We note that $|V(G)| = 2n + k - 2$. Let $u_1, u_2, \ldots, u_n$ be the vertices of first copy of cycle C_n and $v_1, v_2, \ldots, v_n$ be the vertices of second copy of cycle C_n. Let $w_1, w_2, \ldots, w_k$ be the vertices of path P_k with $u_1 = w_1$ and $v_1 = w_k$.

To define labeling $f : V(G) \to \{1, 2, \ldots, 2n, 2n + 1, \ldots, 2n + k - 2\}$ following cases are to be considered.

Case 1: If $n \equiv 0(mod2)$

Subcase I: $k \equiv 0(mod2)$

$$
\begin{aligned}
f(u_i) &= i; 1 \le i \le n \\
f(v_i) &= n + i; 1 \le i \le n \\
f(w_j) &= 2n + k - j; 1 < j < k
\end{aligned}
$$

Subcase II: $k \equiv 1(mod2)$

$$
\begin{aligned}
f(u_i) &= i; 1 \le i \le n \\
f(v_i) &= n + i; 1 \le i \le n \\
f(w_j) &= 2n + k - j; 1 < j < k
\end{aligned}
$$

Case 2: If $n \equiv 1(mod2)$
Subcase I: $k \equiv 0(mod2)$

$$f(u_i) = i; 1 \le i \le n$$
$$f(v_i) = n + i + 1; 1 \le i \le n$$
$$f(w_2) = n + 1$$
$$f(w_j) = 2n + k - j + 1; 2 < j < k$$

Subcase II: $k \equiv 1(mod2)$

In this case $|\alpha(G)| < \lfloor n/2 \rfloor$. Then as proved in (Fu, H., 1994, p.181-186) the graph is not a prime graph.

Thus we derived that the graph obtained by joining two copies of cycle C_n by a path P_k is a prime graph except n and k both are odd.

Illustration 2.11 Consider a graph G obtained by joining two copies of the cycle C_{10} by a path P_3. It is the case related to $n \equiv 0(mod2)$ and $k \equiv 1(mod2)$. The labeling is as shown in *Fig 5*.

3. Concluding Remarks

Labeled graph is the topic of current interest due to its diversified applications. We investigate five new results on prime labeling. It is an effort to relate the prime labeling and some graph operations. This approach is novel as it provides prime labeling for the larger graph resulted due to certain graph operations on a given graph.

Open problems

(1) It is possible to investigate similar results for other graph families.

(2) There is a scope to derive similar results corresponding to different graph labeling techniques.

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Figure 1. Fusion of v_1 with v_6 in C_{11} and Prime labeling

Figure 2. Vertex duplication in C_6 and Prime labeling

Figure 3. Vertex switching in C_9 and Prime labeling

Figure 4. Path union of C_{10} and Prime labeling

Figure 5. Path joining two copies of C_{10} and Prime labeling