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On Statistical Convergence in Metric Spaces

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Abstract

The statistical convergence in metric spaces is considered. Its equivalence to the statistical fundamentality in complete metric spaces is proved. Introduced the concept of p-strong convergence, and proved its equivalence to the statistical convergence. Tauberian theorems concerning statistical convergence in metric spaces are given.

Keywords: statistical convergence, p-strong convergence, a metric space, Tauberian theorems

1. Introduction

The idea of statistical convergence was first proposed by A.Zigmund (Zygmund, 1979) in his famous monograph where he talked about "almost convergence". The first definition of it was given by H. Fast (Fast, 1951) and H. Steinhaus (Steinhaus, 1951). Later, this concept has been generalized in many directions. More details on this matter and on applications of this concept can be found in (T.C. Brown and A.R. Freedman, 1990; B.J. Connor, 1988; R. Erdös and G. Tenenbaum, 1989; A.R. Freedman and J.J. Sember, 1981; J.A. Fridy, 1985; J.A. Fridy and M.K. Khan, 1998; M. Kuchukaslan, U. Deger and O. Dovgoshey, 2012; M. Kuchukaslan and U. Deger, 2012; I.J. Maddox, 1988; D. Maharam, 1976; G.D. Maio and L.D.R. Kocinac, 2008; H.I.Miller, 1995; G.M. Peterson, 1966; I.J. Schoenberg, 1959). It should be noted that the methods of non-convergent sequences have long been known and they include e.g. Cesaro method, Abel method and etc. These methods are used in different areas of mathematics. For the applicability of these methods is very important that the considered space has a linear structure. Therefore, the study of statistical convergence in metric spaces is of special scientific interest. Different aspects of this problem is devoted in M. Kuchukaslan, U. Deger and O. Dovgoshey, 2014; M. Kuchukaslan and U. Deger, 2012. Statistical convergence is currently actively used in many areas of mathematics such as summation theory (B.J. Connor, 1988; A.R. Freedman and J.J. Sember, 1981; J.A. Fridy, 1985), number theory (R. Erdös and G. Tenenbaum, 1989), trigonometric series (A. Zygmund, 1979), probability theory (J.A. Fridy and M.K. Khan, 1998) measure theory (H.I. Miller, 1995), optimization (S. Pehlivan and M.A. Mamedov, 2000), approximation theory (A.D. Gadjiev and C. Orhan, 2002; A.D. Gadjiev, 2011), fuzzy theory, etc. Generalization of statistical convergence to the continuous case have done in (Bilalov, Sadigova).

It should be noted that the concept of *statistical fundamentality* (*stat fundamentality*) was first introduced by J.A. Fridy (J.A. Fridy, 1985) who proved its equivalence to statistical convergence with respect to numerical sequences. This problem was raised in (G.D. Maio and L.D.R. Kocinac, 2008) concerning uniform space $(X; U)$. It is proved that if the sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ *stat-convergent*, then it is *stat-fundamental.* In the same paper raised the Problem 2.16 of the validity of converse statement.

In this paper we consider the statistical convergence in metric spaces. Statistical fundamentality is defined, and in a complete metric space it is proved that the statistical fundamentality is equivalent to the statistical convergence. Concept p-strong convergence in metric spaces is introduced and prove its equivalence to the one of statistical convergence. Some Tauberian theorems concerning statistical convergence in metric spaces are introduced. It should be noted that the issue of statistical convergence in metric spaces considered in (M. Kuchukaslan, U. Deger and O. Dovgoshey, 2014; M. Kuchukaslan and U. Deger, 2012). In these papers the statistical boundedness, the statistical equivalence of sequences in metric spaces and their relationship to the statistical convergence are considered.

2. Needful Information

Let $(X;\rho(\cdot;\cdot))$ be a metric space with a metric ρ. Denote by $O_\varepsilon(a)$ the open ball in X centered at the point $a\in X$ and with a radius ε: $O_\varepsilon(a)\equiv\{x\in X:\rho(x;a)<\varepsilon\}$. A^C means the complement of the set $A\subset\mathbb{N}$ to $\mathbb{N}: A^C=\mathbb{N}\backslash A$, where $\mathbb{N}$ is a set of all positive numbers. $\chi_A(\cdot)$ is the characteristic function of A; $\Rightarrow$ will be a quantifier which means "follows"; $\wedge$ will be a quantifier which means "and"; $\bar{M}$ will stand for the closure of M.

Let $A\subset\mathbb{N}$ be some set. Assume $\delta_n(A)=\frac{1}{n}\sum_{k=1}^{n}\chi_A(k)$. If $\exists\lim\limits_{n\to\infty}\delta_n(A)=\delta(A)$, then $\delta(A)$ is called statistical density of the set A.

Accept the following

Definition 1. *We say that $\{x_n\}_{n\in\mathbb{N}}\subset X$ statistically converges (st-converges) to $x\in X$, if $\delta(A_\varepsilon)=0$, where $A_\varepsilon\equiv\{n\in\mathbb{N}:\rho(x_n;x)\geq\varepsilon\}$, and this kind of convergence is denoted as st-$\lim\limits_{n\to\infty}x_n=x$.*

Put

$$\mathscr{K}\equiv\{K\subset\mathbb{N}:\delta(K)=1\}.$$

In the sequel, we will need the following easily provable

Lemma 1. *Let $K_j\in\mathscr{K}$, $j=1;2\Rightarrow K_1\cap K_2\in\mathscr{K}$.*

Proof. In fact, let $I_n\equiv\{1;...;n\}$. We have

$$K_1\cap K_2=(K_1\cup K_2)\backslash[(K_2\backslash K_1)\cup(K_1\backslash K_2)].$$

Consequently

$$K_1\cap K_2\cap I_n=[(K_1\cup K_2)\cap I_n]\backslash[((K_2\backslash K_1)\cup(K_1\backslash K_2))\cap I_n]. \tag{1}$$

As

$$((K_2\backslash K_1)\cup(K_1\backslash K_2))\cap I_n=((K_2\backslash K_1)\cap I_n)\cup((K_1\backslash K_2)\cap I_n),$$

taking into account

$$(K_2\backslash K_1)\cap I_n\subset K_1^C\cap I_n\Rightarrow\frac{|(K_2\backslash K_1)\cap I_n|}{|I_n|}\leq\frac{\left|K_1^C\cap I_n\right|}{|I_n|}\to 0\,,\ n\to\infty,$$

$$\frac{|(K_1\backslash K_2)\cap I_n|}{|I_n|}\to 0\,,\ n\to\infty,$$

we get

$$\frac{|((K_2\backslash K_1)\cup(K_1\backslash K_2))\cap I_n|}{|I_n|}\to 0,\ n\to\infty.$$

From $(K_1\cap I_n)\subset(K_1\cup K_2)\cap I_n$ and $K_1\in\mathscr{K}$ it follows

$$\frac{|(K_1\cup K_2)\cap I_n|}{|I_n|}\to 1\,,\ n\to\infty,$$

and hence $K_1\cup K_2\in\mathscr{K}$. Then from (1) we directly obtain

$$\frac{|K_1\cap K_2\cap I_n|}{|I_n|}=\frac{|(K_1\cup K_2)\cap I_n|}{|I_n|}-\frac{|((K_2\backslash K_1)\cup(K_1\backslash K_2))\cap I_n|}{|I_n|}\to 1,\ n\to\infty,$$

i.e. $K_1\cap K_2\in\mathscr{K}$.

3. Statistical Fundamentality

Accept the following

Definition 2. *We say that $\{x_n\}_{n\in\mathbb{N}}\subset X$ is statistically fundamental (st-fundamental) in $(X;\rho)$, if $\forall\varepsilon>0$, $\exists n_\varepsilon\in\mathbb{N}:\ \delta(\Delta_{n_\varepsilon})=0$, where*

$$\Delta_{n_\varepsilon}\equiv\{n\in\mathbb{N}:\ \rho(x_n;x_{n_\varepsilon})\geq\varepsilon\}.$$

Let $x_n\xrightarrow{st}x$ in X, and $\varepsilon>0$ be an arbitrary number. Put

$$A_\varepsilon\equiv\{n:\ \rho(x_n;x)\geq\varepsilon\}.$$

It is absolutely clear that $\delta\left(A^C_{\varepsilon/2}\right) = 1$. Take $\forall n_\varepsilon \in A^C_{\varepsilon/2} : \rho\left(x_{n_\varepsilon}; x\right) < \frac{\varepsilon}{2}$. We have

$$\left\{n : \rho\left(x_n; x\right) < \frac{\varepsilon}{2}\right\} \subset \left\{n : \rho\left(x_n; x_{n_\varepsilon}\right) < \varepsilon\right\},$$

i.e. $A^C_{\varepsilon/2} \subset \Delta^C_{n_\varepsilon}$, where $\Delta_{n_\varepsilon} \equiv \{n \in \mathbb{N} : \rho\left(x_n; x_{n_\varepsilon}\right) \geq \varepsilon\}$. Hence, $\delta\left(\Delta^C_{n_\varepsilon}\right) = 1 \Rightarrow \delta\left(\Delta_{n_\varepsilon}\right) = 0$.
Thus, the following lemma is true.

Lemma 2. *Let $x_n \xrightarrow{st} x$ in $(X;\rho)$. Then the sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ is st-fundamental in $(X;\rho)$.*

Let $(X;\rho)$ be complete metric space, and the sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ be *st- fundamental* in $(X;\rho)$. Then $\exists n_j \in \mathbb{N}$: $\delta\left(K_j\right) = 1$, where $K_j \equiv \left\{n : \rho\left(x_n; x_{n_j}\right) \leq 2^{1-j}\right\}$, $j = 1, 2$. By Lemma 1 we obtain $K_1 \cap K_2 \in \mathscr{K}$. Put

$$M_1 \equiv \overline{O_1\left(x_{n_1}\right)} \cap \overline{O_{2^{-1}}\left(x_{n_2}\right)}.$$

It is obvious that $x_n \in M_1$, $\forall n \in (K_1 \cap K_2) \equiv K_{(1)}$. Thus, $\exists n_3 \in \mathbb{N} : K_3 \in \mathscr{K}$, where $K_3 \equiv \left\{n : \rho\left(x_n; x_{n_3}\right) \leq 2^{-2}\right\}$.
Let $K_{(2)} = K_{(1)} \cap K_3$. It is clear that $K_{(2)} \in \mathscr{K}$. Now let

$$M_2 \equiv M_1 \cap \overline{O_{2^{-2}}\left(x_{n_3}\right)}.$$

Denote by $d(M)$ the diameter of the set M, i.e.

$$d(M) = \sup_{x,y\in M} \rho(x; y).$$

Continuing in the same way, we obtain a sequence of closed sets $\{M_n\}_{n\in\mathbb{N}}$, $M_1 \supset M_2 \supset ...$, whose diameters tend to zero: $d(M_n) \leq 2^{-n+1} \to 0$, $n \to \infty$. Moreover, $K_{(n)} \in \mathscr{K}$, where $K_{(n)} \equiv \{k \in \mathbb{N} : x_k \in M_n\}$. Take $\forall \tilde{x}_n \in M_n$. We have

$$\rho\left(\tilde{x}_n; \tilde{x}_{n+p}\right) \leq d(M_n) \to 0, \; n \to \infty, \; \forall p \in \mathbb{N}.$$

Hence, the sequence $\{\tilde{x}_n\}_{n\in\mathbb{N}}$ is fundamental in $(X;\rho)$ and as $(X;\rho)$ is complete, it is clear that $\exists x \in X : \tilde{x}_n \to x$, $n \to \infty$. It is absolutely clear that $x \in \bigcap_n M_n$, i.e. $\bigcap_n M_n$ is non-empty. From $d(M_n) \to 0$, $n \to \infty$, it directly follows that $\{x\} = \bigcap_n M_n$, i.e. $\bigcap_n M_n$ consists of one element. As $K_{(m)} \in \mathscr{K}$, then $\exists \{n_m\}_{m\in\mathbb{N}} \subset \mathbb{N}$; $n_1 < n_2 < ...$:

$$\frac{1}{n}\left|\left\{k \in I_n : k \in K^C_{(m)}\right\}\right| < \frac{1}{m}, \; \forall n > n_m,$$

where $I_n \equiv \{1; ...; n\}$. Assume

$$\mathbb{N}_0 \equiv \left\{k \in \mathbb{N} : n_m < k \leq n_{m+1} \wedge k \in K^C_{(m)}\right\},$$

and

$$y_k = \begin{cases} x, & if \;\; k \in \mathbb{N}_0 \wedge (k > n_1) \;; \\ x_k, & if \;\; othervise. \end{cases}$$

Take $\forall \varepsilon > 0$. If $k \in \mathbb{N}_0 \wedge (k > n_1)$, then $y_k = x$, and, as a result $0 = \rho(y_k; x) < \varepsilon$. If $k \notin \mathbb{N}_0 \Rightarrow k \in K_{(m)} \Rightarrow x_k \in M_m \Rightarrow \rho(x_k; x) \leq \rho\left(x_k; x_{n_m}\right) + \rho\left(x_{n_m}; x\right) \leq 2^{-m+2} < \varepsilon$, for sufficiently great values of m. Consequently, $\lim_{k\to\infty} y_k = x$.
Let $\tilde{K} \equiv \{k \in \mathbb{N} : x_k \neq y_k\}$. Let us show that $\delta\left(\tilde{K}\right) = 0$. Put $n_m < n < n_{m+1}$. Let us prove that

$$\{k \leq n : x_k \neq y_k\} \subset \left\{k \leq n : k \in K^C_{(m)}\right\}.$$

Let $k \leq n \wedge x_k \neq y_k$. Consequently, $k \in \mathbb{N}_0 \Rightarrow k \in K^C_{(m)}$. Thus

$$\frac{1}{n}\left|\{k \leq n : x_k \neq y_k\}\right| \leq \frac{1}{n}\left|\left\{k \leq n : k \in K^C_{(m)}\right\}\right| < \frac{1}{m}.$$

It is clear that if $n \to \infty$, then $m \to \infty$. Then from the previous relation we get

$$\lim_{n\to\infty} \frac{\left|\{k \leq n : x_k \neq y_k\}\right|}{n} = 0. \tag{2}$$

Consequently, $\{k \le n : x_k \ne y_k\}^C \in \mathscr{K}$ and $\lim_{n\to\infty} y_n = x$. Let us show that $st\text{-}\lim_{n\to\infty} x_n = x$. Take $\forall\varepsilon > 0$. We have

$$\{k \le n : \rho(x_k; x) \ge \varepsilon\} \subset \{k \le n : x_k \ne y_k\} \cup \{k \le n : \rho(y_k; x) \ge \varepsilon\}. \quad (3)$$

As, $\lim_{k\to\infty} y_k = x$ in $(X;\rho)$, then $\exists n_\varepsilon \in \mathbb{N} : \rho(y_k; x) < \varepsilon,\ \forall k \ge n_\varepsilon$. Consequently

$$|\{k \le n : \rho(y_k; x) \ge \varepsilon\}| \le n_\varepsilon \Rightarrow \frac{1}{n}|\{k \le n : \rho(y_k; x) \ge \varepsilon\}| \le \frac{n_\varepsilon}{n} \to 0,\ n \to \infty.$$

Then, using (2), from (3) we obtain

$$\frac{1}{n}|\{k \le n : \rho(x_k; x) \ge \varepsilon\}| \le \frac{1}{n}|\{k \le n : x_k \ne y_k\}| +$$

$$+\frac{1}{n}|\{k \le n : \rho(y_k; x) \ge \varepsilon\}| \to 0,\ n \to \infty.$$

So, $st\text{-}\lim_{n\to\infty} x_n = x$. Thus, we have proved the following theorem.

Theorem 1. *Let $(X;\rho)$ be a complete metric space and $\{x_n\}_{n\in\mathbb{N}} \subset X$ be some system. Then the following statements are equivalent to each other:*

1) $\exists st - \lim_{n\to\infty} x_n$;

2) $\{x_n\}_{n\in\mathbb{N}}$ is st-fundamental;

3) $\exists \{y_n\}_{n\in\mathbb{N}} \subset X : \exists \lim_{n\to\infty} y_n \wedge \{n \in \mathbb{N} : x_n = y_n\} \in \mathscr{K}$.

This theorem immediately implies the following

Corollary 1. *Let $\{x_n\}_{n\in\mathbb{N}} \subset X$ and $\exists st\text{-}\lim_{n\to\infty} x_n = x$. Then $\exists \{n_k\}_{k\in\mathbb{N}} \subset \mathbb{N} : n_1 < n_2 < \ldots,\ \lim_{k\to\infty} x_{n_k} = x \wedge \delta(\{n_k\}_{k\in\mathbb{N}}) =$ 1.*

4. p-strong Convergence

Let $(X;\rho)$ be a metric space, and $p \in (0, +\infty)$ be some number. Following (A. Alotaibi and A.M. Alroqi, 2012) we accept the following

Definition 3. *The sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ is called p-strong convergent to $x \in X$, if $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \rho^p(x_k; x) = 0$, and this kind of limit is denoted as $p\text{-}\lim_{n\to\infty} x_n = x$.*

The following theorem is true.

Theorem 2. *It holds: i) If $p\text{-}\lim_{n\to\infty} x_n = x$ then $\exists st\text{-}\lim_{n\to\infty} x_n \wedge st\text{-}\lim_{n\to\infty} x_n = x$; ii) If $\exists st\text{-}\lim_{n\to\infty} x_n = x$ and $\exists O_r(x_0) \subset X : x_n \in O_r(x_0),\ \forall n \in \mathbb{N}$, then $\exists p\text{-}\lim_{n\to\infty} x_n = x$.*

Proof. *i*) Let $p\text{-}\lim_{n\to\infty} x_n = x$. Take $\forall\varepsilon > 0$, and put $K_\varepsilon^{(n)} = \{n \in I_n : \rho(x_n; x) \ge \varepsilon\}$. We have

$$\varepsilon^p \frac{|K_\varepsilon|}{n} \le \frac{1}{n}\sum_{k\in K_\varepsilon^{(n)}} \rho^p(x_k; x) \le \frac{1}{n}\sum_{k=1}^{n} \rho^p(x_k; x) \to 0,\ n \to \infty,$$

i.e. $\delta(K_\varepsilon) = 0 \Rightarrow st\text{-}\lim_{n\to\infty} x_n = x$.

ii) Let $\exists st\text{-}\lim_{n\to\infty} x_n = x$ and $\exists O_r(x_0) \subset X : x_n \in O_r(x_0),\ \forall n \in \mathbb{N}$. We have

$$\rho(x_n; x) \le \rho(x_n; x_0) + \rho(x_0; x) \le r + \rho(x_0; x) = M.$$

Thus

$$\frac{1}{n}\sum_{k=1}^{n} \rho^p(x_k; x) = \frac{1}{n}\sum_{k\in I_n\setminus K_\varepsilon^{(n)}} \rho^p(x_k; x) + \frac{1}{n}\sum_{k\in K_\varepsilon} \rho^p(x_k; x),$$

where $K_\varepsilon^{(n)} = \{k \in I_n : \rho(x_k; x) \geq \varepsilon\}$, $\varepsilon > 0$ is an arbitrary number. So

$$\frac{1}{n} \sum_{k \in I_n \setminus K_\varepsilon^{(n)}} \rho^p(x_k; x) \leq \varepsilon^p;$$

$$\frac{1}{n} \sum_{k \in K_\varepsilon} \rho^p(x_k; x) \leq M^p \frac{|K_\varepsilon|}{n} \to 0, \; n \to \infty.$$

From the arbitrariness of ε we obtain that

$$\frac{1}{n} \sum_{k=1}^{n} \rho^p(x_k; x) = 0 \Rightarrow p - \lim_{n\to\infty} x_n = x.$$

Definition 4. *The function $\mu : [0, \infty) \to [0, \infty)$ is called a modulus, if: (i) $\mu(0) = 0 \Leftrightarrow x = 0$; (ii) $\mu(x + y) \leq \mu(x) + \mu(y)$, $\forall x, y \in [0, \infty)$; (iii) μ is a monotone nondecreasing function; (iv) $\mu(+0) = 0$.*

According to (I.J. Maddox, 1988) we accept the following

Definition 5. *Let $(X; \rho)$ be a metric space and μ be a modulus. $\{x_n\}_{n\in\mathbb{N}} \subset X$ is called μ-convergent to $x \in X$, if*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mu(\rho(x_k; x)) = 0,$$

and this kind of convergence is denoted as $\mu\text{-}\lim_{n\to\infty} x_n = x$.

Similarly to Theorem 2 we prove the following

Theorem 3. *Let $\exists \mu\text{-}\lim_{n\to\infty} x_n = x$. Then $\exists st\text{-}\lim_{n\to\infty} x_n \wedge st\text{-}\lim_{n\to\infty} x_n = x$.*

5. Tauberian Theorems in Metric Spaces

Let $(X; \rho)$ be a metric space and $\{x_n\}_{n\in\mathbb{N}} \subset X$ be some sequence. Let $\Delta\rho_n = \rho(x_n; x_{n+1})$, $\forall n \in \mathbb{N}$. The following theorem is true.

Theorem 4. *Let $st\text{-}\lim_{n\to\infty} x_n = x$ and $\Delta\rho_n = \overline{o}\left(\frac{1}{n}\right)$. Then $\exists \lim_{n\to\infty} x_n \wedge \lim_{n\to\infty} x_n = x$.*

Proof. We will follow (J.A. Fridy, 1985). Let $st\text{-}\lim_{n\to\infty} x_n = x$. Then, by Theorem 1 $\exists \{y_n\}_{n\in\mathbb{N}} \subset X : \lim_{n\to\infty} y_n = x \wedge \{n : x_n = y_n\} \in \mathscr{K}$. Every $k \in \mathbb{N}$ can be represented as $k = m_k + p_k$, where

$$m_k = \begin{cases} \max\{i \leq k : x_i = y_i\}, & A_k \neq \emptyset, \\ -1, & A_k = \emptyset, \end{cases}$$

$A_k = \{i \leq k : x_i = y_i\}$. As proved in (A.R. Freedman and J.J. Sember, 1981), it holds that $\lim_{k\to\infty} \frac{p_k}{m_k} = 0$. It is clear that $\exists M > 0 : \Delta\rho_n \leq \frac{B}{n}$, $\forall n \in \mathbb{N}$.

We have

$$\rho(y_{m_k}; x_k) = \rho(x_{m_k}; x_k) = \rho\left(x_{m_k}; x_{m_k+p_k}\right) \leq$$

$$\leq \sum_{i=m_k}^{m_k+p_k-1} \rho(x_i; x_{i+1}) \leq \sum_{i=m_k}^{m_k+p_k-1} \Delta\rho_i \leq M \frac{p_k}{m_k} \to 0, \; k \to \infty.$$

As, $\lim_{k\to\infty} y_{m_k} = x$ in $(X; \rho)$, it directly follows that $\lim_{k\to\infty} x_k = x$.

We say that $\{x_k\}_{k\in\mathbb{N}} \subset X$ is a gap sequence if $\Delta\rho_k = 0$ except for certain indices k which occur at wide intervals or gaps.

The following Tauberian theorem is true.

Theorem 5. *Let $\{k(i)\}_{i\in\mathbb{N}} \subset \mathbb{N}$ be an increasing sequence such that $\liminf_i \frac{k(i+1)}{k(i)} > 1$ and let $\{x_k\}_{k\in\mathbb{N}} \subset X$ be corresponding gap sequence: $\Delta\rho_n = 0$ if $k \neq k(i)$ for $i \in \mathbb{N}$. If $st\text{-}\lim_{n\to\infty} x_n = x$, then $\lim_{n\to\infty} x_n = x$.*

Proof. Quite similarly proved in (J.A. Fridy, 1985) For completeness of the exposition we present it here. Let $\liminf_i \frac{k(i+1)}{k(i)} = 1 + 2\delta > 1$. Then for sufficiently great values of i we have

$$\frac{k(i+1)}{k(i)} > 1 + \delta > 1, \tag{4}$$

i.e.

$$k(i+1) - k(i) > \delta k(i).$$

This means that the number of terms in the $(i+1)$-st block (throughout which x_k is constant) is greater than $\delta k(i)$. Now, let us assume that $\lim_{n\to\infty} x_n \neq x$. Take $\varepsilon > 0$. Let $k \in \mathbb{N}$ be sufficiently great, such that $\rho(x_k; x) \geq \varepsilon$. Thus if such a k is chosen from the $(i+1)$-st block, where i is large enough to ensure that (4) holds, we have

$$\frac{1}{k(i+1)} |\{k \leq k(i+1) : \rho(x_k; x) \geq \varepsilon\}| > \frac{k(i+1) - k(i)}{k(i+1)} > \frac{\delta}{1+\delta}.$$

Thus, $\frac{1}{n} |\{k \leq n : \rho(x_k; x) \geq \varepsilon\}|$ does not tend to zero, so $st\text{-}\lim_{n\to\infty} x_n \neq x$.

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