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On the $\mathbb{D}$-Stability Criterion of Matrices

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Abstract

Root clustering problems of matrices are considered. Here the conditions are given for eigenvalues of matrices to lie in a prescribed subregion $\mathbb{D}$ of the complex plane. The region $\mathbb{D}$ (stability region) is defined by rational functions. A simple necessary and sufficient condition for stability of a single matrix is obtained. For the commuting polynomial family, a necessary and sufficient condition in terms of a common solution to a set of Lyapunov inequalities is derived. A simple sufficient condition for the Hurwitz stability of a commuting quadratic polynomial family is given.

Keywords: Stability region, Lyapunov inequality, root clustering

1. Introduction

Matrix root clustering, also known as $\mathbb{D}$-stability, is an important problem in control theory. The problem of root clustering has attracted a great deal of consideration in the past. Previous work on matrix root clustering has employed different approaches. First, many authors have presented matrix root clustering problem via the Generalized Lyapunov Theorem (GLT) approach (see, e.g., Gutman and Jury, 1981; Gutman, 1990; Yedevalli, 1993; Yedevalli, 1993; Yedevalli, 1985; Juang, Hong, Wang, 1989; Juang, 1991; Jabbari, 1991; Horng, Horng, Chou, 1993). Extensive works on the subject have led to a study by (Gutman and Jury, 1981), and continuously do so (Gutman, 1990; Yedevalli, 1993; Yedevalli, 1993; Yedevalli, 1985; Juang, Hong, Wang, 1989; Juang, 1991; Jabbari, 1991; Horng, Horng, Chou, 1993). The authors considered in (Gutman and Jury, 1981) the root clustering of a matrix called Γ into the unit circle. There are also issues on Ω transformable region which is defined by mapping from two points in Ω in to the left half plane. (Yedevalli, 1985, 1993) presented some explicit bounds on uncertainty for root clustering of linear state space models in terms of GLT approach and the Kronecker approach. Following this, (Wang, 1994, 2000, 2003) analysed robust root clustering in a specific region in which the GLT is not valid. This clustering region is an intersection of a ring and horizontal strip, located in the left half plane which is the non Ω transformable region providing good ride quality for aircraft.

A further approach was discussed by Chilali and Gahinet in (Chilali, Gahinet, 1996, 1999). In particular, sufficient conditions were derived for a class of convex regions of complex plane via linear matrix inequalities (LMI). Recently Bosche (Bosche, Bachelier, Mehdi, 2005; Rejichi, Bachelier, Chaabane, Mehdi, 2007) addressed the problem of matrix root clustering analysis in EEMI (Extended Ellipsoidal Matrix Inequality) regions which express the set of some non- connected regions.

This paper is organized as follows. In section 2, we present a necessary and sufficient condition for $\mathbb{D}$-stability of a matrix. In section 3, we give the necessary and sufficient conditions for $\mathbb{D}$-stability of the commuting family. Finally, our conclusions are presented in section 4.

Now, we present a number of notations and results that will be needed in the following sections.

Let $\mathbb{R}^n$ be the set of real n vectors, $\mathbb{R}^{n\times n}$ ($\mathbb{C}^{n\times n}$) be the set of $n\times n$ real (complex) matrices. For $P \in \mathbb{R}^{n\times n}$ ($\mathbb{C}^{n\times n}$) the symbol $P > 0$ means that P is symmetric (Hermitian) and positive definite. Let the subregion $\mathbb{D}$ of the complex plane $\mathbb{C}$ be defined as

$$\mathbb{D} = \left\{ z \in \mathbb{C} : Re f_j(z)\bar{g}_j(z) < 0, j = 1, 2, \ldots, m \right\}, \tag{1}$$

where $f_j(z)$ and $g_j(z)$ are polynomials with real coefficients, $\bar{g}$ is the complex conjugate of g . The inequality $Ref_j(z)\bar{g}_j(z) < 0$ is equivalent from the inequality $Rer_j(z) < 0$, where $r_j(z) = \frac{f_j(z)}{g_j(z)}$.

The region $\mathbb{D}$ (1) will also be referred to as the stability region. It is a generalization of the known stability regions:

If m=1, f(z)=z, g(z)=1 it is Hurwitz stability region,
If m=1, f(z)=z+1, g(z)=z-1 it is Schur stability region,
If m=2, f_1(z)=z, f_2(z)=-z^2, g_i(z)=1 ($i = 1, 2$) it is $\frac{\pi}{4}$ left sector stability region,
If m=2, f_1(z)=z+a, f_2(z)=-z-b, g_1(z)=z-a, g_2(z)=z-b it is the ring $\{z \in \mathbb{C} : b < |z| < a\}$.

By the Lyapunov theorem, the matrix $A \in \mathbb{R}^{n\times n}$ ($\mathbb{C}^{n\times n}$) is Hurwitz stable if and only if there exists $P \in \mathbb{R}^{n\times n}$ ($\mathbb{C}^{n\times n}$), $P > 0$ such that

$$\begin{array}{l} A^T P + PA < 0 \\ (A^* P + PA < 0) \end{array} \tag{1}$$

where $A^T (A^*)$ denotes the transpose (conjugate transpose) of A.

In (Mori, Mori, Kokame 2001) the following result is obtained (Mori, Mori, Kokame 2001, Theorem 1).

Theorem 1.1 (Mori, Mori, Kokame 2001) *Let the stability region* Ω *be defined as*

$$\Omega = \left\{ z \in \mathbb{C} : Ref_j(z) < 0, j = 1, 2, \ldots, m \right\}, \tag{2}$$

where $f_j(z)$ *are polynomials. Then the matrix* $A \in \mathbb{R}^{n\times n}$ *is* Ω *-stable if and only if there exists a matrix* $P \in \mathbb{R}^{n\times n}$, $P > 0$ *such that for all* $j = 1, 2, \ldots, m$

$$\left[f_j(A) \right]^T P + P \left[f_j(A) \right] < 0 \tag{3}$$

In (Narendra, Balakrishnan 1994), the following result on the existence of a common $P > 0$ for commuting matrices $A_1, A_2, \ldots, A_k$ is given (Narendra, Balakrishnan 1994, Theorem 2).

Theorem 1.2 (Mori, Mori, Kokame 2001) *Let* $A_i \in \mathbb{R}^{n\times n}$ *(*$i = 1, 2, \ldots, k$*) be Hurwitz stable and commute pairwise. Then there exists* $P \in \mathbb{R}^{n\times n}$ *,* $P > 0$ *such that for all* $i = 1, 2, \ldots, k$

$$A_i^T P + PA_i < 0. \tag{4}$$

Note that in (Mori, Mori, Kokame 2001) an explicit method of generating a common P is also presented.

In this work by using Theorems 1.1 and 1.2 we prove a simple criterion for $\mathbb{D}$-stability of a matrix A. We show that $\mathbb{D}$-stability of a matrix A is equivalent to the Hurwitz stability of the matrices $f_1(A)g_1^{-1}(A), \ldots, f_m(A)g_m^{-1}(A)$ (Theorem 2.1).

2. Stability of a Single Matrix

In this section we give a criterion for the $\mathbb{D}$-stability of the matrix $A \in \mathbb{R}^{n\times n}$.

Lemma 2.1 *Let* $f(z)$ *and* $g(z)$ *be polynomials,* $A \in \mathbb{R}^{n\times n}$*. If* $g(A)$ *is invertible then* $f(A)$ *and* $g^{-1}(A)$ *commute.*

The proof follows from the equality $f(A)g(A) = g(A)f(A)$.

Lemma 2.2 *Let* $f_j(z)$, $g_j(z)$ *(*$j = 1, 2, \ldots, m$*) are polynomials and* $g_j(A)$ *are invertible for all* $j = 1, 2, \ldots, m$*. Then the matrices* $r_j(A) = f_j(A).\ g_j^{-1}(A)$ *(*$j = 1, 2, \ldots, m$*) are commutative.*
Proof. By Lemma 2.1, the following is true

$$\left[g_j(A)g_i(A) \right]^{-1} f_i(A)f_j(A) = f_j(A)f_i(A) \left[g_i(A)g_j(A) \right]^{-1}. \tag{5}$$

Carrying out suitable multiplications in (6), the commutativity of $r_j(A)$ follows.

Lemma 2.3 *If* $f(z)$ *and* $g(z)$ *are polynomials,* $A \in \mathbb{R}^{n\times n}$*,* λ *is an eigenvalue of* A*,* $g(A)$ *is invertible then* $g(\lambda) \neq 0$ *and* $\frac{f(\lambda)}{g(\lambda)}$ *is an eigenvalue of* $f(A)g^{-1}(A)$.
Proof. $g(\lambda)$ is an eigenvalue of $g(A)$. Since $g(A)$ is invertible then $g(\lambda) \neq 0$. There exists $x \in \mathbb{C}^{n\times 1}$, $x \neq 0$ such that

the following can be written:

$$\begin{aligned} Ax &= \lambda x \\ f(A)x &= f(\lambda)x \\ g(A)x &= g(\lambda)x \\ g^{-1}(A)x &= \frac{1}{g(\lambda)}x \\ f(A)g^{-1}(A)x = f(A)\frac{1}{g(\lambda)}x &= \frac{f(\lambda)}{g(\lambda)}x \end{aligned}$$

Lemma 2.4 *Let $f(z)$ and $g(z)$ be polynomials and, $g(A)$ is invertible. If μ is an eigenvalue of $f(A)g^{-1}(A)$ then there exists an eigenvalue λ of A such that $g(\lambda) \neq 0$ and $\mu = \frac{f(\lambda)}{g(\lambda)}$.*

Proof. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be eigenvalues of A. Then $g(\lambda_i) \neq 0 \ (i = 1, 2, \ldots, n)$ and by Lemma 2.3 $\frac{f(\lambda_i)}{g(\lambda_i)}$ are eigenvalues of $f(A)g^{-1}(A)$. Therefore, there exists i such that $\mu = \frac{f(\lambda_i)}{g(\lambda_i)}$.

Theorem 2.1 *Let $A \in \mathbb{R}^{n\times n}$ and the stability region $\mathbb{D}$ (1) be given. Then the following assertions are equivalent :*
i) A is $\mathbb{D}$-stable.
ii) $g_j(A)$ are invertible and $r_j(A) = f_j(A)g_j^{-1}(A)$ are Hurwitz stable $(j = 1, 2, \ldots, m)$.
iii) $g_j(A)$ are invertible and there exists $P \in \mathbb{R}^{n\times n}$, $P > 0$ such that

$$\left[r_j(A)\right]^T P + P\left[r_j(A)\right] < 0 \quad (j = 1, 2, \ldots, m). \tag{6}$$

Proof. The implication iii)⟹ii) follows from the Lyapunov theorem.
ii)⟹i) : Let λ be an arbitrary eigenvalue of A. Then $g_j(\lambda) \neq 0$ and $\frac{f_j(\lambda)}{g_j(\lambda)}$ are eigenvalues of $r_j(A)$ $(j = 1, 2, \ldots, m)$. Since $r_j(A)$ is Hurwitz stable, then $Re\frac{f_j(\lambda)}{g_j(\lambda)} < 0$ or $Ref_j(\lambda)\bar{g}_j(\lambda) < 0$ $(j = 1, 2, \ldots, m)$. Thus $\lambda \in \mathbb{D}$.

i)⟹iii) : Fix arbitrary j. Let μ be an arbitrary eigenvalue of $g_j(A)$. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be eigenvalues of A. Then $g(\lambda_1), \ldots, g(\lambda_n)$ are eigenvalues of $g_j(A)$. Therefore there exists i such that $\mu = g_j(\lambda_i)$. Since A is $\mathbb{D}$-stable then $g_j(\lambda_i) = \mu \neq 0$. On the other hand μ is an arbitrary eigenvalue of $g_j(A)$. Consequently $g_j(A)$ is invertible.

By Lemmas 2.2 and 2.4 the matrices $r_j(A)$ are Hurwitz and commute $(j = 1, 2, \ldots, m)$. Then by Theorem (1.2), there exists $P > 0$ such that (7) is true.

Example 2.1 (Mori, Mori, Kokame 2001) Let A be given as

$$A = \begin{bmatrix} -94.7 & -47.1 & -41.1 & -2.3 \\ 15.2 & -46.9 & 3.0 & -7.6 \\ 121.0 & 77.9 & 46.3 & 9.1 \\ -116.9 & 65.2 & -54.6 & -4.7 \end{bmatrix}$$

and the region Ω is as shown in Figure 1.

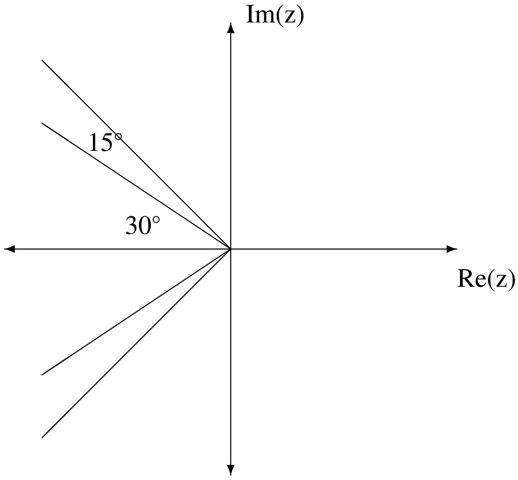

Figure 1. Sector Ω for Example 2.1

This region can be expressed as $\Omega = \{z \in \mathbb{C} : Ref_i(z) < 0, j = 1, 2, 3\}$, where $f_1(z) = z$, $f_2(z) = -z^2$, $f_3(z) = -z^3$. The matrices

$$A = \begin{bmatrix} -94.7 & -47.1 & -41.1 & -2.3 \\ 15.2 & -46.9 & 3.0 & -7.6 \\ 121.0 & 77.9 & 46.3 & 9.1 \\ -116.9 & 65.2 & -54.6 & -4.7 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} -3547.9 & -3317.7 & -1973.5 & -212.57 \\ 900.88 & -1221.9 & 211.56 & -384.5 \\ 5736.1 & 5152.5 & 3092.6 & 491.78 \\ -6004.3 & 2111.7 & -2728.8 & 701.42 \end{bmatrix},$$

$$-A^3 = \begin{bmatrix} 71614. & 1.5511 \times 10^5 & 56100.0 & 16415. \\ -33339. & 6285.4 & -9902.9 & 10947. \\ -1.4818 \times 10^5 & -2.3885 \times 10^5 & -1.0396 \times 10^5 & -26521. \\ 1.8852 \times 10^5 & 16922. & 88469. & -30368. \end{bmatrix}$$

are Hurwitz . Therefore by Theorem 2.1 the matrix A is Ω-stable. In (Mori, Mori, Kokame 2001) this stability is established by finding a common solution $P > 0$ for (4).

Example 2.2 Let A be given as

$$A = \begin{bmatrix} 0 & -0.01 & 0.5 \\ 1 & 0 & -1.5 \\ -0.01 & 1 & 1.9 \end{bmatrix}$$

and $\mathbb{D} = \{z \in \mathbb{C} : Ref_i(z).\bar{g}_i(z) < 0, j = 1, 2\}$, where $f_1(z) = z + 1$, $g_1(z) = z - 1$, $f_2(z) = -z - \frac{1}{2}$, $g_2(z) = z - \frac{1}{2}$. The region $\mathbb{D}$ is the ring $\left\{(x, y) : \frac{1}{4} < x^2 + y^2 < 1\right\}$. Here

$$r_1(A) = \begin{bmatrix} -11.48 & -10.48 & -10.712 \\ 18.409 & 19.616 & 20.8 \\ -20.592 & -20.802 & -20.008 \end{bmatrix},$$

$$r_2(A) = \begin{bmatrix} -8.1846 & -4.6161 & -2.3799 \\ 12.438 & 5.2416 & 2.2451 \\ -8.9358 & -4.4912 & -3.3351 \end{bmatrix}$$

are Hurwitz stable. Therefore A is $\mathbb{D}$-stable.

3. Stability of a Commuting Family

In this section, for a commuting family, we give $\mathbb{D}$-stability criterion in terms of the existence of a common positive definite solution to a set of Lyapunov inequalities.
The following lemma is taken from (Cohen, Lewcowicz, Rodman, 1997; Cohen, Lewcowicz, 1997).

Lemma 3.1 (Cohen, Lewcowicz, Rodman, 1997; Cohen, Lewcowicz, 1997) *Let $\mathcal{B} \subset \mathbb{C}^{n\times n}$ be a compact set of Hurwitz upper triangular matrices. Then there exist $\alpha > 0$ and positive diagonal matrix D such that*

$$A^*D + DA \leq -\alpha I$$

for all $A \in \mathcal{A}$ where I is the identity matrix.

Theorem 3.1 *Let $\mathcal{F} \subset \mathbb{R}^{n\times n}$ be a compact commuting family. Then $\mathcal{F}$ is Hurwitz stable if and only if there exists $P \in \mathbb{R}^{n\times n}$, $P > 0$ such that for all $F \in \mathcal{F}$*

$$F^T P + PF < 0 \tag{7}$$

Proof. $\Longrightarrow$) : By the corollary of the Schur's triangularization theorem (Horn, Johnson, 1985) there exists an unitary matrix U such that the family $\mathcal{B} = \{U^*FU : F \in \mathcal{F}\}$ is upper triangular (and is Hurwitz stable). Applying to $\mathbb{A}$ Lemma 3.1 and setting $Q = UDU^*$, we obtain

$$F^T Q + QF < 0 \tag{8}$$

for all $F \in \mathcal{F}$. Since $Q \in \mathbb{C}^{n\times n}$,$Q > 0$. Then $Q = P + jL$, where $P, L \in \mathbb{R}^{n\times n}$ and $P > 0$. Then from (9) follows (8). Implication $\Longleftarrow$) follows from the Lyapunov theorem.

As follows from above for matrix $A \in \mathbb{R}^{n\times n}$, $\mathbb{D}$-stability is equivalent to invertebility of all $g_j(A)$ and Hurwitz stability of all $f_j(A)g_j^{-1}(A)$ $(i = 1, 2, \dots, m)$ (Theorem 2.1). Then from Theorem 3.1 and (9), we obtain the following Corollary.

Corollary *Let $\mathcal{F} \subset \mathbb{R}^{n\times n}$ be a compact commuting family. Then $\mathcal{F}$ is $\mathbb{D}$-stable if and only if there exists $P \in \mathbb{R}^{n\times n}$, $P > 0$ such that for all $F \in \mathcal{F}$ $(j = 1, 2, \dots, m)$*

$$\left[f_j(F)\, g_j^{-1}(A)\right]^T P + P\left[f_j(F)\, g_j^{-1}(A)\right] < 0.$$

4. Conclusion

In this article, we aimed at analyzing the stability of matrices in the region $\mathbb{D}$ described by rational functions. We determined the necessary and sufficient conditions for a commuting family to be $\mathbb{D}$-stable.

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