Journal of Mathematics Research; Vol. 6, No. 4; 2014
ISSN 1916-9795 E-ISSN 1916-9809
Published by Canadian Center of Science and Education

On Cartesian Products of Cyclic Orthogonal Double Covers of Circulants

Ramadan El-Shanawany[1] & Ahmed El-Mesady[1]

[1] Department of Physics and Engineering Mathematics, Faculty of Electronic Engineering, Menoufiya University, Menouf, Egypt

Correspondence: Ahmed. El-Mesady, Department of Physics and Engineering Mathematics, Faculty of Electronic Engineering, Menoufiya University, Menouf, Egypt. E-mail: ahmed_mesady88@yahoo.com

Received: August 26, 2014 Accepted: September 15, 2014 Online Published: November 10, 2014
doi:10.5539/jmr.v6n4p118 URL: http://dx.doi.org/10.5539/jmr.v6n4p118

Abstract

A collection $\mathcal{G}$ of isomorphic copies of a given subgraph G of T is said to be *orthogonal double cover* (ODC) of a graph T by G, if every edge of T belongs to exactly two members of $\mathcal{G}$ and any two different elements from $\mathcal{G}$ share at most one edge. An ODC $\mathcal{G}$ of T is *cyclic* (CODC) if the cyclic group of order $|V(T)|$ is a subgroup of the automorphism group of $\mathcal{G}$. In this paper, the CODCs of infinite regular circulant graphs by certain infinite graph classes are considered, where the circulant graphs are labelled by the Cartesian product of two abelian groups.

Keywords: orthogonal double cover, orthogonal labelling, circulant graph

1. Introduction

A generalization of notion of an orthogonal double cover (ODC) to arbitrary underlying graphs is as follows. Let T be an arbitrary graph with n vertices and let $\mathcal{G} = \{G_0, G_1, \ldots, G_{n-1}\}$ be a collection of n spanning subgraphs of T. $\mathcal{G}$ is called an orthogonal double cover (ODC) of T if there exists a bijective mapping $\phi : V(T) \to \mathcal{G}$ such that:

(1) Every edge of T is contained in exactly two of the graphs $G_0, G_1, \ldots, G_{n-1}$.

(2) For every choice of different vertices a, b of T,

$$|E(\phi(a)) \cap E(\phi(b))| = \begin{cases} 1 & \text{if} \quad \{a,b\} \in T, \\ 0 & \text{otherwise.} \end{cases}$$

Where $E(\phi(a))$ and $E(\phi(b))$ refer to the edge sets of the graphs $\phi(a)$ and $\phi(b)$ respectively, generally $E(G)$ refers to the edge set of the graph G.

An automorphism of an orthogonal double cover (ODC) $\mathcal{G} = \{G_0, G_1, G_2, \ldots, G_{n-1}\}$ of T is a permutation $\pi : V(T) \longrightarrow V(T)$ such that $\{\pi(G_0), \pi(G_1), \ldots, \pi(G_{n-1})\} = \mathcal{G}$, where for $i \in \{0, 1, 2, \ldots, n-1\}$, $\pi(G_i)$ is a subgraph of T with $V(\pi(G_i)) = \{\pi(v) : v \in V(G_i)\}$ and $E(\pi(G_i)) = \{\{\pi(u), \pi(v)\} : \{u, v\} \in E(G_i)\}$. An orthogonal double cover (ODC) $\mathcal{G}$ of T is cyclic orthogonal double cover (CODC) if the cyclic group of order $|V(T)|$ is a subgroup of the automorphism group of $\mathcal{G}$, the set of all automorphisms of $\mathcal{G}$. Let $\Gamma = \{\gamma_0, \gamma_1, \ldots, \gamma_{n-1}\}$ be an (additive) abelian group of order n. The vertices of $K_{n,n}$ will be labeled by the elements of $\Gamma \times \mathbb{Z}_2$. Namely, for $(v, i) \in \Gamma \times \mathbb{Z}_2$ we will write v_i for the corresponding vertex and define $\{w_i, u_j\} \in E(K_{n,n})$ if and only if $i \neq j$, for all $w, u \in \Gamma$ and $i, j \in \mathbb{Z}_2$.

Let G be a spanning subgraph of $K_{n,n}$ and let $a \in \Gamma$. Then the graph G with $E(G + a) = \{(u + a, v + a) : (u, v) \in E(G)\}$ is called the a-translate of G. The length of an edge $e = (u, v) \in E(G)$ is defined by $d(e) = v - u$.

G is called a half starter with respect to Γ if $|E(G)| = n$ and the lengths of all edges in G are different, i.e. $\{d(e) : e \in E(G)\} = \Gamma$. The following three results were established in (El-Shanawany, 2002).

Theorem 1. *(El-Shanawany, 2002) If G is a half starter, then the union of all translates of G forms an edge decomposition of $K_{n,n}$, i.e. $\bigcup_{a\in\Gamma} E(G + a) = E(K_{n,n})$.*

Here, the half starter will be represented by the vector: $v(G) = (v_{\gamma_0}, v_{\gamma_1}, \ldots, v_{\gamma_{n-1}})$, where $v_{\gamma_i} \in \Gamma$ and $(v_{\gamma_i})_0$ is the unique vertex $((v_{\gamma_i}, 0) \in \Gamma \times \{0\})$ that belongs to the unique edge of length γ_i.

Two half starter vectors $v(G_0)$ and $v(G_1)$ are said to be orthogonal if $\{v_\gamma(G_0) - v_\gamma(G_1) : \gamma \in \Gamma\} = \Gamma$.

Theorem 2. *(El-Shanawany, 2002) If two half starters $v(G_0)$ and $v(G_1)$ are orthogonal, then $G = \{G_{a,i} : (a,i) \in \Gamma \times \mathbb{Z}_2\}$ with $G_{a,i} = G_i + a$ is an ODC of $K_{n,n}$.*

The subgraph G_s of $K_{n,n}$ with $E(G_s) = \{\{u_0, v_1\} : \{v_0, u_1\} \in E(G)\}$ is called the symmetric graph of G. Note that if G is a half starter, then G_s is also a half starter.

A half starter G is called a symmetric starter with respect to Γ if $v(G)$ and $v(G_s)$ are orthogonal.

Theorem 3. *(El-Shanawany, 2002) Let n be a positive integer and let G be a half starter represented by $v(G) = (v_{\gamma_0}, v_{\gamma_1}, \ldots, v_{\gamma_{n-1}})$. Then G is symmetric starter if and only if $\{v_\gamma - v_{-\gamma} + \gamma : \gamma \in \Gamma\} = \Gamma$.*

Definition 4. *(El-Shanawany, 2002) Let $G = (\Gamma \times \mathbb{Z}_2, E(G))$ be a symmetric starter, let $\{a, a\}$ be the edge in G with length zero. The graph $H = (\Gamma, E(H))$ is called corresponding graph of G, where $\{a, b\} \in E(H)$ if and only if $(a, b) \in E(G)$ with $a \neq b$.*

Remark 5. *(El-Shanawany, 2002) Note that $|E(G) - \{a, a\}| = n - 1 = |E(H)|$ the number of edges of the graph H.*

$|\cup_{a\in\Gamma} E(G + a) - \cup_{a\in\Gamma}\{a, a\}| = n^2 - n = n(n - 1) = \cup_{a\in\Gamma} E(H + a) = |E(\mathcal{G})|$ the number of edges of an *ODC* of K_n group generated by H.

Theorem 6. *(El-Shanawany, 2002) Let n be a positive integer. Let G be a symmetric starter of $K_{n,n}$ and let H be the corresponding graph of G. Then H is an orthogonal double cover (ODC) − generating graph with respect to Γ.*

The author of (Gronau, Mullin, & Rosa, 1997) introduced the notion of an *orthogonal labelling*. Given a graph $G = (V, E)$ with $n - 1$ edges, a $1 - 1$ mapping $\psi : V(G) \longrightarrow \mathbb{Z}_n$ is an orthogonal labelling of G if:

(1) For every $l \in \{1, 2, \ldots, \lfloor\frac{(n-1)}{2}\rfloor\}$, G contains exactly two edges of length l, and exactly one edge of length $n/2$ if n is even, and

(2) $\{r(l) : l \in \{1, 2, \ldots, \lfloor\frac{(n-1)}{2}\rfloor\}\} = \{1, 2, \ldots, \lfloor\frac{(n-1)}{2}\rfloor\}$.

The following theorem of (Gronau et al., 1997) relates cyclic orthogonal double covers (CODCs) of K_n and the orthogonal labelling.

Theorem 7. *(Gronau et al., 1997). A cyclic orthogonal double covers (CODC) of K_n by a graph G exists if and only if there exists an orthogonal labelling of G.*

The following theorem of Sampathkumar and Srinivasan is a generalization of Theorem 7

Theorem 8. *(Sampathkumar, et al., 2011). A cyclic orthogonal double cover (CODC) of $Circ(n; \{d_1, d_2, \ldots, d_k\})$ by a graph G exists if and only if there exists an orthogonal $\{d_1, d_2, \ldots, d_k\}$-labelling of G.*

For results on orthogonal double cover of circulant graphs, see (Sampathkumar & Srinivasan, 2011), (Higazy, 2013), and (El-Shanawany & Shabana, 2014). In (Gronau et al., 1997), (Gronau, Hartmann, Grüttmüller, U. Leck, & V. Leck, 2002), (Scapellato, El-Shanawany, & Higazy, 2009), and (El-Shanawany, Higazy, & El-Mesady, 2013), other results of ODCs by different graph classes can be found. In (Balakrishnan & Ranganathan, 2012), the other terminology not defined here can be found.

(El-Shanawany et al., 2013) were the first ones introduced the cartesian product method as a recursive constructing method for orthogonal double covers of complete bipartite graphs , and this idea is potentially promising for treating various graph lift, see (Shang, 2012), and (Linial & Puder, 2010).

2. On Cartesian Products of Cyclic Orthogonal Double Covers of Circulants

Hereafter, we will use the operation $\star$ for the usual multiplication and $\times$ for cartesian product and if there is no danger of ambiguity, if $(i, j) \in \mathbb{Z}_n \times \mathbb{Z}_m$ we can write (i, j) as ij. The above two Theorems 7, 8 motivated us to the following:

Using the fact that there exists a bijective mapping $\Phi : \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \longrightarrow \mathbb{Z}_{n_1 \star n_2}$ defined by $\Phi(ij) = n_2 i + j : i \in \mathbb{Z}_{n_1}, j \in \mathbb{Z}_{n_2}$, and hence we consider $xy > pq$ if $x > p$ or if $x = p$ and $y > q$ where $xy, pq \in \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}$ and $x \star y, p \star q \in \mathbb{Z}_{n_1 \star n_2}$. The circulant graph $Circ(n_1 \star n_2; X)$ has a vertex set $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}$, where $\mathbb{Z}_{n_1} = \{0, 1, \ldots, n_1 - 1\}$, $\mathbb{Z}_{n_2} = \{0, 1, \ldots, n_2 - 1\}$, and $X \subset \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}$. Two vertices ab and cd are adjacent if and only if $ab - cd = \pm(\alpha\beta)$, where $\alpha\beta \in X$, and a, c, and α are calculated inside $\mathbb{Z}_{n_1}$ and b, d, and β are calculated inside $\mathbb{Z}_{n_2}$. For an edge $\{ab, cd\}$ in $Circ(n_1 \star n_2; X)$, the length of $\{ab, cd\}$ is $min\{|ab - cd|, n_1 n_2 - |ab - cd|\}$. Given two edges $e_1 = \{ab, cd\}$ and $e_2 = \{fg, uv\}$ of the same length $\alpha\beta$ in $Circ(n_1 \star n_2; X)$, the rotation-distance $r(\alpha\beta)$ between e_1 and e_2 is $r(\alpha\beta) = min\{wz, st : \{ab + wz, cd + wz\} = e_2, \{fg + st, uv + st\} = e_1\}$, where addition and difference for a, c, f, and u are calculated inside $\mathbb{Z}_{n_1}$ and for b, d, g, and v are calculated inside $\mathbb{Z}_{n_2}$. Note that if $r(\alpha\beta) = \alpha\beta$, then the edges e_1 and e_2 are adjacent; if $r(\alpha\beta) \neq \alpha\beta$, then the edges e_1 and e_2 are nonadjacent. Consider the Cayley graph $Cay(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}, X) = Circ(n_1 \star n_2; X)$. Given a subgraph G of $Circ(n_1 \star n_2; X)$, a $1 - 1$ mapping $\Psi : V(G) \longrightarrow \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}$ is an orthogonal X-labelling of G if one of the following cases in the proof of the following theorem is verified.

Theorem 9. *A cyclic orthogonal double cover (CODC) of $Circ(n_1 \star n_2; X)$ by a graph G exists if and only if there exists an orthogonal X-labelling of G.*

Proof. **Case 1**. Let n_1 be even and $n_2 > 1$ be odd.

Subcase 1.1. For $n_1 > 2$, we find that:

(a) For every $\alpha\beta \in X_1$:

$$X_1 = \begin{cases} 0\beta & : \quad 1 \leq \beta \leq \lfloor \frac{n_2}{2} \rfloor, \\ \alpha\beta & : \quad 1 \leq \alpha \leq \frac{n_1}{2} - 1, \beta \in \mathbb{Z}_{n_2}, \\ \frac{n_1}{2}\beta & : \quad 1 \leq \beta \leq \lfloor \frac{n_2}{2} \rfloor. \end{cases}$$

G contains exactly two edges of length $\alpha\beta$, and exactly one edge of length $X_2 = \{\frac{n_1}{2}0\}$,

(b) $\{ r(\alpha\beta) : \alpha\beta \in X_1\} = X_1$, then $X = X_1 \cup X_2$.

Subcase 1.2. For $n_1 = 2$, we find that:

(a) For every $\alpha\beta \in X_1$:

$X_1 = \{ \alpha\beta \; : \; \alpha \in \mathbb{Z}_2, 1 \leq \beta \leq \lfloor \frac{n_2}{2} \rfloor \}$.

G contains exactly two edges of length $\alpha\beta$, and exactly one edge of length $X_2 = \{10\}$,

(b) $\{ r(\alpha\beta) : \alpha\beta \in X_1\} = X_1$, then $X = X_1 \cup X_2$.

Case 2. Let $n_1 > 1$ be odd and n_2 be even.

Subcase 2.1. For $n_2 > 2$, we find that:

(a) For every $\alpha\beta \in X_1$:

$$X_1 = \begin{cases} \alpha 0 & : \quad 1 \leq \alpha \leq \lfloor \frac{n_1}{2} \rfloor, \\ \alpha\beta & : \quad 0 \leq \alpha \leq \frac{1}{2}(n_1 - 1), 1 \leq \beta \leq \frac{n_2}{2} - 1, \\ n_1 n_2 - \alpha\beta & : \quad \frac{1}{2}(n_1 - 1) < \alpha < n_1, 1 \leq \beta \leq \frac{n_2}{2} - 1, \\ \alpha\frac{n_2}{2} & : \quad 1 \leq \alpha \leq \lfloor \frac{n_1}{2} \rfloor. \end{cases}$$

G contains exactly two edges of length $\alpha\beta$, and exactly one edge of length $X_2 = \{0\frac{n_2}{2}\}$,

(b) $\{ r(\alpha\beta) : \alpha\beta \in X_1\} = X_1$, then $X = X_1 \cup X_2$.

Subcase 2.2. For $n_2 = 2$, we find that:

(a) For every $\alpha\beta \in X_1$: $\quad X_1 = \{ \alpha\beta \;\text{ if }\; 1 \leq \alpha \leq \lfloor \frac{n_1}{2} \rfloor, \beta \in \mathbb{Z}_2 \}$.

G contains exactly two edges of length $\alpha\beta$, and exactly one edge of length $X_2 = \{01\}$,

(b) $\{ r(\alpha\beta) : \alpha\beta \in X_1\} = X_1$, then $X = X_1 \cup X_2$.

Case 3. Let n_1 and n_2 be odd.

Subcase 3.1. For $n_1 > 3$ and $n_2 > 3$, we find that:

(a) For every $\alpha\beta \in X_1$:

$$X_1 = \begin{cases} 0\beta & : \quad 1 \le \beta \le \lfloor \frac{n_2}{2} \rfloor, \\ \alpha\beta & : \quad 1 \le \alpha \le \lfloor \frac{n_1}{2} \rfloor - 1, \beta \in \mathbb{Z}_{n_2}, \\ \lfloor \frac{n_1}{2} \rfloor \beta & : \quad \beta \in \mathbb{Z}_{n_2}. \end{cases}$$

G contains exactly two edges of length $\alpha\beta$.

(b) $\{ r(\alpha\beta) : \alpha\beta \in X_1\} = X_1,$ then $X = X_1$.

Subcase 3.2. For $n_1 = 3$ and $n_2 > 3$, we find that:

(a) For every $\alpha\beta \in X_1$:

$$X_1 = \begin{cases} 0\beta & : \quad 1 \le \beta \le \lfloor \frac{n_2}{2} \rfloor, \\ \lfloor \frac{n_1}{2} \rfloor \beta & : \quad \beta \in \mathbb{Z}_{n_2}. \end{cases}$$

G contains exactly two edges of length $\alpha\beta$.

(b) $\{ r(\alpha\beta) : \alpha\beta \in X_1\} = X_1,$ then $X = X_1$.

Subcase 3.3. For $n_1 > 3$ and $n_2 = 3$, we find that:

(a) For every $\alpha\beta \in X_1$:

$$X_1 = \begin{cases} \alpha 0 & : \quad 1 \le \alpha \le \lfloor \frac{n_1}{2} \rfloor, \\ \alpha \lfloor \frac{n_1}{2} \rfloor & : \quad \alpha \in \mathbb{Z}_{n_1}. \end{cases}$$

G contains exactly two edges of length $\alpha\beta$.

(b) $\{ r(\alpha\beta) : \alpha\beta \in X_1\} = X_1,$ then $X = X_1$.

Subcase 3.4. For $n_1 = 3$ and $n_2 = 3$ is trivial case.

Case 4. Let n_1 and n_2 be even.

Subcase 4.1. For $n_1 > 2$ and $n_2 > 2,$ we find that:

(a) For every $\alpha\beta \in X_1$:

$$X_1 = \begin{cases} \alpha\beta & : \quad \alpha \in \{0, \frac{n_1}{2}\}, 1 \le \beta \le \frac{n_2}{2} - 1, \\ \alpha\beta & : \quad 1 \le \alpha \le \frac{n_1}{2} - 1, \beta \in \mathbb{Z}_{n_2}. \end{cases}$$

G contains exactly two edges of length $\alpha\beta,$ and exactly one edge of length

$X_2 = \left\{ \frac{n_1}{2} 0, 0 \frac{n_2}{2}, \frac{n_1}{2} \frac{n_2}{2} \right\},$

(b) $\{ r(\alpha\beta) : \alpha\beta \in X_1\} = X_1$, then $X = X_1 \cup X_2$.

Subcase 4.2. For $n_1 = 2$ and $n_2 > 2,$ we find that:

(a) For every $\alpha\beta \in X_1$:

$X_1 = \left\{ \alpha\beta : \alpha \in \mathbb{Z}_2, 1 \le \beta \le \frac{n_2}{2} - 1 \right\}.$

G contains exactly two edges of length $\alpha\beta,$ and exactly one edge of length

$X_2 = \left\{ 10, 0\frac{n_2}{2}, 1\frac{n_2}{2} \right\},$

(b) $\{ r(\alpha\beta) : \alpha\beta \in X_1\} = X_1,$ then $X = X_1 \cup X_2$.

Subcase 4.3. For $n_1 > 2$ and $n_2 = 2$, we find that:

(a) For every $\alpha\beta \in X_1$:

$X_1 = \left\{ \alpha\beta : 1 \le \alpha \le \frac{n_1}{2} - 1, \beta \in \mathbb{Z}_2 \right\}.$

G contains exactly two edges of length $\alpha\beta,$ and exactly one edge of length

$X_2 = \left\{ 01, \frac{n_1}{2} 0, \frac{n_1}{2} 1 \right\},$

(b) $\{ r(\alpha\beta) : \alpha\beta \in X_1\} = X_1$, then $X = X_1 \cup X_2$.

Subcase 4.4. For $n_1 = 2$ and $n_2 = 2$ is trivial case. □

For all positive integers m with $\gcd(m, 3) = 1$, $k = l + m$, and $l \in \mathbb{Z}_m,$ then for the following Theorem let H_1^m be a graph with the following edges set: $E\left(H_1^m\right) = \{(0l, 0l + ij) : ij \in Y_1 \backslash \{00\}\} \cup \{(0k, 0k + ij) : ij \in Y_2\} \cup \{(1l, 1l + ij) : ij \in Y_3\} \cup \{(1k, 1k + ij) : ij \in Y_4\}$, where $Y_1 = A_1 \times A_2,\ Y_2 = A_1 \times A_4, Y_3 = A_3 \times A_2, Y_4 = A_3 \times A_4,\ A_1 = \{0, 2\},\ A_2 = \{l, l + 2m\},\ A_3 = \{1, 3\},\ A_4 = \{l + m, l + 3m\}$.

Theorem 10. *For all positive integers m with* $\gcd(m, 3) = 1$*, there exists a CODC of* $Circ(16m; X)$ *by* H_1^m *with respect to* $\mathbb{Z}_4 \times \mathbb{Z}_{4m}$.

Proof. Let us define $\Psi : V(H_1^m) \longrightarrow \mathbb{Z}_4 \times \mathbb{Z}_{4m}$ by

$$\Psi(v_\alpha) = \begin{cases} 0\alpha & : \quad 0 \le \alpha \le 2m - 1, \\ 0\delta & : \quad 2m \le \alpha \le 3m - 1, \delta = 2(\alpha - m), \\ 1\gamma & : \quad 3m \le \alpha \le 5m - 1, \gamma = \alpha - 3m, \\ 2w & : \quad 5m \le \alpha \le 7m - 1, w = 2(\alpha - 5m). \end{cases}$$

Hence, the edge of length $0l$ is $(\Psi(v_l), \Psi(v_{2l}))\backslash(\Psi(v_0), \Psi(v_0))$, the edge of length $2l$ is $(\Psi(v_l), \Psi(v_{l+5m}))$, the edge of length $0(l + 2m)$ is $(\Psi(v_l), \Psi(v_{l+2m}))$, the edge of length $2(l + 2m)$ is $(\Psi(v_l), \Psi(v_{l+6m}))$, the edge of length $0(l + m)$ is $(\Psi(v_{l+m}), \Psi(v_{l+2m}))$, the edge of length $0(l + 3m)$ is $(\Psi(v_{l+m}), \Psi(v_{2l}))$, the edge of length $2(l + m)$ is $(\Psi(v_{l+m}), \Psi(v_{l+6m}))$, the edge of length $2(l + 3m)$ is $(\Psi(v_{l+m}), \Psi(v_{l+5m}))$, the edge of length$1l$ is $(\Psi(v_{l+3m}), \Psi(v_{l+5m}))$, the edge of length $1(l + 2m)$ is $(\Psi(v_{l+3m}), \Psi(v_{l+6m}))$, the edge of length $3l$ is $(\Psi(v_{l+3m}), \Psi(v_{2l}))$, the edge of length $3(l + 2m)$ is $(\Psi(v_{l+3m}), \Psi(v_{l+2m}))$, the edge of length $1(l + m)$ is $(\Psi(v_{l+4m}), \Psi(v_{l+6m}))$, the edge of length $1(l + 3m)$ is $(\Psi(v_{l+4m}), \Psi(v_{l+5m}))$, the edge of length $3(l + m)$ is $(\Psi(v_{l+4m}), \Psi(v_{l+2m}))$, and the edge of length $3(l + 3m)$ is $(\Psi(v_{l+4m}), \Psi(v_{2l}))$.

From the edges set of H_1^m, and according to Subcase 4.1 of Theorem 9, we can find that:

(a) For every $\alpha\beta \in X_1$;

$$X_1 = \begin{cases} \alpha\beta & : \quad \alpha \in \{0, 2\}, 1 \le \beta \le 2m - 1, \\ \alpha\beta & : \quad \alpha = 1, \beta \in \mathbb{Z}_{4m}. \end{cases}$$

H_1^m contains exactly two edges of length $\alpha\beta$, and exactly one edge of length $X_2 = \{20, 0\gamma, 2\gamma : \gamma = 2m\}$,

(b) $\{\ r(\alpha\beta) : \alpha\beta \in X_1\} = X_1$, then $X = X_1 \cup X_2$. □

For an illustration of Theorem 10, let $m = 2$, then there exists a CODC of $Circ(32;\ X)$ by H_1^2 with respect to $\mathbb{Z}_4 \times \mathbb{Z}_8$, where $X = \{01, 02, 03, 21, 22, 23, 10, 11, 12, 13, 14, 15, 16, 17, 20, 04, 24\}$, see Figure 1.

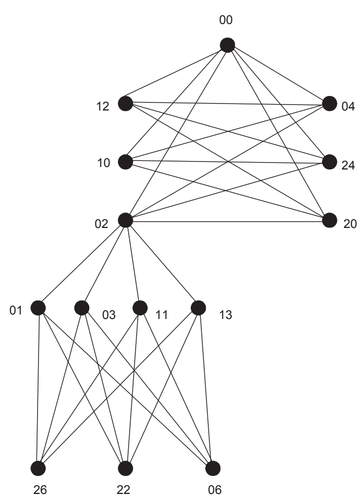

Figure 1

3. Conclusion

In conclusion, in the future we can get the orthogonal double covers of circulant graphs by new graph classes where the circulant graphs are labelled by the Cartesian product of two abelian groups.

References

Balakrishnan, R., & Ranganathan, K. (2012). A Textbook of Graph Theory. Universitext, chapter 1, Springer, New York, NY, USA, 2nd edition. http://dx.doi.org/10.1007/978-1-4419-8505-7_1.

El-Shanawany, R., Higazy, M., & El-Mesady, A. (2013). On Cartesian Products of Orthogonal Double Covers. *International Journal of Mathematics and Mathematical Sciences,* 4. http://dx.doi.org/1155/2013/265136.

El-Shanawany, R., & Shabana, H. (2014). On Orthogonal Double Covers of Circulant Graphs. *British Journal of Mathematics* & *Computer Science, 4*(3), 394-401. http://dx.doi.org/10.9734/BJMCS/2014/5999.

El-Shanawany, R. (2002). Orthogonal double covers of complete bipartite graphs (Unpublished doctoral dissertation). Universitat Rostock, Germany.

Gronau, H. -D .O. F., Hartmann, S., Grüttmüller, M., Leck, U., & Leck, V. (2002). On orthogonal double covers of graphs. *Des. Codes Cryptogr, 27*, 49-91. http://dx.doi.org/10.1023/A:1016546402248.

Gronau, H. -D. O. F., Mullin, R. C., & Rosa, A. (1997). Orthogonal double covers of complete graphs by trees. *Graphs Combin, 13*, 251-262. http://dx.doi.org/10.1007/BF03353002.

Higazy, M. (2013). On Cyclic Orthogonal Double Covers of Circulant Graphs using Infinite Graph Classes. *British Journal of Mathematics* & *Computer Science, 3*(3), 425-436. http://dx.doi.org/10.9734/BJMCS/2013/4165.

Linial, N., & Puder, D. (2010). Words maps and spectra of random graph lifts. *Random Structure and Algorithms, 37*, 100-135. http://dx.doi.org/10.1002/rsa.20304.

Sampathkumar, R., & Srinivasan, S. (2011). Cyclic orthogonal double covers of regular circulant graphs. *Discrete Mathematics, 311*, 2417-2422. http://dx.doi.org/10.1016/j.disc.2011.06.021.

Scapellato, R., El-Shanawany, R., & Higazy, M. (2009). Orthogonal double covers of Cayley graphs. *Discrete Appl. Math, 157*, 3111-3118. http://dx.doi.org/10.1016/j.dam.2009.06.005.

Shang, Y. (2012). Random lifts of graphs: network robustness based on the Estrada index. *Applied Mathematics E-Notes, 12*, 53-61. http://www.math.nthu.edu.tw/_amen/

Copyrights