Strongly Hopfian and Strongly Cohopfian Objects in the Category of Complexes of Left A-Modules

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Abstract

The object of this paper is the study of *strongly hopfian*, *strongly cohopfian*, *quasi-injective*, *quasi-projective*, *Fitting* objects of the category of complexes of A-modules.

In this paper we demonstrate the following results:

a) If C is a strongly hopfian chain complex (respectively strongly cohopfian chain complex) and E a subcomplex which is direct summand then E and C/E are both strongly Hopfian (respectively strongly coHopfian) then C is strongly Hopfian (respectively strongly coHopfian).

b)Given a chain complex C, if C is quasi-injective and strongly-hopfian then C is strongly cohopfian.

c)Given a chain complex C, if C is quasi-projective and strongly-cohopfian then C is strongly hopfian.

In conclusion the main result of this article is the following theorem:

Any *quasi-projective* and *strongly-hopfian* or *quasi-injective* and *strongly-cohofian* chain complex of *A*-modules is a *Fitting* chain complex.

Keywords: chain complex, strongly-hopfian, strongly-cohopfian, quasi-injective, quasi-projective, fitting chain complex

1. Introduction

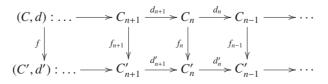
In this paper, we will study *strongly hopfian*, *strongly cohopfian*, *quasi-injective*, *quasi-projective* objects of the category of complexes of A-modules denoted by COMP.

We also study Fitting objects of the category of complexes of A-modules.

The objects of COMP are chain complexes and the morphisms are maps of chains.

A chain complex (C, d): ... $\rightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \dots$ is a sequence (d_n) of homomorphisms of left *A*-modules verifying $d_n \circ d_{n+1} = 0$, for all $n \in \mathbb{Z}$.

A chain map f of (C, d) into (C', d') is defined by:



with $d'_{n+1} \circ f_{n+1} = f_n \circ d_{n+1}, \forall n \in \mathbb{Z}$.

Given (C, d) an object of *COMP* and *f* an endomorphism of (C, d).

C is said to be hopfian (respectively cohopfian) if any epimorphism (respectively monomorphism) is an isomorphism.

A chain complex $(C, d): \ldots \to C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \ldots$ is said to be fully invariant if and only if each *A*-module C_n is fully invariant, $\forall n \in \mathbb{Z}$.

A chain complex $(C, d): \ldots \to C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \ldots$ is said essential if and only if each A- module C_n is essential, $\forall n \in \mathbb{Z}$.

A chain complex (C, d) is said *Fitting* if for any endomorphism f of (C, d), there exists an integer n such $C = Imf^n \oplus Kerf^n$.

After (C, d) is noted by C.

In this paper we demonstrate the following results:

a) If *C* is a strongly hopfian chain complex (respectively strongly cohopfian chain complex) and *E* a subcomplex is direct summand then *E* and C/E are both strongly Hopfian (respectively strongly coHopfian) then *C* is strongly Hopfian (respectively strongly coHopfian).

b) If *E* is a fully invariant chain complex such that *E* and C/E are strongly Hopfian (respectively strongly coHopfian) then *C* is strongly Hopfian (respectively strongly coHopfian).

c) Given $C = \oplus C^i$ such C^i is fully invariant, if C^i is a strongly hopfian (respectively strongly cohopfian) chain complex then C is a strongly hopfian (respectively strongly cohopfian) chain complex.

d) Given a chain complex $C: \ldots \to C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \ldots, C$ is *quasi – injective*, if and only if, C_n is *quasi – injective*, for all $n \in \mathbb{Z}$.

e) A chain complex $C: \ldots \to C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \ldots$, is *quasi – projective*, if and only if, C_n is *quasi – projective*, for all $n \in \mathbb{Z}$.

f) Given a chain complex C, if C is quasi-injective and strongly- hopfian then C is strongly cohopfian.

g) Given a chain complex C, if C is quasi-projective and strongly- cohopfian then C is strongly hopfian.

h) Any *quasi-projective* and *strongly-hopfian* or *quasi-injective* and *strongly-cohofian* chain complex of *A*-modules is a *Fitting* chain complex.

In this paper A denotes a not inevitably commutative, unitarian associative ring and M a left unifere module.

2. Definitions and Preliminary Results on the Category COMP

Definition 1 Given (C, d) a chain complex of left A modules and f an endomorphism of (C, d) such that:

$$(C,d):\ldots \longrightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \longrightarrow \cdots$$

$$f \downarrow \qquad f_{n+1} \downarrow \qquad f_n \downarrow \qquad f_{n-1} \downarrow$$

$$(C,d):\ldots \longrightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \longrightarrow \cdots$$

We call $f \circ f$ the chain map compounded of f by itself denoted f^2 . We also define $(f^2)_n = f_n \circ f_n$, for all $n \in \mathbb{Z}$:

We also define f^k such $f_n^k = f_n \circ f_n \circ \ldots \circ f_n$, with *k* factors, for all $n \in \mathbb{Z}$.

Proposition 1 *Considering* (C, d) *a chain complex of* A*– modules and* f *an endomorphism of* (C, d)*:*

$$(C, d): \dots \longrightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \longrightarrow \cdots$$

$$f \downarrow \qquad f_{n+1} \downarrow \qquad f_n \downarrow \qquad f_{n-1} \downarrow$$

$$(C, d): \dots \longrightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \longrightarrow \cdots$$

Given

$$\Delta_{n+1}^k: Kerf_{n+1}^k \to Kerf_n^k$$
$$x \mapsto d_{n+1}(x)$$

where $f_{n+1}^k = f_{n+1} \circ f_{n+1} \circ \ldots \circ f_{n+1}$ with k factors, then Δ_{n+1}^k is the induced morphism by Δ_{n+1}^{k+1} . *Proof.* Considering

$$\Delta_{n+1}^{k}: Kerf_{n+1}^{k} \to Kerf_{n}^{k}$$

$$x \mapsto d_{n+1}(x)$$

$$\Delta_{n+1}^{k+1}: Kerf_{n+1}^{k+1} \to Kerf_{n}^{k+1}$$

$$x \mapsto d_{n+1}(x).$$

and

We obtain $Kerf_{n+1}^k \subseteq Kerf_{n+1}^{k+1}$ et $Kerf_n^k \subseteq Kerf_n^{k+1}$ therefore Δ_{n+1}^k is the induced by Δ_{n+1}^{k+1} . **Proposition 2** Let be (C, d) a chain complex of A-modules and f an endomorphism of (C, d) such that:

$$(C,d):\ldots \longrightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \longrightarrow \cdots$$

$$f \downarrow \qquad f_{n+1} \downarrow \qquad f_n \downarrow \qquad f_{n-1} \downarrow$$

$$(C,d):\ldots \longrightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \longrightarrow \cdots$$

Given

$$\begin{aligned} \delta^k_{n+1} &: Imf^k_{n+1} &\to Imf^k_n \\ & x &\mapsto d_{n+1}(x) \end{aligned}$$

Then δ_{n+1}^{k+1} is the induced morphism by δ_{n+1}^k .

Proof. Given

$$\delta_{n+1}^{k}: Imf_{n+1}^{k} \to Imf_{n}^{k}$$

$$x \mapsto d_{n+1}(x)$$

$$\delta_{n+1}^{k}: Imf_{n+1}^{k+1} \to Imf_{n}^{k+1}$$

$$x \mapsto d_{n+1}(x)$$

and

then $Imf_{n+1}^{k+1} \subseteq Imf_{n+1}^k$ et $Imf_n^{k+1} \subseteq Imf_n^k$ d'o δ_{n+1}^{k+1} is the morphism induced by δ_{n+1}^k .

Definition 2 Let $C: \ldots \to C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \ldots$ a chain complex and $E: \ldots \to E_{n+1} \xrightarrow{u_{n+1}} E_n \xrightarrow{u_n} E_{n-1} \xrightarrow{u_{n-1}} \ldots$ a subcomplex of C. Then E is direct summand, if and only if, E_n is direct summand.

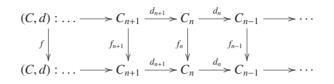
Definition 3 Let *N* a sub-module of a *A*- module *M*. Then *N* is said to be fully invariant, if for any endomorphism *f* of *M* we have $f(N) \subseteq N$.

Definition 4 Given $C : \ldots C_{n+1} \to C_n \to C_{n-1} \to \ldots$ a chain complexes of A- modules and $E : \ldots E_{n+1} \to E_n \to E_{n-1} \to \ldots$ a subcomplex of C. Then E is said to be subcomplex fully invariant in C, if for all $n \in \mathbb{Z}$, E_n is fully invariant.

3. Strongly Hopfian Objects and Strongly CoHopfian Objects of the Category of Complexes of Left *A*-Modules

3.1 Strongly Hopfian and Strongly CoHopfian

Definition 5 Let be f an endomorphism of chain complex (C, d):



Then:

a) the chain complex

$$(Kerf^k): \ldots \to Kerf^k_{n+1} \xrightarrow{\Delta^k_{n+1}} Kerf^k_n \xrightarrow{\Delta^k_n} Kerf^k_n \xrightarrow{\Delta^k_{n-1}} \ldots$$

stabilizes, if and only if, it exists $k_0 \in \mathbb{N}^*$ such that $(Kerf^{k_0}) = (Kerf^{k_0+s})$, for all $s \in \mathbb{N}$. b) the chain complex

$$(Imf^k):\ldots \to Imf_{n+1}^k \xrightarrow{\delta_{n+1}^k} Imf_n^k \xrightarrow{\delta_n^k} Imf_{n-1}^k \xrightarrow{\delta_{n-1}^k} \ldots$$

stabilizes if it exists $k_0 \in \mathbb{N}^*$ such that $(Imf^{k_0}) = Imf^{k_0+s}$, for all $s \in \mathbb{N}$.

Proposition 3 *Given f an endomorphism of a chain complex* (*C*, *d*):

$$(C,d): \dots \longrightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \longrightarrow \cdots$$

$$f \downarrow \qquad f_{n+1} \downarrow \qquad f_n \downarrow \qquad f_{n-1} \downarrow \qquad (C,d): \dots \longrightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \longrightarrow \cdots$$

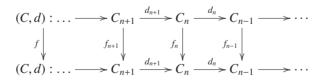
Then $(Kerf^k)$ stabilizes if and only if $(Kerf^k_n)$ stabilizes for all $n \in \mathbb{Z}$.

Proof. If $(Kerf^k)$ stabilizes then it exists $k_0 \in \mathbb{N}^*$ such that $(Kerf^{k_0}) = (Kerf^{k_0+s})$, for all $s \in \mathbb{N}$ hence $Kerf_n^{k_0} = Kerf_n^{k_0+s}$ for all $n \in \mathbb{Z}$. This implies that que $(Kerf_n^k)$ stabilizes for all $n \in \mathbb{Z}$.

Reciprocally suppose that $(Ker f_n^k)$ stabilizes for all $n \in \mathbb{Z}$ then it exists $k_0 \in \mathbb{N}^*$ such that for all $s \in \mathbb{N}$ then $Ker f_n^{k_0} = Ker f_n^{k_0+s}$ for all $n \in \mathbb{Z}$.

So $(Kerf^k)$ stabilizes for all positive integer k.

Proposition 4 *Given* f *an endomorphism of a chain complex* (C, d)*:*



Then (Imf^k) stabilizes if and only if (Imf^k_n) stabilizes for all $n \in \mathbb{Z}$.

Proof. Suppose (Imf^k) stabilizes then it exists $k_0 \in \mathbb{N}^*$ such that $(Imf^{k_0}) = (Imf^{k_0+s})$, for all $s \in \mathbb{N}$ hence $Imf_n^{k_0} = Imf_n^{k_0+s}$ for all $n \in \mathbb{Z}$. Hence (Imf_n^k) stabilizes for all $n \in \mathbb{Z}$.

Reciprocally suppose that (Imf_n^k) stabilizes for all $n \in \mathbb{Z}$ then there is $k_0 \in \mathbb{N}^*$ such that for all $s \in \mathbb{N}$ so $Imf_n^{k_0} = Imf_n^{k_0+s}$ for all $n \in \mathbb{Z}$. That prove (Imf^k) stabilizes for all k.

Definition 6 An A- module M is said to be strongly hopfian (respectively strongly cohopfian) if for any endomorphism f of M the sequence $(Kerf^n)$ stabilizes: $kerf^2 \subseteq kerf^3 \ldots \subseteq Kerf^n$ (respectively $Imf \supseteq Imf^2 \ldots \supseteq Imf^n \ldots$).

Definition 7 A chain complex *C* of *A*-modules is said to be strongly hopfian(respectively strongly cohopfian), if for any endomorphism f of (C, d), $(Kerf^k)$ (respectively (Imf^k)) stabilizes.

Proposition 5 A chain complex C is strongly hopfian if it exists $k \in \mathbb{N}$ such that $Ker f_n^k \cap Im f_n^k = 0$, for all $n \in \mathbb{Z}$.

Proof. Using the Proposition 2.5 (see Hmaimou, 2007) and let be $k \in \mathbb{N}$ such that $Kerf_n^k \cap Imf_n^k = 0$ then $(Kerf_n^k)$ stabilizes for all $n \in \mathbb{Z}$. That prove $(Kerf^k)$ stabilizes, so *C* is a strongly hopfian chain complex.

Proposition 6 *A chain complex C:*

 $(C,d): \dots \longrightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \longrightarrow \cdots$ $f \downarrow \qquad f_{n+1} \downarrow \qquad f_n \downarrow \qquad f_{n-1} \downarrow \qquad (C,d): \dots \longrightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \longrightarrow \cdots$

is strongly cohopfian, if it exists a positive integer k such that:

$$Imf_n^{\kappa} + Kerf_n^{\kappa} = C_n$$
, for all $n \in \mathbb{Z}$.

Proof. Using the Proposition 2.6 (see Hmaimou, 2007), we can suppose it exists a positive integer k such that: $Imf_n^k + Kerf_n^k = C_n$, for all $n \in \mathbb{Z}$ then (Imf_n^k) stabilizes for all $n \in \mathbb{Z}$ hence (Imf^k) stabilizes so C is a strongly cohopfian complex.

Theorem 1 Considering C a chain complex and E a subcomplex of C. If C is strongly hopfian (respectively is strongly cohopfian) and E is direct summand then E et C/E both are strongly hopfian (respectively strongly cohopfian).

Proof. Let us demonstrate at first that E is strongly hopfian (respectively strongly cohopfian).

Suppose $C = E \oplus K$ and let be g a chain map of E in itself which can be extended to C such as $f = g \oplus 0$, with 0 the zero morphism of K.

Given *C* is strongly hopfian (respectively strongly cohopfian), then $(Kerf^k)$ (respectively (Imf^k)) stabilizes then it exists $k_0 \in \mathbb{N}^*$ such that $(Kerf^{k_0}) = (Kerf^{k_0+s})$ (respectively $(Imf^{k_0}) = (Imf^{k_0+s})$), for all $s \in \mathbb{N}$, then $Kerf_n^{k_0} = Kerf_n^{k_0+s}$ (respectively $Imf_n^{k_0+s}$ pour tout $n \in \mathbb{Z}$), therefore C_n is strongly hopfian (respectively strongly cohopfian) and E_n is direct summand.

Hence C_n and C_n/E_n for all $n \in \mathbb{Z}$ are both strongly hopfian (respectively strongly cohopfian).

That prove *E* and C/E are strongly hopfian (respectively strongly cohopfian).

Theorem 2 Considering C a chain complex and E a subcomplexe of C, if E is fully invariant with E and C/E both strongly hopfian (respectively strongly cohopfian) then C is strongly hopfian (respectively strongly cohopfian).

Proof. Let be (C, d) a chain complex of A-modules and f an endomorphism of (C, d) verifying:

$$(C,d):\ldots \longrightarrow C_{m+1} \xrightarrow{d_{m+1}} C_m \xrightarrow{d_m} C_{m-1} \longrightarrow \cdots$$

$$f \downarrow \qquad f_{m+1} \downarrow \qquad f_m \downarrow \qquad f_{m-1} \downarrow$$

$$(C,d):\ldots \longrightarrow C_{m+1} \xrightarrow{d_{m+2}} C_m \xrightarrow{d_m} C_{m-1} \longrightarrow \cdots$$

Given *E* is fully invariant in *C* then *f* induces a chain map *h* of *E* in itself such that:

$$(E,d): \dots \longrightarrow E_{m+1} \xrightarrow{d_{m+1}} E_m \xrightarrow{d_m} E_{m-1} \longrightarrow \dots$$

$$h \downarrow \qquad h_{m+1} \downarrow \qquad h_m \downarrow \qquad h_{m-1} \downarrow$$

$$(E,d): \dots \longrightarrow E_{m+1} \xrightarrow{d_{m+1}} E_m \xrightarrow{d_m} E_{m-1} \longrightarrow \dots$$

f induces also a chain map g of C/E in itself such that:

$$(C/E, u) : \dots \longrightarrow C_{m+1}/E_{m+1} \xrightarrow{u_{m+1}} C_m/E_m \xrightarrow{u_m} C_{m-1}/E_{m-1} \longrightarrow \cdots$$

$$g \downarrow \qquad g_{m+1} \downarrow \qquad g_m \downarrow \qquad g_{m-1} \downarrow$$

$$(C/E, u) : \dots \longrightarrow C_{m+1}/E_{m+1} \xrightarrow{u_{m+1}} C_m/E_m \xrightarrow{u_m} C_{m-1}/E_{m-1} \longrightarrow \cdots$$

Given *E* and *C*/*E* are both strongly cohopfian, hence $Imh^n = Imh^{n+k}$ and $Img^n = Img^{n+k}$, therefore for all $m \in \mathbb{Z}$, we have $Imh_m^n = Imh_m^{n+k}$ and $Img_m^n = Img_m^{n+k}$.

If p = 2n, for $x \in C_m$, we obtain $g_m^n(x + E_m) = g_m^{n+1}(y + E_m)$ for some $y \in E_m$. Then $t = f_m^n(x) - f_m^n(y) \in E_m$, hence $f_m^n(t) = f_m^{p+1}(z)$ for some $z \in E_m$, therefore $f_m^p(x) = f_m^{p+1}(y + z)$, so *C* is strongly cohopfian.

Suppose that *E* and *C*/*E* are both hopfian, then $Kerh^n = Kerh^{n+k}$ and $Kerg^n = Kerg^{n+k}$, hence for all $m \in \mathbb{Z}$, we have $Kerh_m^n = Kerh_m^{n+k}$ and $Kerg_m^n = Kerg_m^{n+k}$.

Let be $x \in Kerf_m^{2n}$, then $g^{2n+1}(x + E_m) = 0$, so $y = f_m^n(x) \in E_m$ and $f_m^{n+1}(y) = 0$. Therefore $y \in Kerh_m^{n+1} = Kerh_m^n$, then $x \in Kerf_m^{2n}$, so *C* is strongly hopfian.

Corollary 1 Let be $C = \oplus C^i$ where C^i is a subcomplex fully invariant of C for all $i \in I$. If C^i is strongly hopfian (respectively strongly cohopfian), then C is strongly hopfian (respectively strongly cohopfian).

Proof. Let be f an endomorphism of chain complex C. Then it exists a unique family (f^i) such that $i \in I$ where $f^i = f|_{C^i}$ and $m = \sum n_i$. f^i is denoted by $[f^i]$. C^i is strongly hopfian then $Ker[f^i]^{n_i} = Ker[f^i]^{n_i+1}$, therefore $\oplus Ker[f^i]^m = \oplus Ker[f^i]^{n_i+1}$. So $Ker(\oplus [f^i])^m = Ker(\oplus [f^i])^{m+1}$, then C is strongly hopfian.

Let be f an endomorphism of C. Then it exists a unique family (f^i) where $i \in I$ $f^i = f|_{C^i}$ and $m = \sum n_i$. C^i is strongly cohopfian, then $Im[f^i]^{n_i} = Im[f^i]^{n_i+1}$, therefore $\oplus Im[f^i]^m = \oplus Im[f^i]^{n_i+1}$. So $Im(\oplus [f^i])^m = Im(\oplus [f^i])^{m+1}$, then C is strongly cohopfian.

3.2 Quasi-Injective Chain Complex

Definition 8 An *A*-module *M* is *quasi-injective*, if for any monomorphism *g* of *A*-module *N* into *M* and for any morphism *f* of *N* into *M*, there exists an endomorphism *h* of *M* such: $f = h \circ g$.

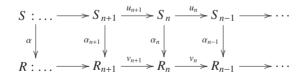


Definition 9 Let us considering two chains complexes of *A*-modules: *C* and *E*. *C* is said to be quasi-injective, if for any monomorphism *g* of *E* into *C* and for any chain map *f* of *E* into *C*, there exists a chain map *h* of *C* into *C* verifying $f = h \circ g$.

Theorem 3 Given $C: \ldots \to C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \to \ldots$ a chain complex of A-modules. C is quasi-injective, if and only if for all $n \in \mathbb{Z}$, C_n a quasi-injective A-module.

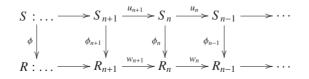
Proof. Suppose that *R* is quasi-injective chain complex and $f: M \to N$ a monomorphism of *A*– modules and $\phi_n: M \to R_n$ a morphism of *A*– modules.

Considering S and R two chains complexes of A – modules and α a chain of S into R such:



where for all $n \in \mathbb{Z}$. $S_n = M$ and $u_n = Id_M$ with $R_n = N$ and $v_n = Id_N \alpha_n$ is a monomorphism of A-modules, for all $n \in \mathbb{Z}$.

Let ϕ a chain map of *S* into *R* such:



where for any morphismes ϕ_k : $M \to R_k$ of A-modules.

Given that *R* is quasi-injective then it exists ψ a chain map:

$$R: \dots \longrightarrow S_{n+1} \xrightarrow{v_{n+1}} S_n \xrightarrow{v_n} S_{n-1} \longrightarrow \cdots$$

$$\psi \downarrow \qquad \psi_{n+1} \downarrow \qquad \psi_n \downarrow \qquad \psi_{n-1} \downarrow$$

$$R: \dots \longrightarrow R_{n+1} \xrightarrow{w_{n+1}} R_n \xrightarrow{w_n} R_{n-1} \longrightarrow \cdots$$

such: $\psi \circ \alpha = \phi$ hence $\psi_n \circ f = \phi_n$. Then it exists $\psi_n \colon R_n \to R_n$ such $\psi_n \circ f = \phi_n$ then R_n is quasi-injective.

Reciprocally suppose that for all $n \in \mathbb{Z}$, R_n is an A-quasi-injective module and let us prove that R quasi-injective chain complex.

Let be γ a monomorphism of chains complexes:

$$S: \dots \longrightarrow S_{n+1} \xrightarrow{u_{n+1}} S_n \xrightarrow{u_n} S_{n-1} \longrightarrow \cdots$$

$$\gamma \bigvee_{\gamma \to 1} \gamma_{n+1} \bigvee_{\gamma \to 1} \gamma_n \bigvee_{\gamma \to 1} \gamma_{n-1} \bigvee_{\gamma \to 1}$$

$$R: \dots \longrightarrow R_{n+1} \xrightarrow{w_{n+1}} R_n \xrightarrow{w_n} R_{n-1} \longrightarrow \cdots$$

Considering β a chain map of complexes such:

$$R: \dots \longrightarrow S_{n+1} \xrightarrow{u_{n+1}} S_n \xrightarrow{u_n} S_{n-1} \longrightarrow \cdots$$

$$\beta \left| \begin{array}{c} \beta_{n+1} \\ \beta_{n+1} \\ \end{array} \right| \begin{array}{c} \beta_n \\ \beta_n \\ \end{array} \right| \begin{array}{c} \beta_{n-1} \\ \beta_{n-1} \\ \end{array}$$

$$R: \dots \longrightarrow R_{n+1} \xrightarrow{w_{n+1}} R_n \xrightarrow{w_n} R_{n-1} \longrightarrow \cdots$$

Then given for all $n \in \mathbb{Z}$, R_n is quasi-injective and γ_n is a monomorphism of A-modules hence it exists $\lambda_n \colon R_n \to R_n$ verifying $\beta_n = \lambda_n \circ \gamma_n$.

Let be λ such:

$$R: \dots \longrightarrow R_{n+1} \xrightarrow{\nu_{n+1}} R_n \xrightarrow{\nu_n} R_{n-1} \longrightarrow \cdots$$

$$\lambda \downarrow \qquad \lambda_{n+1} \downarrow \qquad \lambda_n \downarrow \qquad \lambda_{n-1} \downarrow$$

$$R: \dots \longrightarrow R_{n+1} \xrightarrow{w_{n+1}} R_n \xrightarrow{w_n} R_{n-1} \longrightarrow \cdots$$

Let us demonstrate λ is a chain map:

It is enough that: $w_{n+1} \circ \lambda_{n+1} = \lambda_n \circ v_{n+1}$.

We have: $w_{n+1} \circ \beta_{n+1} = \beta_n u_{n+1}$ but β is chain map, because $\beta_n = \lambda_n \circ \gamma_n$. Then $w_{n+1} \circ (\lambda_{n+1} \circ \gamma_{n+1}) = (\lambda_n \circ \gamma_n) \circ u_{n+1}$. Therefore $(w_{n+1} \circ \lambda_{n+1}) \circ \gamma_{n+1} = \lambda_n \circ (\gamma_n \circ u_{n+1})$. So $(w_{n+1}) \circ \lambda_{n+1} = \lambda_n \circ (v_{n+1} \circ \gamma_{n+1})$. Finally $(w_{n+1} \circ \lambda_{n+1}) \circ \gamma_{n+1} = (\lambda_n \circ v_{n+1}) \circ \gamma_{n+1}$. But γ_{n+1} is a monomorphism of *A*-modules then $w_{n+1} \circ \lambda_{n+1} = \lambda_n \circ v_{n+1}$, for all $n \in \mathbb{Z}$. so λ is achain map.

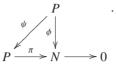
Let us verify that: $\lambda \circ \gamma = \beta$. We know that for all $n \in \mathbb{Z}$, $\beta_n = \lambda_n \circ \gamma_n$ with $\beta = (\beta_n)$, $\gamma = (\gamma_n)$ and $\lambda = (\lambda_n)$ so $\lambda \circ \gamma = \beta$. That prove *R* is a quasi-injective complex.

Theorem 4 Given C a chain complex of A-modules. If C is quasi-injective and strongly hopfian, then C is strongly cohopfian.

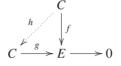
Proof. Suppose that *C* is *quasi-injective* and *strongly hopfian*, then C_n is quasi-injective using Theorem 3 and (Imf_n^k) stabilizing this implies that C_n is quasi-injective and $(kerf^k)$ stabilizes, so *C* is strongly cohopfian.

3.3 Quasi Projective Chain Complex

Definition 10 An *A*-module *P* is said to be quasi-projective if for any *A*-module *N* and any epimorphism $\pi: P \to N$ and any homomorphism $\phi: P \to N$, here exists an endomorphism $\psi: P \to P$ such $\pi \circ \psi = \phi$ illustrated by the following commutative diagramm:



Definition 11 Given *C* and *E* two chains complexes of *A*-modules. *C* is said to be quasi-projective chain complex if for any epimorphism: $C \rightarrow E$ and for any morphism $f: C \rightarrow E$, there exists a chain map $h: C \rightarrow C$ verifying: $f = g \circ h$, illustrated by the following commutative diagramm:



Theorem 5 Let $C: \ldots \to C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \to \ldots$ a chain complex of A-modules. C is quasi-projective if and only if for all $n \in \mathbb{Z}$, C_n is a quasi-projective module.

Proof. Suppose that *E* is quasi-projective.

Considering $f: N \to M$ an epimorphism of A-modules and $\phi_n: E_n \to M$ a morphisme of A-modules.

Let *S* and *E* two chains complexes and α a chain map of *E* into *S* such:

$$E: \dots \longrightarrow E_{n+1} \xrightarrow{v_{n+1}} E_n \xrightarrow{v_n} E_{n-1} \longrightarrow \cdots$$

$$a \bigvee_{q} \qquad a_{n+1} \bigvee_{q} \qquad a_n \bigvee_{q} \qquad a_{n-1} \bigvee_{q}$$

$$S: \dots \longrightarrow S_{n+1} \xrightarrow{u_{n+1}} S_n \xrightarrow{u_n} S_{n-1} \longrightarrow \cdots$$

where $E_n = M$ et $v_n = Id_M$. $S_n = N$ and $u_n = Id_N$.

Given α_n an epimorphism of A- modules. Let ϕ a chain of E into S verifying:

For any morphisms of *A*-modules $\phi_k: E_k \to M$.

Given *E* is quasi-projective then it exists a chain map ψ such:

$$E: \dots \longrightarrow E_{n+1} \xrightarrow{w_{n+1}} E_n \xrightarrow{w_n} E_{n-1} \longrightarrow \cdots$$

$$\psi \left| \qquad \psi_{n+1} \right| \qquad \psi_n \left| \qquad \psi_{n-1} \right|$$

$$R: \dots \longrightarrow R_{n+1} \xrightarrow{v_{n+1}} R_n \xrightarrow{v_n} R_{n-1} \longrightarrow \cdots$$

where: $\alpha \circ \psi = \phi$ so for all $n \in \mathbb{Z}$, $\alpha_n \circ \psi_n = \phi_n$. Then it exists $\psi_n : E_n \to R_n$ such $f \circ \psi_n = \phi_n$. So E_n is quasi-projective.

Reciprocally suppose that for all $n \in \mathbb{Z}$, E_n is quasi-projective A- and let us demonstrate that E is quasi-projective chain complex.

Considering γ an epimorphism such:

$$E: \dots \longrightarrow E_{n+1} \xrightarrow{v_{n+1}} E_n \xrightarrow{v_n} E_{n-1} \longrightarrow \cdots$$

$$\gamma \bigvee_{\gamma_{n+1}} \bigvee_{\gamma_n} \bigvee_{\gamma_n} \bigvee_{\gamma_{n-1}} \bigvee_{\gamma_{n-1}} \bigvee_{\gamma_{n-1}} \bigvee_{\gamma_{n-1}} \sum_{\gamma_{n-1}} S_n \xrightarrow{u_n} S_{n-1} \longrightarrow \cdots$$

Let β a chain map complex verifying:

$$E: \dots \longrightarrow E_{n+1} \xrightarrow{w_{n+1}} E_n \xrightarrow{w_n} E_{n-1} \longrightarrow \dots$$

$$\beta \bigvee \qquad \beta_{n+1} \bigvee \qquad \beta_n \bigvee \qquad \beta_{n-1} \bigvee \qquad \beta_{n-1} \bigvee$$

$$S: \dots \longrightarrow S_{n+1} \xrightarrow{u_{n+1}} S_n \xrightarrow{u_n} S_{n-1} \longrightarrow \dots$$

Then given for all $n \in \mathbb{Z}$, E_n is projective and γ_n is an epimorphism of A-modules so it exists $\lambda_n: E_n \to R_n$ verifying $\gamma_n \circ \lambda_n$.

Considering λ such:

$$E: \dots \longrightarrow E_{n+1} \xrightarrow{w_{n+1}} E_n \xrightarrow{w_n} E_{n-1} \longrightarrow \cdots$$

$$\lambda \downarrow \qquad \lambda_{n+1} \downarrow \qquad \lambda_n \downarrow \qquad \lambda_{n-1} \downarrow$$

$$R: \dots \longrightarrow R_{n+1} \xrightarrow{v_{n+1}} v_n \xrightarrow{v_n} R_{n-1} \longrightarrow \cdots$$

Let us demonstrate that λ is a chain map

It is enough that: $\lambda_{n+1} \circ w_{n+1} = v_{n+1} \circ \lambda_n$. But $\beta_{n+1} \circ w_{n+1} = u_{n+1} \circ \beta_n$, because β is a chain map.

Given $\gamma_n \circ \lambda_n = \beta_n$, then: $\gamma_{n+1} \circ \lambda_{n+1} \circ w_{n+1} = u_{n+1} \circ (\gamma_n \circ \lambda_n)$. So $(\gamma_{n+1} \circ \lambda_{n+1}) \circ w_{n+1} = u_{n+1} \circ (\gamma_n \circ \lambda_n)$. Then $\gamma_{n+1} \circ (\lambda_{n+1} \circ w_{n+1}) = \gamma_{n+1} \circ (\nu_{n+1} \circ \lambda_n)$. But γ_{n+1} is an epimorphism hence $\lambda_{n+1} \circ w_{n+1} = \nu_{n+1} \circ \lambda_n$ What justifies that λ is a chain complex.

Let us verify that: $\gamma \circ \lambda = \beta$. We know for all $n \in \mathbb{Z}$, $\beta_n = \gamma_n \circ \lambda_n$ so $\beta = \gamma \circ \lambda$. What justifies *E* is a quasi-projective chain complex.

Theorem 6 *Given C a chain complex of A-modules. If C is quasi-projective and strongly cohop fian then C is strongly hop fian.*

Proof. Suppose that *C* is quasi-projective and *strongly cohop fian*, then C_n is quasi-projective using Theorem 4 and $(kerf_n^k)$ stabilizing this implies that C_n is quasi-projective and $(kerf^k)$ stabilizes so *C* is strongly hopfian. \Box

3.4 Fitting Chains Complexes

Definition 12 An *A*-module *M* is said to be *FITTING* module if for any endomorphism *f* of *M*, there exists a positive integer $n \ge 1$ such: $M = Kerf^n \oplus Imf^n$.

Definition 13 A chain complex of A-modules is said to be *FITTING* chain complex if for any endomorphism f of C, it exists $n \ge 1$ such $C = Kerf^n \oplus Imf^n$.

Theorem 7 Considering C a chain complex of A-modules such C: ... $\rightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \dots$ C is a FITTING chain complex if and only if for all $n \in \mathbb{Z}$, C_n is A-FITTING module.

Proof. Suppose that *C* is a *FITTING* chain complex of *A*-modules then it exists an positive integer *k* such $C = Kerf^k \oplus Imf^k$ donc $Kerf^k \sqcap Imf^k = 0$ and $C = Kerf^k + Imf^k$ hence for all $n \in \mathbb{Z}$, $C_n = Kerf_n^k + Imf_n^k$ so C_n is strongly hopfian and strongly cohpfian then C_n is a *FITTING A*-module. Reciprocally suppose that C_n is an *A*-*FITTING* module then $(Kerf_n^k)$ and (Imf_n^k) stabilizes so $(Kerf^k)$ stabilizes and (Imf^k) also. Which prove that *C* is a *FITTING* chain complex.

Theorem 8 Any quasi-projective and strongly-hopfian or quasi-injective and strongly-cohofian chain complex of A-modules is a Fitting chain complex.

Proof. Suppose that *C* is *quasi-projective* and *cohopfian* then using the previous theorem we can say *C* is *hopfian*, so *C* is *cohopfian* and *hopfian*, then is a *FITTING* chain complex.

Suppose that *C* is *quasi-injective* and *hopfian* then using the previous theorem we can say *C* is *cohopfian*, so *C* is *cohopfian* and *hopfian*, then is a *FITTING* chain complex. \Box

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