

On Intersections of the Hyperbolicity Domain With Straight Lines

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Abstract

We consider the family of polynomials $x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_n$, $a_i \in \mathbf{R}$, and its *hyperbolicity domain* Π_n , i.e. the set of values of the coefficients a_i for which the polynomial is with real roots only. We prove that for $0 \leq k \leq n - 2$ there exist generic straight lines in $\mathbf{R}^n \cong Oa_1 \dots a_n$ intersecting Π_n along k segments and two half-lines.

Keywords: hyperbolic polynomial, hyperbolicity domain

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1. Introduction

1.1 The Hyperbolicity Domain

A real non-constant polynomial in one variable is (*strictly*) *hyperbolic* if it has only real (and distinct) roots. Consider the general family of monic real degree n polynomials

$$P(x, a) := x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_n.$$

Its *hyperbolicity domain* $\Pi_n \subset \mathbf{R}^n \cong Oa_1 \dots a_n$ is the set of values of the coefficients a_i for which the polynomial is hyperbolic. In what follows we allow a slight abuse of terminology by identifying the points of the domain Π_n with the corresponding hyperbolic polynomials. We set

$$\Pi_{n,0} := \Pi_n \cap \{a_1 = 0\} \subset \mathbf{R}^{n-1} \cong Oa_2 \dots a_n \text{ and } \Pi_n^* := \Pi_{n,0} \cap \{a_2 = -1\}.$$

The set $\Pi_{n,0}$ is of interest to us because a polynomial P can be transformed by the shift $x \mapsto x - a_1/n$ into one with $a_1 = 0$. The sets Π_n and $\Pi_{n,0}$ are invariant under the one-parameter groups of quasi-homogeneous dilatations

$$x \mapsto e^\tau x, \quad a_j \mapsto e^{j\tau} a_j, \quad \tau \in \mathbf{R}. \quad (1)$$

Therefore a hyperbolic polynomial P with $a_1 = 0$, $a_2 \neq 0$ can be transformed into one with $a_1 = 0$, $a_2 = -1$ (If $a_1 = 0$ and $a_2 \neq 0$, then $a_2 < 0$ because $a_2 = -(x_1^2 + \dots + x_n^2)/2$, where x_i are the roots of P . The only hyperbolic polynomial with $a_1 = a_2 = 0$ is x^n).

Definition 1 We use only the letters x and y to denote independent variables. We denote by $\Delta[P]$ the *discriminant* of the family of polynomials P , i.e. the set of values of the coefficients a_i for which P has a multiple root in x . A similar notation is used for the discriminants of all other polynomial families used in the paper. The discriminant $\Delta[P]$ is defined as the zero set of a polynomial in the coefficients a_i . It can be considered in the context of real or of complex coefficients.

In the present paper we are interested in pencils of real monic polynomials of the form $tQ + (1 - t)R$, $t \in \mathbf{R}$, $\deg Q = \deg R = n$. We consider the question how many times can such a pencil intersect the hyperbolicity domain. A classical result (see Obrechhoff, 1963/2003) says that all polynomials of the pencil are strictly hyperbolic if and only if both Q and R are strictly hyperbolic and their roots interlace.

Remark 2 (1) The discriminant $\Delta[P]$ is defined as the set of zeros of the determinant D of the Sylvester matrix $S(P, P')$ of the polynomials P and P' . The matrix is of size $(2n - 1) \times (2n - 1)$. Suppose that all coefficients a_i are affine functions $\mu_i s + \nu_i$ of a (real or complex) parameter s . The first column of $S(P, P')$ does not depend on s while

in any of its other columns there are at least two entries which are affine functions of s . Therefore it is to be expected that typically $\deg D = 2n - 2$. One can show that the coefficient of s^{2n-2} in D is a non-trivial homogeneous degree $2n - 2$ polynomial in the coefficients μ_i . Therefore a generic straight line intersects $\Delta[P]$ at $2n - 2$ points. In the case when all coefficients are real part (or none or all) of the points are real, the remaining ones form complex pairs. In the context of $\Pi_{n,0}$ (i. e. if one sets $a_1 = 0$) one has $\deg D = 2n - 3$ (because two columns of $S(P, P')$ are independent of s). In the same way for $a_1 = 0, a_2 = -1$ (i.e. in the context of Π_n^*) one has $\deg D = 2n - 4$. Hence in these cases a generic line intersects $\Delta[P]$ at $2n - 3$ and $2n - 4$ points respectively.

(2) For more details about the hyperbolicity domain see, for example, Kostov (1989, 2011), Méguerditchian (1991a, 1991b, 1992). An impetus to study its properties was given by the articles (Arnold, 1986) and (Givental, 1987). Two particular classes of hyperbolic polynomials and the corresponding domains are studied in Kostov (2004, 2005a, 2005b, 2007). See also Chebotarev (1942), Meiman (1938, 1940) and Meiman and Chebotarev (1934).

1.2 Examples

In the present subsection we show how the hyperbolicity domain looks like for small values of n . We discuss what can be the intersection of a straight line with the hyperbolicity domain.

(1) The domains Π_1 and $\Pi_{1,0}$ are just \mathbf{R}^1 and its origin.

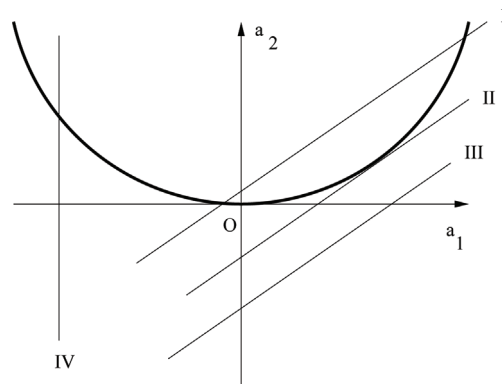


Figure 1. The hyperbolicity domain for $n = 2$ and the lines intersecting it

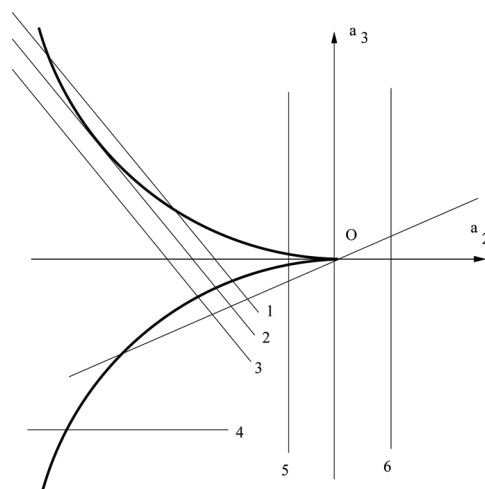


Figure 2. The hyperbolicity domain for $n = 3, a_1 = 0$, and the lines intersecting it

(2) One has $\Pi_2 = \{a_1^2 - 4a_2 \geq 0\}$, see Figure 1. A generic straight line in \mathbf{R}^2 either belongs to the interior of Π_2 (see line III) or intersects Π_2 along two half-lines (line I). A non-generic line is either tangent to the parabola $a_1^2 = 4a_2$ (line II) or is parallel to the a_2 -axis and hence intersects Π_2 along a half-line (line IV). The domain $\Pi_{2,0}$ is the half-line $\{a_2 \leq 0\} \subset \mathbf{R}^1 \cong Oa_2$.

(3) The domain $\Pi_{3,0}$ is the part of \mathbf{R}^2 delimited by the semi-cubic parabola $4a_2^3 + 27a_3^2 = 0$ and containing the half-line $a_2 \leq 0$, see Figure 2. Any generic line is neither vertical nor horizontal hence intersects it along a half-line or a half-line and a segment, see lines 1 and 3 on Figure 2. Within the class of non-generic lines there is the subclass of “least non-generic” ones. It consists of the vertical lines intersecting the domain along segments (line 5), the vertical lines not intersecting it at all (line 6), the horizontal lines not passing through the origin (line 4, they intersect the domain along half-lines), the non-horizontal lines passing through the origin (line 7, they intersect the domain along a half-line and at the origin) and the lines tangent to the semi-cubic parabola, but not at the origin (line 2). Finally there are the “most non-generic” lines—the two coordinate axes.

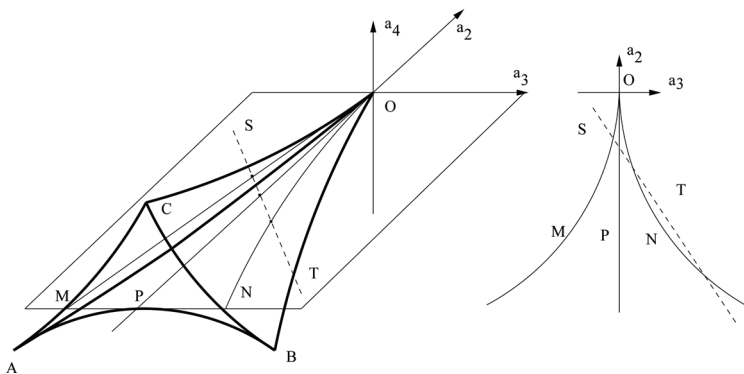


Figure 3. The hyperbolicity domain for $n = 4$, $a_1 = 0$, and the lines intersecting it

(4) The domain $\Pi_{4,0}$ is shown on the left of Figure 3. This is the curvilinear pyramid $OABC$ having a transversal self-intersection along the open arc CO and semi-cubic singularities in the transversal cross-sections at the interior points of the open arcs AO and BO . The concavity of the faces of the pyramid is as shown on Figure 3. The intersection of $\Pi_{4,0}$ with the coordinate plane Oa_2a_3 is the closure of the domain in Oa_2a_3 delimited by the semi-cubic parabola MON and containing the negative a_2 -half-axis (i.e. OP).

The right part of Figure 3 shows the Oa_2a_3 -plane and the semi-cubic parabola MON . A straight line ST belonging to the Oa_2a_3 -plane and as shown on Figure 3 intersects $\Pi_{4,0}$ along a segment and a half-line. The segment contains inside itself a point at which the line ST is tangent to the border of $\Pi_{4,0}$; this point is in fact $ST \cap OP$. Hence when the line ST is shifted slightly while remaining parallel to itself so that the a_2 - and a_3 -coordinates of its points remain the same while the a_4 -coordinate decreases, the intersection $ST \cap \Pi_{4,0}$ will consist of two segments and a half-line.

We are not giving the complete description of the possible cases of intersection of a line with $\Pi_{4,0}$. However we prove the following proposition:

Proposition 3 Any straight line not intersecting $\Pi_{4,0}$ either belongs to a plane $\{a_2 = \text{const}\}$ or its intersection with the half-space $\{a_2 \leq 0\}$ belongs to the set $\{a_2 \leq 0, a_3 = 0, a_4 < 0\}$.

Remark 4 Suppose that a straight line belongs to a plane $\{a_2 = r\}$. If $r > 0$, then any such straight line does not intersect $\Pi_{4,0}$ because $\Pi_{4,0} \subset \{a_2 \leq 0\}$. If $r \leq 0$, then the line does not intersect $\Pi_{4,0}$ exactly when it avoids the compact set $\Pi_{4,0} \cap \{a_2 = r\}$. This set is obtained from Π_4^* by a linear transformation, see formula (1). Hence it is necessary and sufficient for the line to avoid the (rectilinear) triangle ABC , see Figure 3.

Proof of Proposition 3. If a line does not belong to a plane $\{a_2 = \text{const}\}$, then it can be parametrized as follows:

$$a_2 = -s, \quad a_3 = \alpha s + \beta, \quad a_4 = \gamma s + \delta, \quad s \in \mathbf{R}. \tag{2}$$

If $\gamma > 0$ or $\gamma = 0 < \delta$, then the absolute values of the a_3 - and a_4 -coordinates of a point of the line increase at most linearly in s . Consider a half-disk $\{a_2 = -1, a_3^2 + a_4^2 \leq \varepsilon, a_4 \geq 0\}$, where $\varepsilon > 0$ is so small that the half-disk belongs to Π_4^* . Consider the image of the half-disk under the one-parameter group of diffeomorphisms defined by formula (1). This image belongs to $\Pi_{4,0}$. Set $s = e^t$. Then the a_3 - and a_4 -coordinates of the image of each point of the interior of the half-disk increase respectively as $s^{3/2}$ and s^2 . Hence for s sufficiently large the image of the half-disk contains the point of the straight line.

For $\gamma = \delta = 0$ and $\alpha \neq 0$ or $\alpha = 0 \neq \beta$ the reasoning is much the same. For s large enough the point of the line

belongs to the border of the image of the half-disk, but is not at its centre (i.e. not on the a_2 -axis), hence the point belongs to the interior of $\Pi_{4,0}$.

Suppose that $\gamma < 0$, $\alpha \neq 0$ or $\gamma = 0 > \delta$, $\alpha \neq 0$. The set Π_4^* contains a domain of the form $\{a_2 = -1, |a_3| \leq \varepsilon, 0 \geq a_4 \geq -ga_3^2\}$ for some $\varepsilon > 0$ and $g > 0$. Indeed, the arc APB has an ordinary tangency with the line MN (see Kostov, 2011). The image of this domain under the one-parameter group of diffeomorphisms (1) is the domain $\{a_2 = -s, |a_3| \leq \varepsilon s^{3/2}, 0 \geq a_4 \geq -sga_3^2\}$. For $s > 0$ sufficiently large this domain contains the corresponding point of the straight line, see (2).

For $\gamma < 0$, $\alpha = 0 \neq \beta$ or $\gamma = 0 > \delta$, $\alpha = 0 \neq \beta$ the point of the line belongs to the border of the image of the domain, but is not on the a_2 -axis, hence the point belongs to the interior of $\Pi_{4,0}$.

Hence the line might not intersect $\Pi_{4,0}$ only for $\alpha = \beta = 0$, i.e. when the straight line belongs to the Oa_2a_4 -plane. The line must not intersect the set $\{a_2 \leq 0, a_3 = 0, a_4 \geq 0\}$. Indeed, in this case it will intersect $\Pi_{4,0}$ because it will either intersect OP (the negative a_2 -half-axis) or the arc OC (the a_4 -coordinate of a point of the arc OC increases as s^2 while the a_4 -coordinate of the point of the line increases at most linearly in s). Hence the intersection of the line with the half-space $\{a_2 \leq 0\}$ belongs to the set $\{a_2 \leq 0, a_3 = 0, a_4 < 0\}$. \square

1.3 The Basic Result

The main result of the paper is the following theorem:

Theorem 5 For $n \geq 3$ the following statements hold true:

(1) For $a_1 = 0$ a generic straight line in $\mathbf{R}^{n-1} \cong Oa_2 \dots a_n$ intersects $\Pi_{n,0}$ along k segments and a half-line, where $0 \leq k \leq n - 2$, and for any such k there exists such a generic straight line.

(2) For $0 \leq k \leq n - 2$ there exist generic straight lines in $\mathbf{R}^n \cong Oa_1 \dots a_n$ intersecting Π_n along k segments and two half-lines.

To prove or disprove the following conjecture would also be of interest:

Conjecture 6 Any straight line in \mathbf{R}^n which is not parallel to any of the coordinate hyperplanes intersects the interior of Π_n .

2. Proof of Theorem 5

For $n = 3$ part (1) of the theorem can be checked directly, see part (3) of Subsection 1.2.

We prove part (1) by induction on n . Suppose that it is proved for $n - 1$ and that $n \geq 4$. The domain Π_n^* is compact (as $x_1^2 + \dots + x_n^2 = 2$, all roots x_k and all coefficients a_i are bounded). This domain contains the polynomial $U := (x - b)^{n-1}(x + (n - 1)b)$, $b = \sqrt{2/n(n - 1)}$.

In a neighbourhood of the point $U \in \mathbf{R}^n$ (respectively $U \in \mathbf{R}^{n-1}$) one can present any polynomial in the form $(x - u)G(x, \lambda)$ (respectively $(x - \lambda_1)G$), where $G(x, \lambda) := x^{n-1} + \lambda_1 x^{n-2} + \dots + \lambda_{n-1}$ and u and λ_1 are close to $-(n - 1)b$.

Lemma 7 In a neighbourhood of the point $U \in \mathbf{R}^n$ (respectively $U \in \mathbf{R}^{n-1}$) one can choose as local coordinates the quantities $(\lambda_1, \lambda_2, \dots, \lambda_{n-1}, u)$ (respectively $(\lambda_1, \lambda_2, \dots, \lambda_{n-1})$).

The lemma is proved in the next section.

Set $y := (x - b)/v$, where $v > 0$ is a parameter. Hence $v^{n-1}U = y^{n-1}(nb + vy)$. To choose $v > 0$ small means to magnify the local picture around the point U .

Consider the family of polynomials in y

$$v^{n-1}\tilde{U} := (nb + vy)V, \text{ where } V := y^{n-1} + c_2 y^{n-3} + \dots + c_{n-1}. \quad (3)$$

Consider the discriminant $\Delta[v^{n-1}\tilde{U}]$. Hence $\Delta[v^{n-1}\tilde{U}]$ can be regarded as a family of hypersurfaces in the space $Oc_2 \dots c_{n-1}$ analytically depending on v .

By inductive assumption for any $0 \leq k \leq n - 3$ there exist generic straight lines in $\mathbf{R}^{n-2} \cong Oc_2 \dots c_{n-1}$ intersecting the hyperbolicity domain of the family of polynomials V along k segments and a half-line. Suppose that for any such k such a generic line L_k is fixed. Genericity of the lines implies that the intersections of these lines with the discriminant $\Delta[V]$ are transversal. Hence the intersections of these lines (when considered as relative to the family $v^{n-1}\tilde{U}$, not V) will be transversal for $v \geq 0$ small enough as well.

For any $\nu > 0$ small enough the straight line L_k defines a straight line M_k in $Oa_2 \dots a_n$. Indeed, the coefficients a_i are affine functions (depending on ν as on a parameter) of the coefficients c_i . One can introduce a parameter $w \in \mathbf{R}$ and express the coordinates of a point of the line L_k in the form $c_i = p_{i,k}w + q_{i,k}$, $p_{i,k}, q_{i,k} \in \mathbf{R}$, $i = 1, \dots, n-1$. Thus the coordinates of a point of the line M_k can also be expressed by affine functions (depending on ν) of w .

The lines L_k and the coordinates c_i can be considered also as complex ones. Hence each line L_k intersects transversally the complex discriminant $\Delta[V]$ at $2n-5$ complex points out of which $2k+1$ are real. Genericity implies that all these points are finite.

Fix ν small enough. The line M_k intersects the domain Π_n^* along k segments whose endpoints are close to the point U and along another segment only one of whose endpoints is close to U . It is true that the line L_k intersects the hyperbolicity domain of the family of polynomials V along a half-line, but the presentation of a polynomial in the form $(x-u)G(x, \lambda)$ (respectively $(x-\lambda_1)G$) is defined only locally, in a neighbourhood of the point U . It implies the existence of the endpoints of the k segments and of the half-line as intersection points of the line L_k with the discriminant $\Delta[V]$. However the line M_k a priori intersects the compact domain Π_n^* only along segments and/or points, not along a half-line. Hence its intersection with Π_n^* consists of $k+1$ segments.

The line M_k belongs to a hyperplane $\{a_2 = \text{const}\}$. It has $2n-4$ intersection points with $\Delta[V]$ out of which $2n-5$ are close to the point U and $2k+2$ are real. Perturb the position of M_k so that it no longer belongs to a hyperplane of the form $\{a_2 = \text{const}\}$. Such a generic perturbation introduces a single “distant” point which is real (otherwise it would be a complex conjugate pair). This point is the endpoint of a half-line of the intersection of M_k with $\Pi_{n,0}$.

To finish the proof of the induction step for part (1) one should prove that there exist generic lines intersecting $\Pi_{n,0}$ along half-lines and no segments. Consider a monic strictly hyperbolic polynomial P with roots $x_1 < x_2 < \dots < x_n$. Consider a polynomial $Q := (x-z_1) \dots (x-z_{n-2})$, where $x_i < z_i < x_{i+1}$, $i = 1, \dots, n-2$. The interlacing of the roots implies that for any $t \in \mathbf{R}$ the polynomial $S_t := P + tQ$ has at least $n-2$ real roots ξ_i such that any interval (x_i, x_{i+1}) and any interval (z_i, z_{i+1}) , $i = 1, \dots, n-3$ contains exactly one of the roots ξ_i .

There exists t_0 such that the rightmost root of the polynomial S_{t_0} is a double one. For $t > t_0$, S_t has a complex conjugate pair of roots. Indeed, as $S_t = S_{t_0} + (t-t_0)Q$ and as $Q > 0$ for $x > z_{n-2}$, the polynomial S_t is positive for $x > z_{n-2}$. On the other hand it has at most $n-1$ and at least $n-2$ roots that are distinct and $\leq z_{n-2}$. For any $t \in \mathbf{R}$ the polynomial S_t is of degree n . Hence for $t > t_0$ the polynomial S_t has exactly $n-2$ real distinct roots.

For $t < t_0$ a similar reasoning shows that S_t has two distinct real roots that are $> z_{n-2}$. As S_t has at least $n-3$ distinct real roots that are $< z_{n-2}$, it has at least $n-1$ distinct real roots. Hence it has n real roots (because $\deg S_t = n$). Its roots that are not greater than z_{n-2} are distinct hence all its roots are distinct. This means that the straight line defined by the family of polynomials S_t has the Property (A): it has a double root only for $t = t_0$, it has n distinct real roots for $t < t_0$ and exactly $n-2$ distinct real roots for $t > t_0$.

The roots z_i are required only to interlace with the roots x_1, \dots, x_{n-2} . Hence any nearby line defines a family S_t which satisfies the Property (A) for some t^* close to t_0 . This means that the straight line defined by the family S_t is generic. This line belongs to $Oa_2 \dots a_n$ because the coefficients of x^{n-1} are 0 in P and in Q .

Part (2) of the theorem follows from part (1). Indeed, suppose that a generic line $M_k \subset \Pi_{n,0}$ as above is constructed ($-1 \leq k \leq n-3$). Perturb this line so that it no longer belongs to a hyperplane of the form $\{a_1 = \text{const}\}$. Such a generic perturbation introduces a single “distant” point which (not being a complex conjugate pair) is real. This point is the endpoint of a half-line of the intersection of M_k with Π_n . \square

3. Proof of Lemma 7

The following formulas connect the quantities a_i with the quantities u and λ_i :

$$a_1 = \lambda_1 - u, \quad a_2 = \lambda_2 - u\lambda_1, \quad \dots \quad a_{n-1} = \lambda_{n-1} - u\lambda_{n-2}, \quad a_n = -u\lambda_{n-1}. \quad (4)$$

Hence the Jacobian matrix of this change of variables equals

$$\begin{pmatrix} -1 & 1 & 0 & \cdots & 0 & 0 \\ -\lambda_1 & -u & 1 & \cdots & 0 & 0 \\ -\lambda_2 & 0 & -u & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -\lambda_{n-2} & 0 & 0 & \cdots & -u & 1 \\ -\lambda_{n-1} & 0 & 0 & \cdots & 0 & -u \end{pmatrix}.$$

To find its determinant add the first row multiplied by u to the second one, then the second row multiplied by u to the third one, . . . then the $(n - 1)$ -st row multiplied by u to the n -th one. The matrix becomes

$$\begin{pmatrix} & & & & -1 & 1 & 0 & \cdots & 0 & 0 \\ & & & & -\lambda_1 - u & 0 & 1 & \cdots & 0 & 0 \\ & & & & -\lambda_2 - \lambda_1 u - u^2 & 0 & 0 & \cdots & 0 & 0 \\ & & & & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ & & & & -\lambda_{n-2} - \lambda_{n-3}u - \cdots - \lambda_1 u^{n-3} - u^{n-2} & 0 & 0 & \cdots & 0 & 1 \\ -\lambda_{n-1} - \lambda_{n-2}u - \lambda_{n-3}u^2 - \cdots - \lambda_1 u^{n-2} - u^{n-1} & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}.$$

Its entry in position $(n, 1)$ equals $-G(u, \lambda)$, so its determinant is $(-1)^{n+1}G(u, \lambda)$. The root u is not close to b , therefore $G(u, \lambda) \neq 0$.

When one sets $u = \lambda_1$, then formulas (4) become

$$a_2 = \lambda_2 - \lambda_1^2, \quad a_3 = \lambda_3 - \lambda_1 \lambda_2, \quad \dots \quad a_{n-1} = \lambda_{n-1} - \lambda_1 \lambda_{n-2}, \quad a_n = -\lambda_1 \lambda_{n-1}.$$

The Jacobian matrix equals

$$\begin{pmatrix} -2\lambda_1 & 1 & 0 & \cdots & 0 & 0 \\ -\lambda_2 & -\lambda_1 & 1 & \cdots & 0 & 0 \\ -\lambda_3 & 0 & -\lambda_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -\lambda_{n-2} & 0 & 0 & \cdots & -\lambda_1 & 1 \\ -\lambda_{n-1} & 0 & 0 & \cdots & 0 & -\lambda_1 \end{pmatrix}.$$

Add the k th row multiplied by λ_1 to the $(k + 1)$ -st one for $k = 1, 2, \dots, n - 2$. This makes disappear all terms $-\lambda_1$ on the diagonal, the entry in position $(n, 1)$ becomes $-G(\lambda_1, \lambda) \neq 0$ (because λ_1 is close to $-(n - 1)b$, not to b); the determinant equals $\pm G(\lambda_1, \lambda)$. \square

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