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Making Holes in the Second Symmetric Product of a Cyclicly Connected Graph

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Abstract

A *continuum* is a connected compact metric space. The *second symmetric product* of a continuum X, $\mathcal{F}_2(X)$, is the hyperspace of all nonempty subsets of X having at most two elements. An element A of $\mathcal{F}_2(X)$ is said to *make a hole with respect to multicoherence degree* in $\mathcal{F}_2(X)$ if the multicoherence degree of $\mathcal{F}_2(X) - \{A\}$ is greater than the multicoherence degree of $\mathcal{F}_2(X)$. In this paper, we characterize those elements $A \in \mathcal{F}_2(X)$ such that A makes a hole with respect to multicoherence degree in $\mathcal{F}_2(X)$ when X is a cyclicly connected graph.

Keywords: continuum, symmetric products, multicoherence degree, make a hole with respect to multicoherence degree

1. Introduction

A *continuum* is a connected compact metric space. Let X be a continuum. For each positive interger n, let $\mathcal{F}_n(X) = \{A \subset X : A$ has at most n elements and $A \neq \emptyset\}$. The hyperspace $\mathcal{F}_n(X)$ is called the n^{th} *symmetric product of* X. It is known that each hyperspace $\mathcal{F}_n(X)$ is a continuum (see Borsuk & Ulam, 1931, pp. 876, 877) and (Michael, 1951, Theorem 4.10, p. 165).

If Z is any topological space, let $b_0(Z)$ denote the number of components of Z minus one if this number is finite and $b_0(Z) = \infty$ otherwise. Given a connected topological space Y, the *multicoherence degree* of Y, is defined by $r(Y) = \sup\{b_0(K \cap L) : K$ and L are closed connected subsets of Y and $Y = K \cup L\}$. The space Y is said to be *unicoherent* if $r(Y) = 0$. Let $y \in Y$ such that $Y - \{y\}$ is connected, we say that y *makes a hole with respect to multicoherence degree* in Y if $r(Y - \{y\}) > r(Y)$. This is a generalization of the notion of to make a hole in a unicoherent topological space defined in (Anaya, 2007, p. 2000).

In this paper, we are interesting in the following problem.

Problem. Let $\mathcal{H}(X)$ be a hyperspace of a continuum X. For which elements $A \in \mathcal{H}(X)$, A makes a hole with respect to multicoherence degree in $\mathcal{H}(X)$.

In the current paper, we are presenting the solution to this problem when X is a cyclicly connected graph and $\mathcal{H}(X) = \mathcal{F}_2(X)$.

Readers specially interested in this problem are refered to Anaya (2007, 2011), Anaya, Maya and Orozco-Zitli (2010, 2012).

2. Preliminaries

Given a positive interger m, define $\lambda(m) = \{1, 2, \ldots, m\}$. A *map* is a continuous function. The identity map for a topological space Z is denoted by id_Z. An *arc* is any space homeomorphic to $[0, 1]$. A *simple closed curve* is a space which is homeomorphic to the unit circle S^1 in the Euclidean plane $\mathbb{R}^2$. A *theta curve* is a space which is homeomorphic to $S^1 \cup ([-1, 1] \times \{0\})$ in $\mathbb{R}^2$. The symbol $[0, 1]^2$ denotes the space $[0, 1] \times [0, 1]$. The set $\{(u, v) \in [0, 1]^2 \colon u \leq v\}$ is denoted by Δ. A *graph* is a continuum which can be written as the union of finitely many arcs any two of which are either disjoint or intersect only in one or both their end points. A point y in a connected topological space Y is called *cut point* (*non-cut point*) if $Y - \{y\}$ is not connected (connected). A space W is said to be *cyclicly connected* provided that every two points of W belong to some simple closed curve in W

(see (Whyburn, 1942, p. 77)). A graph X is a *cyclicly connected graph* if X is a cyclicly connected space.

Given a topological space Y. A subspace Z of Y is said to be:

(a) a *retract of* Y if there exists a map $f\colon Y \to Z$ such that $f(z) = z$ for every $z \in Z$. The map f is called a *retraction*.

(b) a *deformation retract of* Y if there exist a retraction $f\colon Y \to Z$ and a map $g\colon Y \times [0, 1] \to Y$ such that $g(x, 0) = x$ and $g(x, 1) = f(x)$ for every $x \in Y$.

(c) a *strong deformation retract of* Y if there exist f and g as in (b) with the additional property that $g(z, t) = z$ for every $(z, t) \in Z \times [0, 1]$.

Let $y \in Y$. Let β be a cardinal number. We say that *y is of order less than or equal to β in Y*, written $\text{ord}(y, Y) \leq \beta$, provided that for each open subset U of Y containing y, there exists an open subset V of Y such that $y \in V \subset U$ and the cardinality of the boundary of V is less than or equal to β. We say that *y is of order β in Y*, written $\text{ord}(y, Y) = \beta$, provided that $\text{ord}(y, Y) \leq \beta$ and $\text{ord}(y, Y) \nleq \alpha$ for any cardinal number $\alpha < \beta$. Put $E(Y) = \{x \in Y\colon \text{ord}(x, Y) = 1\}$, $O(Y) = \{x \in Y\colon \text{ord}(x, Y) = 2\}$ and $R(Y) = \{x \in Y\colon \text{ord}(x, Y) \geq 3\}$. Define $\mathcal{I}(Y) = \{I \subset Y\colon I \text{ is an arc and } E(I) = I \cap R(Y)\}$, $\mathcal{N}(y, Y) = \{I \in \mathcal{I}(Y)\colon y \notin I\}$, $\mathcal{M}(y, Y) = \{I \in \mathcal{I}(Y)\colon y \in I\}$, $N(y, Y) = \bigcup \mathcal{N}(y, Y)$ and $M(y, Y) = \bigcup \mathcal{M}(y, Y)$. If K and L are nonempty subsets of Y, let $\langle K, L\rangle = \{\{x, y\} \subset Y\colon x \in K, y \in L\}$.

2.1 Auxiliary Results

Lemma 2.1 *If X is a cyclicly connected graph different from a simple closed curve, then the following conditions hold:*

(1) for each simple closed curve S in X, $S \cap R(X)$ has at least two points;

(2) $X = \bigcup \mathcal{I}(X)$;

(3) the set $\mathcal{I}(X)$ is finite;

(4) for each $p \in X$, $M(p, X)$ is a nondegenerate subcontinuum of X.

Proof. In order to prove (1), let S be a simple closed curve in X. Since $S \neq X$, there exists a simple closed curve $S_1 \neq S$ in X such that $S \cap S_1 \neq \emptyset$. So, using (Nadler, Jr., 1992, Proposition 9.5, p. 142), $R(S \cup S_1) \cap S \cap S_1 \neq \emptyset$. Thus, by (Kuratowski, 1968, Theorem 3, p. 278), $R(X) \cap S \cap S_1 \neq \emptyset$. Now, assume that $R(X) \cap S \cap S_1$ consists of precisely one point. Then, there exists a simple closed curve $S_2 \neq S$ in X such that $S_2 \cap (S - S_1) \neq \emptyset$. Applying the previous argument to $S \cup S_2$, we have $R(X) \cap (S - S_1) \cap S_2 \neq \emptyset$. Hence, $S \cap R(X)$ has at least two points.

(2) Follows from (1) and the fact that $R(X)$ is a finite set (see (Nadler, Jr., 1992, Theorem 9.10, p. 144)).

(3) Follows from the fact that $R(X)$ is a finite set (see (Nadler, Jr., 1992, Theorem 9.10, p. 144)).

Finally, to check (4), let $p \in X$. By (2), there exists $I \in \mathcal{I}(X)$ such that $p \in I$. So, since $I \subset M(p, X)$, $M(p, X)$ is nondegenerate set. On the other hand, clearly, $M(p, X)$ is connected. By (3), $M(p, X)$ is closed in X. □

Lemma 2.2 *Let X be a cyclicly connected graph and let $p \in X$. If $\mathcal{N}(p, X) \neq \emptyset$, then $N(p, X)$ is a subcontinuum of X.*

Proof. First, by (3) of Lemma 2.1, $N(p, X)$ is closed in X. We shall prove the connectedness of $N(p, X)$. By (Whyburn, 1942, (9.3), p. 79), $X - \{p\}$ is connected. So, it suffices to prove that $N(p, X)$ is a continuous image of $X - \{p\}$. Consider $F = \bigcup\{E(I)\colon I \in \mathcal{M}(p, X)\} - \{p\}$. By (3) of Lemma 2.1, $\mathcal{M}(p, X)$ is finite. Then, F is discrete. By (4) of Lemma 2.1, $M(p, X) - \{p\}$ is a nonempty set. Now, define $f\colon M(p, X) - \{p\} \to F$ as follows: given $z \in M(p, X) - \{p\}$, let $f(z)$ be the unique element of $F \cap C$ where C is the component of $M(p, X) - \{p\}$ containing z. Clearly, f is surjective. We prove that f is continuous. Let $e \in F$. By the definition of f, it is easy to see that $f^{-1}(\{e\})$ is a component of $M(p, X) - \{p\}$. Thus, since each component of $M(p, X) - \{p\}$ is closed in $M(p, X) - \{p\}$, $f^{-1}(\{e\})$ is closed in $M(p, X) - \{p\}$.

Now, define $\bar{f}\colon X - \{p\} \to N(p, X)$ by

$$\bar{f}(x) = \begin{cases} x, & \text{if } x \in N(p, X), \\ f(x), & \text{if } x \in M(p, X) - \{p\}. \end{cases}$$

Since $N(p, X) \cap M(p, X) = F$ and by the definition of f, $\bar{f}$ is well defined. Clearly, $\bar{f}$ is surjective. The continuity of $\bar{f}$ follows from the continuity of f and the fact that $N(p, X)$ and $M(p, X) - \{p\}$ are closed subsets of $X - \{p\}$. This finishes the proof of that $N(p, X)$ is connected. □

Lemma 2.3 *Let X be a cyclicly connected graph different from a simple closed curve and let p, q be different points in X. If $X - \{p, q\}$ is not connected, there exist a simple closed curve S in X containing p and q and a retract $f\colon X \to S$ such that $f^{-1}(p) = \{p\}$ and $f^{-1}(q) = \{q\}$.*

Proof. Let C_1 and C_2 be different components of $X - \{p, q\}$. Since $C_k \cup \{p, q\}$ is a subcontinuum of X, there exists an arc J_k in $C_k \cup \{p, q\}$ such that $E(J_k) = \{p, q\}$ for each $k \in \{1, 2\}$. Put $S = J_1 \cup J_2$. Clearly, S is a simple closed curve in X and $p, q \in S$.

Now, let $f_0\colon R(X) \to S$ be a function such that $f_0|_{R(X)\cap S} = \mathrm{id}_{R(X)\cap S}$, $f_0(R(X) \cap C_1) \subset J_1$ and $f_0(R(X) \cap C) \subset J_2$ for each component C of $X - \{p, q\}$ with $C \neq C_1$.

Given $I \in \mathcal{I}(X)$, let $f_I : I \to S$ be a one-to-one map such that $f_I|_S = \mathrm{id}_{S\cap I}$, $E(f_I(I)) = f_0(E(I))$, $f_I(I \cap C_1) \subset J_1$ and $f_I(I \cap C) \subset J_2$ for each component C of $X - \{p, q\}$ with $C \neq C_1$. From the fact that f_I is one-to-one, it follows that $f_I(I - \{p, q\}) \subset S - \{p, q\}$.

Define $f\colon X \to S$ as follows: for each $x \in X$, take $I \in \mathcal{I}(X)$ such that $x \in I$ and let $f(x) = f_I(x)$. Notice that $f|_{R(X)} = f_0$. Hence, f is well defined. The continuity of f follows from the fact that each f_I is continuous and, by (2) and (3) of Lemma 2.1. It is easy to see that $f|_S = \mathrm{id}_S$. Thus, f is a retraction.

Finally, since $S - \{p, q\} \subset X - \{p, q\}$ and $f|_S = \mathrm{id}_S$, $S - \{p, q\} \subset f(X - \{p, q\})$. To check that $f(X - \{p, q\}) \subset S - \{p, q\}$, notice that $f(X - \{p, q\}) = \bigcup\{f_I(I - \{p, q\})\colon I \in \mathcal{I}(X)\} \subset S - \{p, q\}$ (see (2). of Lemma 2.1). Thus, $f(X - \{p, q\}) = S - \{p, q\}$. Hence, $f^{-1}(\{p, q\}) = \{p, q\}$. From the fact that $p \neq q$, we have that $f^{-1}(p) = \{p\}$ and $f^{-1}(q) = \{q\}$. □

Lemma 2.4 *Let X be a cyclicly connected graph different from a simple closed curve and let p, q be different points in X. If $X - \{p, q\}$ is connected, there exist a theta curve Y in X containing p and q and a retract $f\colon X \to Y$ such that $f^{-1}(p) = \{p\}$ and $f^{-1}(q) = \{q\}$.*

Proof. By the definition of cyclic connectedness, there exists a simple closed curve S in X such that $p, q \in Y$. Since $X - \{p, q\}$ is connected, there exists an arc J in X such that $S - \{p, q\} \cap J = E(J)$. Put $Y = S \cup J$. Clearly, Y is a theta curve in X containing p and q such that $Y - \{p, q\}$ is connected.

First, consider a function f_0: $R(X) \to Y$ such that $f_0|_Y = \mathrm{id}_{R(X)\cap Y}$. Now, for each $I \in \mathcal{I}(X)$, fix a one-to-one map f_I: $I \to Y$ such that $f_I|_Y = \mathrm{id}_{Y\cap I}$ and $f(I - \{p, q\}) \subset Y - \{p, q\}$.

Define f: $X \to Y$ as follows: for each $x \in X$, take $I \in \mathcal{I}(X)$ such that $x \in I$ and let $f(x) = f_I(x)$. From the fact that $f|_{R(X)} = f_0$, it follows that f is well defined. Since $X = \bigcup \mathcal{I}(X)$ and $\mathcal{I}(X)$ is finite (see (2) and (3) of Lemma 2.1), f is continuous. From the fact that $f|_Y = \mathrm{id}_Y$, it follows that f is a retraction.

We will prove that $f(X - \{p, q\}) = Y - \{p, q\}$. Since $X - \{p, q\} = \bigcup\{I - \{p, q\}\colon I \in \mathcal{I}(X)\}$, $f(X - \{p, q\}) \subset Y - \{p, q\}$. Clearly, $Y - \{p, q\}$ is contained in $f(X - \{p, q\})$. We have that $f^{-1}(\{p, q\}) = \{p, q\}$. Since $p \neq q$, $f^{-1}(p) = \{p\}$ and $f^{-1}(q) = \{q\}$. □

Proposition 2.5 *Let X be a continuum and let K and L be connected subsets (subcontinua) of X. Then $\langle K, L\rangle$ is a connected subset (subcontinuum) of $\mathcal{F}_2(X)$ and, it does not have cut points when K and L are nondegenerate sets.*

Proof. The connectedness of $\langle K, L\rangle$ follows from (Martínez-Montejano, 2002, Lemma 1, p. 230).

In order to prove the second part of this proposition, let $\{p, q\} \in \langle K, L\rangle$. Using K and L are nondegenerate sets and the arguments in (Kuratowski, 1968, Theorem 11, p. 137), it can be shown that $K \times L - \{(p, q), (q, p)\}$ is connected. So, since $\langle K, L\rangle - \{\{p, q\}\}$ is a continuous image of $K \times L - \{(p, q), (q, p)\}$, $\langle K, L\rangle - \{\{p, q\}\}$ is connected. □

Lemma 2.6 *Let I be an arc and let $p \in I - E(I)$. If H and J are subcontinua of I such that $H \cup J \subset I - \{p\}$ and each one of them contains a different end point of I, then $\langle H, I\rangle \cup \langle J, I\rangle$ is a strong deformation retract of $\mathcal{F}_2(I) - \{\{p\}\}$.*

Proof. Put $\Gamma = \Delta - \left\{\left(\frac{1}{2}, \frac{1}{2}\right)\right\}$, $\Gamma_0 = \{(u, v) \in \Gamma\colon u \leq \frac{1}{4}\} \cup \{(u, v) \in \Gamma\colon \frac{3}{4} \leq v\}$ and $\Gamma_1 = \{(u, v) \in \Gamma\colon \frac{1}{4} \leq u, v \leq \frac{3}{4}\}$. First, we are going to prove that Γ_0 is a strong deformation retract of Γ. Define $f\colon \Gamma \to \Gamma_0$ by

$$f(u, v) = \begin{cases} (u, v), & \text{if } (u, v) \in \Gamma_0, \\ \left(\frac{1}{4}, u + v - \frac{1}{4}\right), & \text{if } (u, v) \in \Gamma_1 \text{ and } v \leq 1 - u, \\ \left(u + v - \frac{3}{4}, \frac{3}{4}\right), & \text{if } (u, v) \in \Gamma_1 \text{ and } 1 - u \leq v, \end{cases}$$

and $g\colon \Gamma \times [0, 1] \to \Gamma$ by

$$g((u, v), t) = (1 - t) \cdot (u, v) + t \cdot f(u, v).$$

It is easy to verify that f and g have the required properties.

Finally, let h: $[0,1] \to I$ be a homeomorphism such that $h([0,\frac{1}{4}]) = H$, $h([\frac{3}{4},1]) = J$ and $h(\frac{1}{2}) = p$. Define $\bar{h}$: $\Gamma \to \mathcal{F}_2(I) - \{\{p\}\}$ by $\bar{h}(u,v) = \{h(u), h(v)\}$. It can be proved that $\bar{h}$ is a homeomorphism such that $\bar{h}(\Gamma_0) = \langle H, I\rangle \cup \langle J, I\rangle$. Therefore, $\langle H, I\rangle \cup \langle J, I\rangle$ is a strong deformation retract of $\mathcal{F}_2(I) - \{\{p\}\}$. □

Lemma 2.7 *If X is a graph containing a simple closed curve, then X is not unicoherent.*

Proof. We shall prove that there exist subcontinua K and L of X such that $b_0(K \cap L) > 0$ and $X = K \cup L$. Let S be a simple closed curve in X. By (Nadler, Jr., 1992, Theorem 9.10, p. 144), there exists $x \in S$ such that $\text{ord}(x, X) = 2$. Now, using (Nadler, Jr., 1992, Theorem 9.7, p. 143), it can be proved that there exists an arc J in S which is a neighborhood of x in X. Then, $J - E(J)$ is an open connected subset of X. Now, by (Nadler, Jr., 1992, 9.44, (a), p. 160), $S - (J - E(J))$ is connected. Hence, $X - (J - E(J))$ is a subcontinuum of X. So, $K = J$ and $L = X - (J - E(J))$ satify the requiered properties. □

Theorem 2.8 *If X is a cyclicly connected graph, then $r(\mathcal{F}_2(X)) = 1$.*

Proof. The result follows from (Nadler, Jr., 1992, Theorem 8.25, p. 131), Lemma 2.7 and (Illanes, 1985, Theorem 1.6, p. 16). □

3. Making Holes in the Second Symmetric Product of a Cyclicly Connected Graph

Theorem 3.1 *Let X be a graph and let $p \in O(X)$. Then $\{p\}$ does not make a hole with respect to multicoherence degree in $\mathcal{F}_2(X)$.*

Proof. We will show that $r(\mathcal{F}_2(X) - \{\{p\}\}) = r(\mathcal{F}_2(X))$. Since X is a graph, it is easy to see that $\mathcal{F}_2(X) - \{\{p\}\}$ is a locally connected metric space and, by Proposition 2.5, $\mathcal{F}_2(X) - \{\{p\}\}$ is connected. So, in light of (Eilenberg, 1936, Theorem 4, p. 162) and (Stone, 1950, Theorem 5, p. 472), it suffices to prove that there exists a deformation retract $\mathcal{Z}$ of $\mathcal{F}_2(X) - \{\{p\}\}$ such that $r(\mathcal{Z}) = r(\mathcal{F}_2(X))$.

Since $p \in O(X)$, using (Nadler, Jr., 1992, Lemma 9.7, p. 143), it can be shown that there exists an arc I in X such that I is a neighborhood of p in X. So, clearly, $p \in I - E(I)$. Let H and J be nondegenerate subcontinua of I such that $H \cup J \subset I - \{p\}$ and each one of them contains a different end point of I. Put $Z = (X - I) \cup H \cup J$ and $\mathcal{Z} = \langle X, Z\rangle$. Clearly, $\mathcal{F}_2(X) = \mathcal{Z} \cup \mathcal{F}_2(I)$. Now, by Lemma 2.6, there exist a retraction f: $\mathcal{F}_2(I) - \{\{p\}\} \to \langle H, I\rangle \cup \langle J, I\rangle$ and a map g: $(\mathcal{F}_2(I) - \{\{p\}\}) \times [0,1] \to \mathcal{F}_2(I) - \{\{p\}\}$ such that $g(A, 0) = A$ and $g(A, 1) = f(A)$ for each $A \in \mathcal{F}_2(I) - \{\{p\}\}$ and $g(B, t) = B$ for each $(B, t) \in (\langle H, I\rangle \cup \langle J, I\rangle) \times [0,1]$.

Define $\bar{f}$: $\mathcal{F}_2(X) - \{\{p\}\} \to \mathcal{Z}$ by

$$\bar{f}(A) = \begin{cases} A, & \text{if } A \in \mathcal{Z}, \\ f(A), & \text{if } A \in \mathcal{F}_2(I) - \{\{p\}\}, \end{cases}$$

and $\bar{g}$: $(\mathcal{F}_2(X) - \{\{p\}\}) \times [0,1] \to \mathcal{F}_2(X) - \{\{p\}\}$ by

$$\bar{g}(A, t) = \begin{cases} A, & \text{if } A \in \mathcal{Z}, \\ g(A, t), & \text{if } A \in \mathcal{F}_2(I) - \{\{p\}\}. \end{cases}$$

To check that $\bar{f}$ and $\bar{g}$ are well defined, notice that $\mathcal{Z} \cap \mathcal{F}_2(I) - \{\{p\}\} = \langle H, I\rangle \cup \langle J, I\rangle$ and $f(B) = B = g(B, t)$ for each $(B, t) \in (\langle H, I\rangle \cup \langle J, I\rangle) \times [0,1]$. Now, the continuity of $\bar{f}$ and $\bar{g}$ follows from the continuity of the maps f and g and the fact that $\mathcal{Z}$ and $\mathcal{F}_2(I) - \{\{p\}\}$ are closed in $\mathcal{F}_2(X) - \{\{p\}\}$. It is easy to verify that $\bar{f}$ and $\bar{g}$ have the required properties. Thus, $\mathcal{Z}$ is a deformation retract of $\mathcal{F}_2(X) - \{\{p\}\}$.

Finally, to check that $r(\mathcal{Z}) = r(\mathcal{F}_2(X))$, we shall show that $\mathcal{Z}$ is homeomorphic to $\mathcal{F}_2(X)$. It can be shown that there exists a homeomorphism h: $\mathcal{F}_2(I) \to \langle H, I\rangle \cup \langle J, I\rangle$ such that $h|_{\langle E(I), I\rangle} = \text{id}_{\langle E(I), I\rangle}$. Define $\bar{h}$: $\mathcal{F}_2(X) \to \mathcal{Z}$ by

$$\bar{h}(A) = \begin{cases} h(A), & \text{if } A \in \mathcal{F}_2(I), \\ A, & \text{otherwise.} \end{cases}$$

It is easy to see that $\bar{h}$ is a homeomorphism. Hence, $r(\mathcal{F}_2(X)) = r(\mathcal{Z})$.

This finishes the proof that $\{p\}$ does not make a hole with respect to multicoherence degree in $\mathcal{F}_2(X)$. □

Theorem 3.2 *Let X be a cyclicly connected graph and $p \in R(X)$. Then $\{p\}$ makes a hole with respect to multicoherence degree in $\mathcal{F}_2(X)$.*

Proof. Since $r(\mathcal{F}_2(X)) = 1$ (see Theorem 2.8), we shall show that $r(\mathcal{F}_2(X) - \{\{p\}\}) \geq 2$. So, it suffices to prove that there exist two closed connected subsets $\mathcal{K}$ and $\mathcal{L}$ of $\mathcal{F}_2(X) - \{\{p\}\}$ such that $\mathcal{F}_2(X) - \{\{p\}\} = \mathcal{K} \cup \mathcal{L}$ and $b_0(\mathcal{K} \cap \mathcal{L}) \geq 2$.

Put $\Lambda = \{(u, v) \in [0, 1]^2 - \{\mathbf{0}\}: \frac{u}{2} \leq v \leq 2u\}$, $\Omega = \{(u, v) \in [0, 1]^2 - \{\mathbf{0}\}: v \leq \frac{u}{2}\}$, $\Gamma = \{(u, v) \in [0, 1]^2 - \{\mathbf{0}\}: 2u \leq v\}$ where $\mathbf{0} = (0, 0)$, $m = \text{ord}(p, X)$ and $\mathcal{M}(p, X) = \{I_1, I_2, \ldots, I_m\}$. For each $k \in \lambda(m)$, fix a homeomorphism φ_k: $[0, 1] \to I_k$ such that $\varphi_k(0) = p$. Given elements $k \neq j \in \lambda(m)$, define $\psi_{(k,j)}$: $[0, 1]^2 - \{\mathbf{0}\} \to \langle I_k, I_j\rangle - \{\{p\}\}$ by $\psi_{(k,j)}(s, t) = \{\varphi_k(s), \varphi_j(t)\}$. Since $\varphi_k(0) = \varphi_j(0) = p$ and, φ_k and φ_j are one-to-one, $\psi_{(k,j)}$ is well defined. Using the fact that φ_k and φ_j are surjective, it is easy to prove that $\psi_{(k,j)}$ is surjective. Clearly, for each $k, j \in \lambda(m)$ with $k \neq j$, $\psi_{(k,j)}(\Lambda) = \psi_{(j,k)}(\Lambda)$ and $\psi_{(k,j)}(\Omega) = \psi_{(j,k)}(\Gamma)$.

Consider the following cases.

Case A. $\mathcal{N}(p, X) \neq \emptyset$.

Let $Y = N(p, X)$. By Lemma 2.2, Y is a subcontinuum of X. For each $k \in \lambda(m)$, define

$$\mathcal{K}_k = \langle \varphi_k([\frac{1}{2}, 1]), Y \cup \varphi_k([\frac{1}{2}, 1])\rangle \text{ and } \mathcal{L}_k = \langle \varphi_k([0, \frac{1}{2}]), Y \cup \varphi_k([0, 1])\rangle - \{\{p\}\}.$$

Consider

$$\mathcal{K} = \mathcal{F}_2(Y) \cup \bigcup\{\mathcal{K}_k : k \in \lambda(m)\} \cup \bigcup\{\psi_{(k,j)}(\Lambda) : k, j \in \lambda(m), k \neq j\}$$

$$\text{and } \mathcal{L} = \bigcup\{\mathcal{L}_k : k \in \lambda(m)\} \cup \bigcup\{\psi_{(k,j)}(\Gamma) : k, j \in \lambda(m), k \neq j\}.$$

Clearly, $\mathcal{K}$ and $\mathcal{L}$ are closed subsets of $\mathcal{F}_2(X) - \{\{p\}\}$. To prove $\mathcal{F}_2(X) - \{\{p\}\} = \mathcal{K} \cup \mathcal{L}$, let $\{x, y\} \in \mathcal{F}_2(X) - \{\{p\}\}$. Since $X = M(p, X) \cup Y$, $\mathcal{F}_2(X) = \mathcal{F}_2(M(p, X)) \cup \mathcal{F}_2(Y) \cup \langle M(p, X), Y\rangle$. If $\{x, y\} \in \mathcal{F}_2(Y) \cup \langle M(p, X), Y\rangle$, it is easy to see that $\{x, y\} \in \mathcal{K} \cup \mathcal{L}$. Suppose that $\{x, y\} \in \mathcal{F}_2(M(p, X)) - \{\{p\}\}$. Take $k, j \in \lambda(m)$ such that $x \in I_k$ and $y \in I_j$. First, if $k = j$, then $\{x, y\} \in \mathcal{K}_k \cup \mathcal{L}_k$. Now, without loss of generality, we may assume that $k < j$. Consider $(u, v) \in [0, 1]^2 - \{\mathbf{0}\}$ such that $\psi_{(k,j)}(u, v) = \{x, y\}$. Thus, since $[0, 1]^2 - \{\mathbf{0}\} = \Lambda \cup \Omega \cup \Gamma$, $\psi_{(k,j)}(\Lambda) = \psi_{(j,k)}(\Lambda)$ and $\psi_{(k,j)}(\Gamma) = \psi_{(j,k)}(\Omega)$, $\{x, y\} \in \mathcal{K} \cup \mathcal{L}$.

To show that $\mathcal{K}$ and $\mathcal{L}$ are connected, let $k \neq j \in \lambda(m)$. The connectedness of $\mathcal{K}_k$ and $\mathcal{L}_k$ follows from the fact that $\varphi_k(1) \in Y$ and Proposition 2.5. Without loss of generality, we may assume that $k < j$. The connectedness of $\mathcal{L}$ follows from the connectedness of Ω and the fact that $\psi_{(k,j)}(1, 0) \in \mathcal{L}_k \cap \mathcal{L}_j \cap \psi_{(k,j)}(\Omega)$. Since $\psi_{(k,j)}(\Lambda)$ is connected, $\psi_{(k,j)}(\Lambda) = \psi_{(j,k)}(\Lambda)$ and $\psi_{(k,j)}(1, 1) \in \mathcal{K}_k \cap \psi_{(k,j)}(\Lambda) \cap \mathcal{F}_2(Y)$, $\mathcal{K}$ is connected.

Finally, we will show that $b_0(\mathcal{K} \cap \mathcal{L}) \geq 2$. Put $\Sigma = \{(u, v) \in [0, 1]^2 - \{\mathbf{0}\}: v = 2u\}$. Given $k \in \lambda(m)$. Define $\mathcal{D}_k = \langle\{\varphi_k(\frac{1}{2})\}, Y \cup \varphi_k([\frac{1}{2}, 1])\rangle$ and $C_k = \bigcup\{\psi_{(k,j)}(\Sigma): j \in \lambda(m) - \{k\}\} \cup \mathcal{D}_k$. We are going to prove that $C_1, \ldots, C_m$ are the components of $\mathcal{K} \cap \mathcal{L}$. The connectedness of $\mathcal{D}_k$ follows from the fact that $\varphi_k(1) \in Y$ and Proposition 2.5. Since $\psi_{(k,j)}(\Sigma)$ is connected and $\psi_{(k,j)}(\frac{1}{2}, 1) \in \psi_{(k,j)}(\Sigma) \cap \mathcal{D}_k$ for each $j \in \lambda(m) - \{k\}$, C_k is connected.

We need to prove the following properties,

i) $\mathcal{F}_2(Y) \cap \mathcal{L} = \emptyset$,

ii) $\mathcal{K}_k \cap \mathcal{L}_k = \mathcal{D}_k$ for each $k \in \lambda(m)$,

iii) $\mathcal{K}_k \cap \mathcal{L}_j = \emptyset$ and $\mathcal{K}_k \cap \psi_{(k,j)}(\Gamma) = \{\varphi_k(\frac{1}{2}), \varphi_j(1)\}$ for each $k \neq j \in \lambda(m)$,

iv) $\mathcal{L} \cap \langle I_k, I_j\rangle = \psi_{(k,j)}(\Omega) \cup \psi_{(k,j)}(\Gamma)$ for each $k \neq j \in \lambda(m)$,

v) $\varphi_k([0, 1]) \cap \varphi_j([0, 1]) = \{\varphi_k(0), \varphi_k(1)\} \cap \{\varphi_j(0), \varphi_j(1)\}$ for each $k \neq j \in \lambda(m)$,

vi) $\mathcal{D}_k \cap \mathcal{D}_j = \emptyset$ for each $k \neq j \in \lambda(m)$,

vii) if $k \neq j \in \lambda(m)$, then $\psi_{(k,j)}(\Sigma) \cap \mathcal{D}_l = \emptyset$ for each $l \in \lambda(m) - \{k\}$,

viii) if $k \neq j \in \lambda(m)$, then $\psi_{(k,j)}(\Sigma) \cap \psi_{(l,n)}(\Sigma) = \emptyset$ for each $(l, n) \in ((\lambda(m) - \{k\}) \times (\lambda(m) - \{j\})) - \{(j, k)\}$.

It is easy to see the properties i)-v).

vi) Follows from the facts that $\varphi_k(\frac{1}{2}) \notin Y \cup \varphi_j([\frac{1}{2}, 1])$, $\varphi_j(\frac{1}{2}) \notin Y \cup \varphi_k([\frac{1}{2}, 1])$ and v).

vii) Suppose to the contrary that there exists $l \in \lambda(m) - \{k\}$ such that $\psi_{(k,j)}(\Sigma) \cap \mathcal{D}_l \neq \emptyset$. Consider $(u, 2u) \in \Sigma$ such that $\psi_{(k,j)}(u, 2u) \in \mathcal{D}_l$. Then, either $\varphi_k(u) = \varphi_l(\frac{1}{2})$ or $\varphi_j(2u) = \varphi_l(\frac{1}{2})$. So, by v), $j = l$ and $\varphi_j(2u) = \varphi_l(\frac{1}{2})$. Thus, $u = \frac{1}{4}$ and $\varphi_k(\frac{1}{4}) \in Y \cup \varphi_l([\frac{1}{2}, 1])$, a contradiction.

viii) Suppose to the contrary that there exist $(l,n) \in ((\lambda(m) - \{k\}) \times (\lambda(m) - \{j\})) - \{(j,k)\}$ and $(u,v),(s,t) \in \Sigma$ such that $\psi_{(k,j)}(u,v) = \psi_{(l,n)}(s,t)$. So, since $u > 0$, $s \leq \frac{1}{2}$ and $k \neq l$, by v), $\varphi_k(u) \neq \varphi_l(s)$ and $\varphi_k(u) = \varphi_n(t)$. Then, $\varphi_j(v) = \varphi_l(s)$. Thus, since $0 < u, s \leq \frac{1}{2}$, by v), $k = n$ and $j = l$. Hence, since φ_k and φ_l are one-to-one maps, $u = t$ and $v = s$. Therefore, $(t,s),(s,t) \in \Sigma$, a contradiction.

We are ready to prove that $\mathcal{K} \cap \mathcal{L} = \bigcup\{C_k\colon k \in \lambda(m)\}$. From the fact that $\Sigma = \Lambda \cap \Gamma$ and ii), we have that $\bigcup\{C_k\colon k \in \lambda(m)\} \subset \mathcal{K} \cap \mathcal{L}$. Now, let $\{w,z\} \in \mathcal{K} \cap \mathcal{L}$. If $\{w,z\} \in \mathcal{K}_k \cap \mathcal{L}$ for some $k \in \lambda(m)$, by ii) and iii), $\{w,z\} \in \mathcal{D}_k \subset C_k$. Now, suppose that $\{w,z\} \in \psi_{(k,j)}(\Lambda) \cap \mathcal{L}$ for some $k \neq j \in \lambda(m)$. Since $\psi_{(k,j)}(\Lambda) \subset \langle I_k, I_j\rangle$ and $\psi_{(k,j)}(\Gamma) = \psi_{(j,k)}(\Omega)$, by iv), $\{w,z\} \in (\psi_{(k,j)}(\Lambda) \cap \psi_{(k,j)}(\Omega)) \cup (\psi_{(k,j)}(\Lambda) \cap \psi_{(j,k)}(\Omega))$. So, using $\Sigma = \Lambda \cap \Omega$, $\varphi_{(k,j)}(\Lambda) = \varphi_{(j,k)}(\Lambda)$ and $\varphi_{(k,j)}|_\Sigma, \varphi_{(j,k)}|_\Sigma$ are one-to-one maps, it can be proved $\{w,z\} \in \varphi_{(k,j)}(\Sigma) \cup \varphi_{(j,k)}(\Sigma)$. Hence, $\{w,z\} \in C_k \cup C_j$.

Finally, in order to prove that $C_1, C_2, \ldots, C_m$ are mutually disjoint, let $k \neq j \in \lambda(m)$. By vi)-viii), $C_k \cap C_j = \emptyset$. Thus, $b_0(\mathcal{K} \cap \mathcal{L}) + 1 = m \geq 3$.

Case B. $\mathcal{N}(p,X) = \emptyset$.

Then, $N(p,X) = \emptyset$ and $\varphi_1(1) = \varphi_2(1) = \cdots = \varphi_k(1)$. This case can be proved using similar arguments in the proof of Case A by considering $Y = \{\varphi_1(1)\}$. □

Theorem 3.3 *Let X be a simple closed curve and let $p,q \in X$ such that $p \neq q$. Then $\{p,q\}$ makes a hole with respect to multicoherence degree in $\mathcal{F}_2(X)$.*

Proof. First, we are going to prove that $A = \{(1,0),(-1,0)\} \subset S^1$ makes a hole with respect to multicoherence degree in $\mathcal{F}_2(S^1)$. By Theorem 2.8, $r(\mathcal{F}_2(S^1)) = 1$. So, it suffices to show that there exist two closed connected subsets $\mathcal{K}$ and $\mathcal{L}$ of $\mathcal{F}_2(S^1) - \{A\}$ such that $\mathcal{F}_2(S^1) - \{A\} = \mathcal{K} \cup \mathcal{L}$ and $b_0(\mathcal{K} \cap \mathcal{L}) \geq 2$.

Define $\varphi\colon [0,1] \to S^1$ by $\varphi(t) = (\cos(2\pi t), \sin(2\pi t))$ and $\psi\colon \Delta \to \mathcal{F}_2(S^1)$ by $\psi(t,s) = \{\varphi(t), \varphi(s)\}$. Notice that ψ is well defined and it is surjective. The continuity of ψ follows from that of φ. Put $\Gamma_1 = \{(u,v) \in \Delta - \{(0,\frac{1}{2})\}\colon \frac{1}{2} - u \leq v \leq 1 - u\}$, $\Gamma_2 = \{(u,v) \in \Delta - \{(\frac{1}{2},1)\}\colon \frac{3}{2} - u \leq v\}$, $\Gamma_3 = \{(u,v) \in \Delta - \{(0,\frac{1}{2})\}\colon v \leq \frac{1}{2} - u\}$, $\Gamma_4 = \{(u,v) \in \Delta - \{(\frac{1}{2},1)\}\colon 1 - u \leq v \leq \frac{3}{2} - u\}$, $\mathcal{K} = \psi(\Gamma_1) \cup \psi(\Gamma_2)$ and $\mathcal{L} = \psi(\Gamma_3) \cup \psi(\Gamma_4)$.

It is easy to prove that $\mathcal{K}$ and $\mathcal{L}$ are closed subset of $\mathcal{F}_2(S^1) - \{A\}$. Clearly, $\mathcal{K} \cup \mathcal{L} \subset \mathcal{F}_2(S^1) - \{A\}$. Now, we will prove that $\mathcal{F}_2(S^1) - \{A\} \subset \mathcal{K} \cup \mathcal{L}$. Let $\{x,y\} \in \mathcal{F}_2(S^1) - \{A\}$ and let $t,s \in [0,1]$ such that $\varphi(t) = x$ and $\varphi(s) = y$. Without loss of generality, we may suppose that $t \leq s$. So, since $\psi(t,s) = \{x,y\}$ and $(t,s) \in \Delta - \{(0,\frac{1}{2}),(\frac{1}{2},1)\} = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$, $\{x,y\} \in \psi(\Gamma_1) \cup \psi(\Gamma_2) \cup \psi(\Gamma_3) \cup \psi(\Gamma_4) = \mathcal{K} \cup \mathcal{L}$. Thus, $\mathcal{F}_2(S^1) - \{A\} = \mathcal{K} \cup \mathcal{L}$.

The connectedness of $\mathcal{K}$ and $\mathcal{L}$ follows from the facts that $\psi(0,0) = \psi(0,1) = \psi(1,1) \in \psi(\Gamma_1) \cap \psi(\Gamma_2) \cap \psi(\Gamma_3) \cap \psi(\Gamma_4)$ and each $\psi(\Gamma_i)$ is connected.

Now, we are going to prove that $b_0(\mathcal{K} \cap \mathcal{L}) \geq 2$. Put $\Lambda_1 = \{(u,v) \in \Delta : v = 1 - u\}$, $\Lambda_2 = \{(u,v) \in \Delta - \{(0,\frac{1}{2})\}\colon v = \frac{1}{2} - u\}$ and $\Lambda_3 = \{(u,v) \in \Delta - \{(\frac{1}{2},1)\}\colon v = \frac{3}{2} - u\}$. Notice that $\Lambda_1 = \Gamma_1 \cap \Gamma_4$, $\Lambda_2 = \Gamma_1 \cap \Gamma_3$, $\Lambda_3 = \Gamma_2 \cap \Gamma_4$ and, Λ_1, Λ_2 and Λ_3 are mutually disjoint. It is easy to see that $\psi(\Lambda_1) = \psi(\Gamma_1 \cap \Gamma_4) = \psi(\Gamma_1) \cap \psi(\Gamma_4)$, $\psi(\Lambda_2) = \psi(\Gamma_1 \cap \Gamma_3) = \psi(\Gamma_1) \cap \psi(\Gamma_3)$, $\psi(\Lambda_3) = \psi(\Gamma_2 \cap \Gamma_4) = \psi(\Gamma_2) \cap \psi(\Gamma_4)$ and $\psi(\Gamma_2) \cap \psi(\Gamma_3) = \emptyset$. We will show that $\psi(\Lambda_1)$, $\psi(\Lambda_2)$ and $\psi(\Lambda_3)$ are the components of $\mathcal{K} \cap \mathcal{L}$. First, notice that $\psi(\Lambda_1) \subset \mathcal{K} \cap \mathcal{L}$ since $\psi(\Gamma_1) \subset \mathcal{K}$ and $\psi(\Gamma_4) \subset \mathcal{L}$. Similarly, it can be proved that $\psi(\Lambda_2)$ and $\psi(\Lambda_3)$ is contained in $\mathcal{K} \cap \mathcal{L}$. Now, to verify that $\mathcal{K} \cap \mathcal{L} \subset \bigcup_{i=1}^{3} \varphi(\Lambda_i)$, let $\{x,y\} \in \mathcal{K} \cap \mathcal{L}$. Since $\mathcal{K} = \psi(\Gamma_1) \cup \psi(\Gamma_2)$, either $\{x,y\} \in \psi(\Gamma_1) \cap \mathcal{L}$ or $\{x,y\} \in \psi(\Gamma_2) \cap \mathcal{L}$. From the facts that $\mathcal{L} = \psi(\Gamma_3) \cup \psi(\Lambda_4)$ and $\psi(\Gamma_2) \cap \psi(\Gamma_3) = \emptyset$, we have $\{x,y\} \in (\psi(\Gamma_1) \cap \psi(\Gamma_3)) \cup (\psi(\Gamma_1) \cap \psi(\Gamma_4)) \cup (\psi(\Gamma_2) \cap \psi(\Gamma_4))$. So, $\{x,y\} \in \bigcup_{i=1}^{3} \psi(\Lambda_i)$. Finally, since Λ_1, Λ_2 and Λ_3 are connected and mutually disjoint, $\psi(\Lambda_1)$, $\psi(\Lambda_2)$ and $\psi(\Lambda_3)$ are also connected and mutually disjoint. This proves that $b(\mathcal{K} \cap \mathcal{L}) + 1 = 3$.

So, $\mathcal{K}$ and $\mathcal{L}$ satisfy the required properties.

We are ready to prove that $\{p,q\}$ makes a hole with respect to multicoherence degree in $\mathcal{F}_2(X)$. Since X is a simple closed curve, there exists a homeomorphism $h\colon S^1 \to X$ such that $h(A) = \{p,q\}$. Consider the induced mapping $h_2\colon \mathcal{F}_2(S^1) \to \mathcal{F}_2(X)$ defined by $h_2(B) = h(B)$ for each $B \in \mathcal{F}_2(S^1)$. By (Higuera & Illanes, 2011, Theorem 3.1, p. 369), h_2 is a homeomorphism. Then, since A makes a hole with respect to multicoherence degree in $\mathcal{F}_2(S^1)$ and $h_2(A) = \{p,q\}$, $\{p,q\}$ makes a hole with respect to multicoherence degree in $\mathcal{F}_2(X)$. □

Theorem 3.4 *Let X be a theta curve and let $p,q \in X$ such that $ord(p,X) = ord(q,X) = 2$ and $X - \{p,q\}$ is connected. Then $\{p,q\}$ makes a hole with respect to multicoherence degree in $\mathcal{F}_2(X)$.*

Proof. Clearly, X is a cyclicly connected graph. Then, by Theorem 2.8, $r(\mathcal{F}_2(X)) = 1$. So, to show that $r(\mathcal{F}_2(X) - \{\{p,q\}\}) > 1$, we are going to prove that there exist connected closed subsets $\mathcal{K}$ and $\mathcal{L}$ of $\mathcal{F}_2(X) - \{\{p,q\}\}$ satisfying $\mathcal{F}_2(X) - \{\{p,q\}\} = \mathcal{K} \cup \mathcal{L}$ and $b_0(\mathcal{K} \cap \mathcal{L}) \geq 2$.

Put $\mathcal{I}(X) = \{I_1, I_2, I_3\}$. Without loss of generality, we may assume that $p \in I_1$ and $q \in I_2$. Given $k \in \{1,2,3\}$, fix a homeomorphism φ_k: $[0,1] \to I_k$ such that $\varphi_1(0) = \varphi_2(0) = \varphi_3(0)$. We may assume that $\varphi_1(\frac{1}{2}) = p$ and $\varphi_2(\frac{1}{2}) = q$. Notice that $\varphi_1(1) = \varphi_2(1) = \varphi_3(1)$. Put $w = \varphi_1(0)$ and $z = \varphi_1(1)$. So, $R(X) = \{w, z\}$. Now, for each $k, j \in \{1,2,3\}$, consider π_k: $\Delta \to \mathcal{F}_2(I_k)$ and $\psi_{(k,j)}$: $[0,1]^2 \to \langle I_k, I_j \rangle$ defined by $\pi_k(t,s) = \{\varphi_k(t), \varphi_k(s)\}$ for each $(t,s) \in \Delta$ and $\psi_{(k,j)}(u,v) = \{\varphi_k(u), \varphi_j(v)\}$ for each $(u,v) \in [0,1]^2$. Now, let $\Lambda_1 = \{(u,v) \in [0,\frac{1}{2}] \times [\frac{1}{2},1] - \{\boldsymbol{a}\}$: $\frac{1}{2} \leq v \leq \frac{3-2u}{4}\}$, $\Lambda_2 = (([\frac{1}{2},\frac{3}{4}] \times [\frac{1}{2},1]) \cup ([\frac{1}{2},1] \times [\frac{1}{2},\frac{3}{4}])) - \{\boldsymbol{a}\}$, $\Lambda_3 = \{(u,v) \in [\frac{1}{2},1] \times [0,\frac{1}{2}] - \{\boldsymbol{a}\}$: $v \leq \frac{4-3u}{2}\}$, $\Omega_1 = \{(u,v) \in [0,\frac{1}{2}] \times [\frac{1}{2},1] - \{\boldsymbol{a}\}$: $\frac{3-2u}{4} \leq v\}$, $\Omega_2 = [\frac{3}{4},1] \times [\frac{3}{4},1]$, $\Omega_3 = \{(u,v) \in [\frac{1}{2},1] \times [0,\frac{1}{2}] - \{\boldsymbol{a}\}$: $\frac{3-4u}{2} \leq v\}$ and $\Omega_4 = [0,\frac{1}{2}] \times [0,\frac{1}{2}] - \{\boldsymbol{a}\}$ where $\boldsymbol{a} = (\frac{1}{2},\frac{1}{2})$. Consider $\Gamma_1 = \{(u,v) \in \Delta$: $\frac{1}{2} - u \leq v \leq \frac{3}{4} - u\}$, $\Gamma_2 = \{(u,v) \in \Delta$: $\frac{3}{2} - u \leq v \leq \frac{7}{4} - u\}$, $\Sigma_1 = \{(u,v) \in \Delta$: $v \leq \frac{1}{2} - u\}$, $\Sigma_2 = \{(u,v) \in \Delta$: $\frac{3}{4} - u \leq v \leq \frac{3}{2} - u\}$ and $\Sigma_3 = \{(u,v) \in \Delta$: $\frac{7}{4} - u \leq v\}$. Put $\Lambda = \Lambda_1 \cup \Lambda_2 \cup \Lambda_3$, $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4$, $\Gamma = \Gamma_1 \cup \Gamma_2$ and $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$

For each $k \in \{1,2\}$, let $\mathcal{K}_k = \psi_{(k,3)}([\frac{1}{2},\frac{3}{4}] \times [0,1])$, $\mathcal{L}_k^1 = \psi_{(k,3)}([0,\frac{1}{2}] \times [0,1])$ and $\mathcal{L}_k^2 = \psi_{(k,3)}([\frac{3}{4},1] \times [0,1])$.

Define

$$\mathcal{K} = \psi_{(1,2)}(\Lambda) \cup \pi_1(\Gamma) \cup \pi_2(\Gamma) \cup \mathcal{K}_1 \cup \mathcal{K}_2 \text{ and}$$

$$\mathcal{L} = \psi_{(1,2)}(\Omega) \cup \pi_1(\Sigma) \cup \pi_2(\Sigma) \cup \mathcal{L}_1^1 \cup \mathcal{L}_2^1 \cup \mathcal{L}_1^2 \cup \mathcal{L}_2^2 \cup \mathcal{F}_2(I_3).$$

It is easy to see that $\mathcal{K}$ and $\mathcal{L}$ are closed subset of $\mathcal{F}_2(X) - \{\{p,q\}\}$. In order to prove that $\mathcal{F}_2(X) - \{\{p,q\}\} \subset \mathcal{K} \cup \mathcal{L}$, let $\{x,y\} \in \mathcal{F}_2(X) - \{\{p,q\}\}$. First, since $X = I_1 \cup I_2 \cup I_3$, $\mathcal{F}_2(X) = \mathcal{F}_2(I_1) \cup \mathcal{F}_2(I_2) \cup \mathcal{F}_2(I_3) \cup \langle I_1, I_2 \rangle \cup \langle I_1, I_3 \rangle \cup \langle I_2, I_3 \rangle$. Now, notice that $\mathcal{F}_2(I_1) = \pi_1(\Gamma \cup \Sigma)$, $\mathcal{F}_2(I_2) = \pi_2(\Gamma \cup \Sigma)$, $\mathcal{F}_2(I_3) \subset \mathcal{L}$, $\langle I_1, I_2 \rangle - \{\{p,q\}\} = \psi_{(1,2)}(\Gamma \cup \Omega)$, $\langle I_1, I_3 \rangle = \mathcal{K}_1 \cup \mathcal{L}_1^1 \cup \mathcal{L}_1^2$ and $\langle I_2, I_3 \rangle = \mathcal{K}_2 \cup \mathcal{L}_2^1 \cup \mathcal{L}_2^2$. Hence, $\{x,y\} \in \mathcal{K} \cup \mathcal{L}$. This proves that $\mathcal{F}_2(X) - \{\{p,q\}\} = \mathcal{K} \cup \mathcal{L}$.

To prove that $\mathcal{K}$ and $\mathcal{L}$ are connected, put $\mathfrak{C} = \{\Lambda_1, \Lambda_2, \Lambda_3, \Omega_1, \Omega_2, \Omega_3, \Omega_4\}$, $\mathfrak{D} = \{\Gamma_1, \Gamma_2, \Sigma_1, \Sigma_2, \Sigma_3\}$, $\mathfrak{F} = \{[\frac{1}{2},\frac{3}{4}] \times [0,1], [0,\frac{1}{2}] \times [0,1], [\frac{3}{4},1] \times [0,1]\}$ and $\mathfrak{G} = \mathfrak{C} \cup \mathfrak{D} \cup \mathfrak{F}$. It is easy to see that each element of $\mathfrak{G}$ is connected. So, $\psi_{(1,2)}(\Theta)$, $\pi_k(\Psi)$ and $\psi_{(k,3)}(\Upsilon)$ are connected for each $(\Theta, \Psi, \Upsilon, k) \in \mathfrak{C} \times \mathfrak{D} \times \mathfrak{F} \times \{1,2\}$. Notice that $\{w,p\} \in \pi_1(\Gamma_1) \cap \psi_{(1,2)}(\Lambda_3) \cap \mathcal{K}_1$, $\{p,z\} \in \pi_1(\Gamma_2) \cap \psi_{(1,2)}(\Lambda_2) \cap \mathcal{K}_1$, $\{q,z\} \in \psi_{(1,2)}(\Lambda_2) \cap \pi_2(\Gamma_2) \cap \mathcal{K}_2$ and $\{w,q\} \in \psi_{(1,2)}(\Lambda_1) \cap \pi_2(\Gamma_1) \cap \mathcal{K}_2$. Then, $\mathcal{K}$ is connected. Now, since $\{w\} \in \pi_1(\Sigma_1) \cap \pi_2(\Sigma_1) \cap \psi_{(1,2)}(\Omega_4) \cap \mathcal{L}_1^1 \cap \mathcal{L}_2^1 \cap \mathcal{F}_2(I_3)$, $\{z\} \in \pi_1(\Sigma_3) \cap \pi_2(\Sigma_3) \cap \psi_{(1,2)}(\Omega_2) \cap \mathcal{L}_1^2 \cap \mathcal{L}_2^2 \cap \mathcal{F}_2(I_3)$ and $\{w,z\} \in \pi_1(\Sigma_2) \cap \pi_2(\Sigma_2) \cap \psi_{(1,2)}(\Omega_1) \cap \psi_{(1,2)}(\Omega_3) \cap \mathcal{L}_1^1 \cap \mathcal{L}_1^2 \cap \mathcal{L}_2^1 \cap \mathcal{L}_2^2 \cap \mathcal{F}_2(I_3)$, $\mathcal{L}$ is connected.

Finally, we are going to show that $b_0(\mathcal{K} \cap \mathcal{L}) \geq 2$. Given $(n,m) \in \{1,2\} \times \{1,2,3\}$, let $\Pi_{(n,m)} = \Gamma_n \cap \Sigma_m$. Define $\Upsilon_{(i,j)} = \Lambda_i \cap \Omega_j$ for each $(i,j) \in \{1,2,3\} \times \{1,2,3,4\}$. For each $k \in \{1,2\}$, consider $\mathcal{H}_k = \mathcal{L}_k^1 \cap \mathcal{K}_k$ and $\mathcal{J}_k = \mathcal{L}_k^2 \cap \mathcal{K}_k$. Let $C_1 = \pi_1(\Pi_{(1,1)}) \cup \psi_{(1,2)}(\Upsilon_{(3,4)}) \cup \mathcal{H}_1 \cup \pi_1(\Pi_{(2,2)}) \cup \psi_{(1,2)}(\Upsilon_{(2,1)})$, $C_2 = \pi_2(\Pi_{(1,1)}) \cup \psi_{(1,2)}(\Upsilon_{(1,4)}) \cup \mathcal{H}_2 \cup \pi_2(\Pi_{(2,2)}) \cup \psi_{(1,2)}(\Upsilon_{(2,3)})$ and $C_3 = \pi_1(\Pi_{(1,2)}) \cup \psi_{(1,2)}(\Upsilon_{(3,3)}) \cup \mathcal{J}_1 \cup \pi_1(\Pi_{(2,3)}) \cup \psi_{(1,2)}(\Upsilon_{(2,2)}) \cup \pi_2(\Pi_{(2,3)}) \cup \mathcal{J}_2 \cup \pi_2(\Pi_{(1,2)}) \cup \psi_{(1,2)}(\Upsilon_{(1,1)})$. We are going to prove that C_1, C_2 and C_3 are the components of $\mathcal{K} \cap \mathcal{L}$. □

The following properties are easy to verify,

i) $\Pi_{(n,m)}$ is connected and $\pi_k(\Pi_{(n,m)}) = \pi_k(\Gamma_n) \cap \pi_k(\Sigma_m)$ for each $n, k \in \{1,2\}$ and $m \in \{n, n+1\}$,

ii) if $\Upsilon_{(i,j)} \neq \emptyset$, then $\Upsilon_{(i,j)}$ is connected and $\psi_{(1,2)}(\Upsilon_{(i,j)}) = \psi_{(1,2)}(\Lambda_i) \cap \psi_{(1,2)}(\Omega_j)$,

iii) $\mathcal{H}_k$ and J_k are connected for each $k \in \{1,2\}$,

iv) $\{w,p\} \in \pi_1(\Pi_{(1,1)}) \cap \psi_{(1,2)}(\Upsilon_{(3,4)}) \cap \mathcal{H}_1$,

v) $\{p,z\} \in \mathcal{H}_1 \cap \pi_1(\Pi_{(2,2)}) \cap \psi_{(1,2)}(\Upsilon_{(2,1)})$,

vi) $\{w,q\} \in \pi_2(\Pi_{(1,1)}) \cap \psi_{(1,2)}(\Upsilon_{(1,4)}) \cap \mathcal{H}_2$,

vii) $\{q,z\} \in \mathcal{H}_2 \cap \pi_2(\Pi_{(2,2)}) \cap \psi_{(1,2)}(\Upsilon_{(2,3)})$,

viii) $\{w, \varphi_1(\frac{3}{4})\} \in \pi_1(\Pi_{(1,2)}) \cap \psi_{(1,2)}(\Upsilon_{(3,3)}) \cap \mathcal{J}_1$,

ix) $\{\varphi_1(\frac{3}{4}), z\} \in \mathcal{J}_1 \cap \pi_1(\Pi_{(2,3)}) \cap \psi_{(1,2)}(\Upsilon_{(2,2)})$,

x) $\{\varphi_2(\frac{3}{4}), z\} \in \psi_{(1,2)}(\Upsilon_{(2,2)}) \cap \pi_2(\Pi_{(2,3)}) \cap \mathcal{J}_2$,

xi) $\{w, \varphi_2(\frac{3}{4})\} \in \mathcal{J}_2 \cap \pi_2(\Pi_{(1,2)}) \cap \psi_{(1,2)}(\Upsilon_{(1,1)})$,

xii) $\Pi_{(n_1,m_1)} \cap \Pi_{(n_2,m_2)} = \emptyset$ for each $(n_1,m_1) \neq (n_2,m_2) \in \{1,2\} \times \{1,2,3\}$,

xiii) $\Upsilon_{(i_1,j_1)} \cap \Upsilon_{(i_2,j_2)} = \emptyset$ for each $(i_1, j_1) \neq (i_2, j_2) \in \{1,2,3\} \times \{1,2,3,4\}$,

xiv) $\mathcal{H}_k \cap \mathcal{J}_k = \emptyset$ for each $k \in \{1,2\}$, and

xv) $\mathcal{K} \cap \mathcal{L} = \bigcup\{\pi_k(\Pi_{(n,m)})\colon (k,n,m) \in \{1,2\} \times \{1,2\} \times \{1,2,3\}\} \cup \bigcup\{\psi_{(1,2)}(\Upsilon_{(i,j)})\colon (i,j) \in \{1,2,3\} \times \{1,2,3,4\}\} \cup \bigcup\{\mathcal{H}_k \cup \mathcal{J}_k\colon k \in \{1,2\}\}$.

The connectedness of C_1, C_2 and C_3 follows from i)-xi). Using xii)-xiv), it can be proved that C_1, C_2 and C_3 are mutually disjoint. Finally, from xv), it follows that $\mathcal{K} \cap \mathcal{L} = C_1 \cup C_2 \cup C_3$.

So, $\mathcal{K}$ and $\mathcal{L}$ satisfy the required properties.

Theorem 3.5 *Let X be a cyclicly connected graph and $p, q \in X$. If $p \neq q$, then $\{p,q\}$ makes a hole with respect to multicoherence degree in $\mathcal{F}_2(X)$.*

Proof. In the case that X is a simple closed curve, the result follows from Theorem 3.3. Now, suppose that X is not a simple closed curve. Since X is a graph, by (Borsuk & Ulam, 1931, (a), p. 877), $\mathcal{F}_2(X)$ is a locally connected space. Then, $\mathcal{F}_2(X) - \{\{p,q\}\}$ is a locally connected metric space. So, by (Eilenberg, 1936, Theorem 4, p. 162) and (Stone, 1950, Theorem 5, p. 472), it suffices to show that there exists a retract $\mathcal{Z}$ of $\mathcal{F}_2(X) - \{\{p,q\}\}$ such that $r(\mathcal{Z}) > r(\mathcal{F}_2(X)) = 1$ (see Theorem 2.8). We consider two cases.

Case I. $X - \{p,q\}$ is not connected.

By Lemma 2.3, there exists a simple closed curve S in X containing p and q and a retraction $f\colon X \to S$ such that $f^{-1}(p) = \{p\}$ and $f^{-1}(q) = \{q\}$. Put $\mathcal{Z} = \mathcal{F}_2(S) - \{\{p,q\}\}$. Since S is a cyclicly connected graph, $r(\mathcal{F}_2(S)) = 1$ (see Theorem 2.8). So, by Theorem 3.3, $r(\mathcal{Z}) \geq 2$. Finally, define $\bar{f}\colon \mathcal{F}_2(X) - \{\{p,q\}\} \to \mathcal{Z}$ as follows: for each $A \in \mathcal{F}_2(X) - \{\{p,q\}\}$, let $\bar{f}(A) = f(A)$. Using the fact that $f^{-1}(p) = \{p\}$ and $f^{-1}(q) = \{q\}$, it can be proved that $\bar{f}$ is well defined. Since f is continuous, $\bar{f}$ is continuous. Finally, notice that $\bar{f}(B) = B$ for each $B \in \mathcal{Z}$. Thus, $\bar{f}$ is a retraction.

Case II. $X - \{p,q\}$ is connected.

There exists a theta curve Y in X such that $p, q \in Y$ and a retraction $f\colon X \to Y$ satisfying $f^{-1}(p) = \{p\}$ and $f^{-1}(q) = \{q\}$ (see Lemma 2.4). Since $X - \{p,q\}$ is connected, $Y - \{p,q\}$ is also connected. Put $\mathcal{Z} = \mathcal{F}_2(Y) - \{\{p,q\}\}$. By Theorem 3.4, $r(\mathcal{Z}) \geq 2$ since $r(\mathcal{F}_2(Y)) = 1$ (see Theorem 2.8). Now, define $\bar{f}\colon \mathcal{F}_2(X) - \{\{p,q\}\} \to \mathcal{Z}$ by $\bar{f}(A) = f(A)$ for each $A \in \mathcal{F}_2(X) - \{\{p,q\}\}$. Notice that $\bar{f}$ is well defined since $f^{-1}(p) = \{p\}$ and $f^{-1}(q) = \{q\}$. The continuity of $\bar{f}$ follows from the fact that f is continuous. It is easy to verify that $\bar{f}(B) = B$ for each $B \in \mathcal{Z}$. Thus, $\mathcal{Z}$ is a retract of $\mathcal{F}_2(X) - \{\{p,q\}\}$. □

3.1 Classification

Theorem 3.6 *Let X be a cyclicly connected graph and let $p, q \in X$. Then, $\{p,q\}$ makes a hole with respect to multicoherence degree in $\mathcal{F}_2(X)$ if and only if either $p = q$ and $p \in R(X)$, or $p \neq q$.*

Proof. From (Nadler, Jr., 1992, Theorem 9.10, p. 144; Kuratowski, 1968, Theorem 3, p. 278) and (Nadler, Jr., 1992, Corollary 9.6, p. 142), it follows that $E(X) = \emptyset$. Then, $p, q \in O(X) \cup R(X)$

Assume that $\{p,q\}$ makes a hole in $\mathcal{F}_2(X)$. Now, by Theorem 3.1, either $p = q$ and $p \notin O(X)$, or $p \neq q$. So, eihter $p = q$ and $p \in R(X)$ or $p \neq q$. This proves the necessity.

Finally, the sufficiency follows from Theorems 3.2 and 3.5. □

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