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Some Operation Properties of Adjoint Matrices for Block Matrices

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Abstract

Adjoint matrix is an important matrix, which has very good property. In this paper , to mainly discuss the operation properties of the adjoint matrices of a kind of block matrices and its applications.

Keywords: Block matrices, Adjoint matrices, Operation

Adjoint matrix is an important concepts of matrix, it can deduce the formula for calculating the inverse matrix of square matrix, thus solving the problem of inverse square. At the same time, the properties of the matrix are very important.As a partition of a matrix into rectangular smaller matrices, the operation of a matrix can be transferred to the block matrix, making the operation more convenient. In the following we obtain some operation properties of adjoint matrices of a kind of block matrices by means of basic properties of adjoint matrices.

1. Basic properties of the adjoint matrix

Proposition 1.1 Denotes the adjoint matrix of a matrix A by A^*, then $AA^* = A^*A = |A|E$, and if $|A| \neq 0$, then $A^* = |A|A^{-1}$.

Proposition 1.2 Let A be an n-order square matrix, then $(A^T)^* = (A^*)^T$.

Proposition 1.3 Let A be an $n-$order nonsingular square matrix, then

$$(A^{-1})^* = (A^*)^{-1} = \frac{1}{|A|}A.$$

Proposition 1.4 Let A be an n-order nonsingular square matrix, then

$$[(A^{-1})^T]^* = [(A^*)^T]^{-1} = \frac{1}{|A|}A^T.$$

2. The operation properties of the adjoint matrix of a block matrix

Definition Let A be an n-order matrix. If A only is there non-zero sub-blocks on secondary diagonal, while the others are zero matrices, and the secondary diagonal sub-blocks are square matrices, that is

$$A = \begin{bmatrix} 0 & & & A_1 \\ & & A_2 & \\ & \iddots & & \\ A_s & & & \end{bmatrix},$$

where $A_i\ (i = 1, 2, \ldots, s)$ is a square matrix, then A is called block secondary diagonal matrix.

Next we give some properties of the adjoint matrix of an inverses block secondary diagonal matrix.

Proposition 2.1 Let both A and B be n-order inverses square matrices. Then we have

$$\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}^* = \begin{bmatrix} 0 & (-1)^n|A|B^* \\ (-1)^n|B|A^* & 0 \end{bmatrix}.$$

Proof: Since

$$\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & B^{-1} \\ A^{-1} & 0 \end{bmatrix},$$

and
$$\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} = (-1)^n \begin{bmatrix} B & 0 \\ 0 & A \end{bmatrix} = (-1)^n |B||A|,$$
by $A^* = |A|A^{-1}$, we have

$$\begin{aligned} \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}^* &= \begin{vmatrix} 0 & A \\ B & 0 \end{vmatrix} \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}^{-1} \\ &= (-1)^{n^2}|B||A| \begin{bmatrix} 0 & B^{-1} \\ A^{-1} & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & (-1)^n|B||A|B^{-1} \\ (-1)^n|B||A|A^{-1} & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & (-1)^n|A|B^* \\ (-1)^n|B|A^* & 0 \end{bmatrix}. \end{aligned}$$

Remark: The adjoint matrix of an inverses block secondary diagonal matrix is still a block secondary diagonal matrix, and we only need change sub-blocks of the secondary diagonal matrix into adjoint matrices, multiplied by coefficient and interchange of position.

These results can be generalized to the case of having more sub-blocks on secondary diagonal, such as

$$\begin{bmatrix} 0 & 0 & A \\ 0 & B & 0 \\ C & 0 & 0 \end{bmatrix}^* = \begin{bmatrix} 0 & 0 & (-1)^n|A||B|C^* \\ 0 & (-1)^n|A||C|B^* & 0 \\ (-1)^n|B||C|A^* & 0 & 0 \end{bmatrix},$$

where A, B, C are n-order inverses matrices.

Example 1 Let A, B, C be 2-order inverses matrices, and

$$A^* = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}, B^* = \begin{bmatrix} 2 & 2 \\ 1 & 2 \end{bmatrix}, C^* = \begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix}.$$

Solve $\begin{bmatrix} 0 & 0 & A \\ 0 & B & 0 \\ C & 0 & 0 \end{bmatrix}^*$.

Solution: It is known that $A^* = |A|^{n-1}$, we get $|A| = 3, |B| = 2$ and $|C| = 5$. By using above results we have

$$\begin{aligned} \begin{bmatrix} 0 & 0 & A \\ 0 & B & 0 \\ C & 0 & 0 \end{bmatrix}^* &= \begin{bmatrix} 0 & 0 & (-1)^n|A||B|C^* \\ 0 & (-1)^n|A||C|B^* & 0 \\ (-1)^n|B||C|A^* & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & (-1)^3 3 \cdot 2C^* \\ 0 & (-1)^2 3 \cdot 5B^* & 0 \\ (-1)^2 2 \cdot 5A^* & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 6C^* \\ 0 & 15B^* & 0 \\ 10A^* & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 & 0 & 6 & 6 \\ 0 & 0 & 0 & 0 & 6 & 18 \\ 0 & 0 & 30 & 45 & 0 & 0 \\ 0 & 0 & 15 & 30 & 0 & 0 \\ 10 & -10 & 0 & 0 & 0 & 0 \\ 10 & 20 & 0 & 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

Proposition 2.2 Let A, B be n-order inverses square matrices. Then

$$\left[\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}^T \right]^* = \left[\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}^* \right]^T = \begin{bmatrix} 0 & (-1)^n|B|(A^*)^T \\ (-1)^n|A|(B^*)^T & 0 \end{bmatrix}.$$

Proof: Since A, B are n-order inverses square matrices, so $\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}$ is reversible, by

$$\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}^* = \begin{bmatrix} 0 & (-1)^n|A|B^* \\ (-1)^n|B|A^* & 0 \end{bmatrix},$$

it follows that

$$\left[\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}^T\right]^* = \left[\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}^*\right]^T = \begin{bmatrix} 0 & (-1)^n|A|B^* \\ (-1)^n|B|A^* & 0 \end{bmatrix}^T$$
$$= \begin{bmatrix} 0 & (-1)^n|B|(A^*)^T \\ (-1)^n|A|(B^*)^T & 0 \end{bmatrix}.$$

The above results can be generalized to the case of having more sub-blocks on secondary diagonal, such as

$$\left[\begin{pmatrix} 0 & 0 & A \\ 0 & B & 0 \\ C & 0 & 0 \end{pmatrix}^T\right]^* = \begin{bmatrix} 0 & 0 & (-1)^n|B||C|(A^*)^T \\ 0 & (-1)^n|A||C|(B^*)^T & 0 \\ (-1)^n|A||B|(C^*)^T & 0 & 0 \end{bmatrix},$$

where A, B, C are n-order inverses matrices.

Example 2 Let A, B, C be 3-order inverses matrices, and

$$|A| = 3, |B| = -2, |C| = 5, A^* = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 9 \end{bmatrix}, B^* = \begin{bmatrix} 4 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}, C^* = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Solve $\left[\begin{pmatrix} 0 & 0 & A \\ 0 & B & 0 \\ C & 0 & 0 \end{pmatrix}^T\right]^*$.

Solution: By

$$\left[\begin{pmatrix} 0 & 0 & A \\ 0 & B & 0 \\ C & 0 & 0 \end{pmatrix}^T\right]^* = \begin{bmatrix} 0 & 0 & (-1)^n|B||C|(A^*)^T \\ 0 & (-1)^n|A||C|(B^*)^T & 0 \\ (-1)^n|A||B|(C^*)^T & 0 & 0 \end{bmatrix},$$

we get

$$\left[\begin{pmatrix} 0 & 0 & A \\ 0 & B & 0 \\ C & 0 & 0 \end{pmatrix}^T\right]^* = \begin{bmatrix} 0 & 0 & (-1)^3(-2)\cdot 5|(A^*)^T \\ 0 & (-1)^3 3\cdot 5(B^*)^T & 0 \\ (-1)^3 3\cdot(-2)(C^*)^T & 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & 10(A^*)^T \\ 0 & -15(B^*)^T & 0 \\ 6(C^*)^T & 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 10 & 40 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -10 & 10 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 9 \\ 0 & 0 & 0 & -60 & -15 & -30 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -15 & 15 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -15 & 0 & 0 & 0 \\ 30 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 30 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Proposition 2.3 Let A, B be n-order inverses square matrices. Then we have

$$\left[\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}^{-1}\right]^* = \left[\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}^*\right]^{-1} = \begin{bmatrix} 0 & (-1)^n\frac{1}{|A||B|}A \\ (-1)^n\frac{1}{|A||B|}B & 0 \end{bmatrix}.$$

Proof: In veiw of Proposition 1.3, we have

$$\left[\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}^{-1}\right]^* = \left[\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}^*\right]^{-1} = \frac{1}{\left|\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}\right|}\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} = \frac{1}{(-1)^n|A||B|}\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}$$
$$= \begin{bmatrix} 0 & (-1)^n \frac{1}{|A||B|}A \\ (-1)^n \frac{1}{|A||B|}B & 0 \end{bmatrix}.$$

The above results can be generalized to the case of having more sub-blocks on secondary diagonal, such as

$$\left[\begin{pmatrix} 0 & 0 & A \\ 0 & B & 0 \\ C & 0 & 0 \end{pmatrix}^{-1}\right]^* = \left[\begin{pmatrix} 0 & 0 & A \\ 0 & B & 0 \\ C & 0 & 0 \end{pmatrix}^*\right]^{-1} = \begin{bmatrix} 0 & 0 & (-1)^n \frac{1}{|A||B||C|}A \\ 0 & (-1)^n \frac{1}{|A||B||C|}B & 0 \\ (-1)^n \frac{1}{|A||B||C|}C & 0 & 0 \end{bmatrix},$$

where A, B, C are n-order inverses matrices.

Proposition 2.4 Let A, B be n-order inverses square matrices. Then we have

$$\left(\left[\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}^{-1}\right]^T\right)^* = \left(\left[\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}^*\right]^T\right)^{-1} = \begin{pmatrix} 0 & (-1)^n \frac{1}{|A||B|}B^T \\ (-1)^n \frac{1}{|A||B|}A^T & 0 \end{pmatrix}.$$

Proof: In view of Proposition 1.4, we have

$$\left(\left[\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}^{-1}\right]^T\right)^* = \left(\left[\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}^*\right]^T\right)^{-1} = \frac{1}{\left|\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}\right|}\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}^T$$
$$= (-1)^n\frac{1}{|A||B|}\begin{pmatrix} 0 & B^T \\ A^T & 0 \end{pmatrix} = \begin{pmatrix} 0 & (-1)^n \frac{1}{|A||B|}B^T \\ (-1)^n \frac{1}{|A||B|}A^T & 0 \end{pmatrix}.$$

The above results can be generalized to the case of having more sub-blocks on secondary diagonal, such as

$$\left(\left[\begin{pmatrix} 0 & 0 & A \\ 0 & B & 0 \\ C & 0 & 0 \end{pmatrix}^{-1}\right]^T\right)^* = \left(\left[\begin{pmatrix} 0 & 0 & A \\ 0 & B & 0 \\ C & 0 & 0 \end{pmatrix}^*\right]^T\right)^{-1} = \begin{pmatrix} 0 & 0 & (-1)^n \frac{1}{|A||B||C|}C^T \\ 0 & (-1)^n \frac{1}{|A||B||C|}B^T & 0 \\ (-1)^n \frac{1}{|A||B||C|}A^T & 0 & 0 \end{pmatrix}.$$

Example 3 Let

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}, B = \begin{bmatrix} -1 & 1 \\ -3 & 2 \end{bmatrix}, C = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}.$$

Solve $\left(\left[\begin{pmatrix} 0 & 0 & A \\ 0 & B & 0 \\ C & 0 & 0 \end{pmatrix}^{-1}\right]^T\right)^*$.

Solution: In view of assumption, we have $|A| = 1, |B| = 1, |C| = 1$. It follows that

$$\left(\left[\begin{pmatrix} 0 & 0 & A \\ 0 & B & 0 \\ C & 0 & 0 \end{pmatrix}^{-1}\right]^T\right)^* = \begin{bmatrix} 0 & 0 & (-1)^n \frac{1}{|A||B||C|}C^T \\ 0 & (-1)^n \frac{1}{|A||B||C|}B^T & 0 \\ (-1)^n \frac{1}{|A||B||C|}A^T & 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & C^T \\ 0 & B^T & 0 \\ A^T & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & -1 & -3 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 2 & 3 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

By using the operation properties of adjoint matrix of block matrices, we can easily find adjoint matrix of a block matrix. As in the above operation properties, the sub-blocks on secondary diagonal are in the same order, if not, we can get similar results.

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