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ω-Connectedness and Local ω-Connectedness on an $L\omega$-Space

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Abstract

In this paper, the concepts of the ω-coincidence neighborhood, local ω-connected set and local ω-connected space on an $L\omega$-space are introduced. The characterizations of the concepts are given, such as topological invariant property and good extension.

Keywords: $L\omega$-space, ω-remote neighborhood, ω-connectedness, Local ω-connectedness

1. Introduction

The connectedness is one of the most important notions in topology. In 1988, Wang introduced the concept of the remote-neighborhood and studied the connectedness on an *LF* topology space (Wang, 1988). In 2002, Chen and Dong further generalized the above notions on an *LF* order-preserving operator space (or on an $L\omega$-space) (Chen and Dong, 2002, pp.36-41), then the author discussed the ω-connectedness (Huang, 2003, pp.165-168), the quasi ω-Lindelöf property (Huang, 2004, pp.34-38) and the ω-separation (Huang, 2005, pp.383-388) on an $L\omega$-space respectively. In this paper, some characterizations with respect to the local ω-connectedness on an $L\omega$-space are given.

2. Preliminary definitions

Throughout this paper, L denotes the fuzzy lattice, M denotes the set consisting of all nonzero irreducible element(i.e. so-called molecule) in L. X denotes nonempty crisp set, L^X denotes the set of all L-fuzzy sets on X. A' denotes the pseudo-complement of A. 1_X and 0_X denote the greatest and the least elements in L^X, respectively. $M^*(L^X) = \{x_\alpha | x \in X, \alpha \in M\}$. Other notions and symbols can be obtained from references.

Definition 2.1 Let X be a nonempty set, $\omega : L^X \to L^X$ is called an *LF* order-preserving operator if (1) $\omega(1_X) = 1_X$; (2) For each $A, B \in L^X$, if $A \leq B$, then $\omega(A) \leq \omega(B)$; (3) For each $A \in L^X$, then $A \leq \omega(A)$. Meanwhile, A is called an ω-set in L^X if $A = \omega(A)$. Let $\Omega = \{A \in L^X | A = \omega(A)\}$, then (L^X, Ω) is called an *LF* order-preserving operator space, or an $L\omega$-space.

Definition 2.2 Let (L^X, Ω) be an $L\omega$-space, $x_\alpha \in M^*(L^X)$, $P, A \in L^X$.

(1) P is called an ω-remote neighborhood of x_α if there exists a $Q \in \Omega$, such that $x_\alpha \nleq Q$ and $P \leq Q$. Let $\omega\eta(x_\alpha)$ be the set of all ω-remote neighborhood of x_α.

(2) x_α is called an ω-adherence point of A if for each $P \in \omega\eta(x_\alpha)$, we have $A \nleq P$. Let A^-_ω be the set of all ω-adherence point of A. A is called an ω-closed set of (L^X, Ω) if $A = A^-_\omega$. A is called an ω-open set of (L^X, Ω) if A' is an ω-closed set.

Definition 2.3 Let (L^X, Ω) be an $L\omega-$space, $A, B \in L^X$. If $A^-_\omega \wedge B = A \wedge B^-_\omega = 0_X$, then A and B are called the ω-separated sets.

Definition 2.4 Let (L^X, Ω) be an $L\omega$-space, $A \in L^X$. If there do not exist two nonzero ω-separated sets B and C, such that $A = B \vee C$, then A is called an ω-connected set. Particularly, (L^X, Ω) is called an ω-connected space if 1_X is an ω-connected set.

Definition 2.5 Let (L^X, Ω) be an $L\omega$-space, A is called the maximal ω-connected set if B is an ω-connected set and $B = A$ with $A \leq B$. A is also called an ω-connected component of (L^X, Ω).

Lemma 2.1 Let $A, B \in L^X$, if $A \nleq B$, then $A' \vee B \neq 1_X$.

Proof There exists $x \leq A$, $x \nleq B$, then we have $x \nleq A'$ and $x \nleq B$, this means $x \nleq A' \vee B$, hence $A' \vee B \neq 1_X$.

Lemma 2.2 (Huang, 2003, pp.165-168) If (L^X, Ω) is an $L\omega$-space, $A \in L^X$, then A is an ω-connected set if and only if for

any two molecules a, b of A and for the ω-remote neighborhood $P(x)$ of x in A, there exist finite molecules $x_0,\ x_1,\ \cdots,\ x_n$ in A, such that
$x_0 = a,\ x_n = b$ and $A \nleq P(x_i) \vee P(x_{i+1}),\ (i = 0,\ 1, \cdots,\ n)$
or
$P(x_i) \vee P(x_{i+1}) \vee A' \neq 1_X\ (i = 0,\ 1, \cdots,\ n)$.

3. ω-connectedness on an $L\omega$-space

Theorem 3.1 If $(L^X,\ \Omega)$ is an $L\omega$-space, $A \in L^X$, then the followings are equivalent:

(1) A is an ω-connected set.

(2) There do not exist two nonzero ω-closed sets $F_1,\ F_2$, such that $A \nleq F_1$, $A \nleq F_2$, $A \wedge F_1 \wedge F_2 = 0_X$, $A' \vee F_1 \vee F_2 = 1_X$.

(3) There do not exist two nonzero ω-open sets $Q_1,\ Q_2$, such that $A \wedge Q_1 \neq 0_X$, $A \wedge Q_2 \neq 0_X$, $A \wedge Q_1 \wedge Q_2 = 0_X$, $A' \vee Q_1 \vee Q_2 = 1_X$.

Proof (1)$\Rightarrow$ (2): Suppose that there exist two nonzero ω-closed sets $F_1,\ F_2$, such that $A \nleq F_1$, $A \nleq F_2$, $A \wedge F_1 \wedge F_2 = 0_X$, $A' \vee F_1 \vee F_2 = 1_X$, then for any molecule x of A, only one of $x \nleq F_1$, $x \nleq F_2$ holds, otherwise we get $A \nleq F_1 \vee F_2$, and this contradicts $A' \vee F_1 \vee F_2 = 1_X$. For any molecule x of A, let $P(x) = F_1$ if $x \nleq F_1$ and $P(x) = F_2$ if $x \nleq F_2$. Since $A \nleq F_1$, $A \nleq F_2$, there exist two molecules a, b of A, such that $a \nleq F_1$, $b \nleq F_2$, hence for arbitrary finite molecules $x_0,\ x_1,\ \cdots,\ x_n$ with $x_0 = a$, $x_n = b$, there exists $i(0 \leq i \leq n)$, such that $P(x_i) = F_1$, $P(x_{i+1}) = F_2$. This means that $A' \vee P(x_i) \vee P(x_{i+1}) = 1_X$, or A is not an ω-connected set.

(2)$\Rightarrow$ (3): Let $F_1,\ F_2$ be two nonzero ω-closed sets, such that $A \nleq F_1$, $A \nleq F_2$, $A \wedge F_1 \wedge F_2 = 0_X$, $A' \vee F_1 \vee F_2 = 1_X$, then F_1' and F_2' are two nonzero ω-open sets, and for lemma 2.1, we have $A \wedge F_1' \neq 0_X$, $A \wedge F_2' \neq 0_X$, $A \wedge F_1' \wedge F_2' = 0_X$, $A' \vee F_1' \vee F_2' = 1_X$. Let $Q_1 = F_1'$, $Q_2 = F_2'$, then we have (3).

(3)$\Rightarrow$ (2): Suppose that there exist two nonzero ω-closed sets $F_1,\ F_2$, such that $A \nleq F_1$, $A \nleq F_2$, $A \wedge F_1 \wedge F_2 = 0_X$, $A' \vee F_1 \vee F_2 = 1_X$, let $Q_1 = F_1'$, $Q_2 = F_2'$, then we get $A \wedge Q_1 \wedge Q_2 = 0_X$, $A' \vee Q_1 \vee Q_2 = 1_X$. At the same time, by lemma 2.1, we have $A \wedge Q_1 \neq 0_X$, $A \wedge Q_2 \neq 0_X$. This means that there exist two nonzero ω-open sets $Q_1,\ Q_2$, such that $A \wedge Q_1 \neq 0_X$, $A \wedge Q_2 \neq 0_X$, $A \wedge Q_1 \wedge Q_2 = 0_X$, $A' \vee Q_1 \vee Q_2 = 1_X$, this contradicts (3).

(2)$\Rightarrow$ (1): Suppose that A is not an ω-connected set, then there exist two molecules a, b of A and for the ω-remote neighborhood $P(x)$ of x in A, for arbitrary finite molecules $x_0,\ x_1, \cdots,\ x_n$ with $x_0 = a$, $x_n = b$, there exists $i\ (0 \leq i \leq n)$, such that

$P(x_i) \vee P(x_{i+1}) \vee A' = 1_X$.

We call that a and b cannot connect finitely, let

$W_a = \{x|x$ is a molecule which can connect with a in A finitely$\}$,

$W_b = \{x|x$ is a molecule of A which does not belong to $W_a\}$.

Then for any $c \in W_a$, $d \in W_b$, we have

$P(c) \vee P(d) \vee A' = 1_X$.

Let $F_1 = \wedge\{P(c)|C \in W_a\}$, $F_2 = \wedge\{P(d)|d \in W_b\}$, then one can get $A' \vee F_1 \vee F_2 = \bigwedge_{\substack{c\in W_a\\ d\in W_b}} \{P(c) \vee P(d) \vee A'\} = 1_X$, Obviously, we have $A \wedge F_1 \wedge F_2 = 0_X$. Because of $a \in W_a$, $b \in W_b$, hence $A \nleq F_1$, $A \nleq F_2$, this contradicts (2).

Similarly, we have

Theorem 3.2 If $(L^X,\ \Omega)$ is an $L\omega$-space, $A \in L^X$, then the followings are equivalent:

(1) A is an ω-connected set.

(2) There do not exist two nonzero ω-open sets $Q_1,\ Q_2$, such that $A \nleq Q_1$, $A \nleq Q_2$, $A \wedge Q_1 \wedge Q_2 = 0_X$, $A' \vee Q_1 \vee Q_2 = 1_X$.

(3) There do not exist two nonzero ω-closed sets $F_1,\ F_2$, such that $A \wedge F_1 \neq 0_X$, $A \wedge F_2 \neq 0_X$, $A \wedge F_1 \wedge F_2 = 0_X$, $A' \vee F_1 \vee F_2 = 1_X$.

Theorem 3.3 If $(L^X,\ \Omega)$ is an $L\omega$-space, $A \in L^X$, then A is an ω-connected set if and only if $A_0 = \{x \in X|A(x) > 0\}$ is an ω- connected set.

Proof It is obvious that for any A, $B \in L^X$, $A \wedge B = \phi$ is equivalent to $A_0 \wedge B = \phi$, by theorem 3.1 or theorem 3.2, we have the conclusion.

4. Local ω-connectedness on an $L\omega$-space

Definition 4.1 Let $(L^X,\ \Omega)$ be an $L\omega$-space, $x_\alpha \in M^*(L^X)$, $Q \in \Omega$.

(1) Q is called the ω-open coincidence neighborhood of x_α, if Q' is the ω-closed remote neighborhood of x_α.

(2) $P \in L^X$ is called the ω-coincidence neighborhood of x_α, if P' is the ω-remote neighborhood of x_α.

Definition 4.2 (L^X, Ω) is called a local ω-connected space, if for any $x_\alpha \in M^*(L^X)$ and for any ω-coincidence neighborhood Q of x_α, Q includes an ω-connected coincidence neighborhood P of x_α.

It is obvious that (L^X, Ω) is a local ω-connected space if and only if for any $x_\alpha \in M^*(L^X)$, the ω-coincidence neighborhood base (Chen, 2004, pp.11-16) of x_α is composed of all of the ω-connected coincidence neighborhood of x_α.

Theorem 4.1 If (L^X, Ω) is an $L\omega$-space, then the followings are equivalent:

(1) (L^X, Ω) is a local ω-connected space;

(2) If B is an open ω-connected component, then B is an ω-open set;

(3) (L^X, Ω) includes an ω-base which elements are ω-connected.

Proof (1)⇒ (2): Let $A \in \Omega$ and B is an ω-connected component of A. Now we will prove that B' is an ω-closed set, therefore B is an ω-open set. Let $x_\alpha \in M^*(L^X)$ and $x_\alpha \nleq B'$, then $x_\alpha \nleq A'$, by definition 4.1, A is an ω-coincidence neighborhood of x_α. And for (1), A includes an ω-connected coincidence neighborhood P of x_α. We notice that $x_\alpha \nleq (B \wedge P)'$, that is to say $B \wedge P \neq \phi$. Since B is an ω-connected component of A, $B \wedge P \leq B$, hence $P \leq B$, $B' \leq P'$, so we have $x_\alpha \nleq (B')^-_\omega$, this is $B' = (B')^-_\omega$, or B is an ω-open set.

(2)⇒ (3): Let $A \in \Omega$, then A is the union of all ω- connected components of A, by (2) all of the ω- connected components are ω-open sets, therefore the ω-base of (L^X, Ω) is composed of all of the open ω-connected components.

(3)⇒ (1): Let μ is the ω-base of (L^X, Ω) which elements are ω-connected. It obvious that for any $x_\alpha \in M^*(L^X)$, $\mu(x_\alpha) = \{B \in \mu | x_\alpha \nleq B'\}$ is the ω-coincidence neighborhood of x_α, hence (L^X, Ω) is an local ω-connected space.

Using theorem 3.4 on (Huang, 2003, pp.165-168), we have

Corollary 4.1 Each ω- connected component of local ω-connected space (L^X, Ω) is not only an ω-open set but also an ω-closed set.

Theorem 4.2 Let $(L_i^{X_i}, \Omega_i)(i = 1, 2)$ be two $L\omega$-spaces, $f : L_1^{X_1} \to L_2^{X_2}$ is an (ω_1, ω_2)-continuous, open, full order homomorphism (Chen and Dong, 2002, pp.36-41). If $(L_1^{X_1}, \Omega_1)$ is a local ω_1-connected space, then $(L_2^{X_2}, \Omega_2)$ is a local ω_2- connected space.

Proof We can suppose that β is an ω_1-base of $(L_1^{X_1}, \Omega_1)$ by theorem 4.1, which elements are ω_1-connected. For each $B \in \beta$, $f(B)$ is ω_2-connected (Huang, 2003, pp.165-168). At the same time, since f is an open, full order homomorphism, $f(L_1^{X_1}) = L_2^{X_2}$ is an ω_2-open set, this means that the family $\tilde{\beta} = \{f(B) | B \in \beta\}$ is composed of the ω_2-connected open sets of $(L_2^{X_2}, \Omega_2)$. Now we will prove that $\tilde{\beta}$ is an ω_2-base of $(L_2^{X_2}, \Omega_2)$. If U is an ω_2-open set of $(L_2^{X_2}, \Omega_2)$, then $f^{-1}(U)$ is an ω_1-open set of $(L_1^{X_1}, \Omega_1)$, so there exists $\beta_1 \subset \beta$, such that $f^{-1}(U) = \vee_{B \in \beta_1} B$, hence we have

$$U = f(f^{-1}(U)) = \vee_{B \in \beta_1} f(B),$$

This means that $(L_2^{X_2}, \Omega_2)$ is a local ω_2- connected space.

Corollary 4.2 Local ω- connectedness on an $L\omega$-space has the invariant property of homoeomorphism.

Definition 4.3 Let X be an non-empty set, $P(X)$ is the power set of X. If the operator $\omega : P(X) \to P(X)$ which satisfies the followings: (1) $\omega(X) = X$; (2) For any $A, B \in P(X)$, if $A \subset B$, then $\omega(A) \subset \omega(B)$; (3) For any $A \subset X$, $A \subset \omega(A)$, then ω is called the order-preserving operator of X. Meanwhile, A is called an ω-set of X if $A = \omega(A)$. Let $\triangle = \{A \in P(X) | A = \omega(A)\}$, then $(X, \triangle)$ is called an ω-order-preserving operator space, or an ω-space.

Theorem 4.3 Let $(X, \triangle)$ be an ω-space, $(L^X, \omega_L(\triangle))$ is an $L\omega$-space generated topologically by $(X, \triangle)$ (Huang, 2005, pp.383-388), then $(L^X, \omega_L(\triangle))$ is local ω-connected if and only if $(X, \triangle)$ is local ω-connected.

Proof Let $(L^X, \omega_L(\triangle))$ be local ω-connected, then there exists an ω-base μ which elements are ω-connected, let

$$S = \{A_0 | A \in \mu\}$$

.

It is obvious that S is an ω-base of $\triangle$. By theorem 3.3, the elements in S are ω-connected in $(L^X, \omega_L(\triangle))$, therefore they are ω-connected in $(X, \triangle)$(Huang, 2005, pp.383-388), this means that $(X, \triangle)$ is local ω-connected.

Conversely, if $(X, \triangle)$ is local ω-connected, then there exists an ω-base S in $\triangle$ which elements are ω-connected, so the elements of S are ω-connected in $(L^X, \omega_L(\triangle))$ (Huang, 2005, pp.383-388). It is obvious that $\mu = \{\lambda \wedge A | \lambda \in L, A \in S\}$ is an ω-base of $\omega_L(\triangle)$, when $\lambda \neq 0$, we notice $(\lambda \wedge A)_0 = A_0$, so all the elements of μ are ω-connected by theorem 3.3, hence $(L^X, \omega_L(\triangle))$ is local ω-connected by theorem 4.1.

Corollary 4.3 Local ω-connectedness on an $L\omega$-space has good extension.

5. Conclusions

In this paper, starting with the unified operator which is called an $L\omega$-space containing various closure operators such as θ - closure operator (Chen, 1992), δ-closure operator (Cheng, 1997, pp.38-41), we introduce the concept of the ω-operator, ω-remote neighborhood, ω-coincidence neighborhood and local ω-connected space, discuss the basic properties of the $L\omega$-space, such as the ω-connectedness, the local ω-connectedness and the invariant property of homeomorphism. All the discussions will offer a theoretical foundation in fuzzy operator.

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