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Optimal Control Theory to Solve Production Inventory System in Supply Chain Management

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Abstract

This paper describes how to control the inventory production system with Weibull distributed deterioration items. The model is solved by two methods and a comparison between them is conducted. In the first method the model is solved using the control theory approach. In the second method the model is discretized then the Dynamic Programming (DP) technique is applied. The advantage of second method is easier than the first method in computational and its accuracy can be improved by increasing the number of discretization intervals (sampling).

Keywords: Optimal Control, Production Inventory System, Weibull Distribution, Dynamic Programming.

1 Introduction

(Zaher 2013) Optimal Control theory becomes very useful tool to solve dynamic inventory and production problems .The production system consists of manufacturing plant and finished goods in warehouse to store those products which are fabricated but not at once sold. Excess Inventory on hand will be sold during high demand intervals. The advantage of having products in inventory are: First it will be available to meet demand, second by using warehouse to store excess production. The firm has to evaluate the high production cost s and find the quantity it should be produced in order to maintain the total cost at minimum. The main aim of the paper is to minimize the difference between actual production flow rate and required production flow rate. In this paper we compare between linear quadratic control (LQC) and dynamic programming(DP). Due to using discretization to convert continuous time system to discrete time system LQC is more exact than DP but more complex in computational than DP. (Emamverdi 2011) presented optimal control of production inventory system with deteriorating items in which the deteriorating rate follows the Weibull distribution. They adjust the optimal production rate to minimize total production and inventory costs. (El-Gharry 2009) presented the production inventory system consisting of two stores. The model represented as an optimal control problem with two state variables, the inventory levels in the first store and the same in the second store. The paper considered also the case of three control variables, the manufacturing, and remanufacturing and disposal rates. He used The Pontryagin's minimum principle to find the optimum control of the Holt, Modigiliani, Muth and Simon (HMMS) reverse logistics model of production inventory system with deteriorating items. (Varbie 2009) presented a model where the new policy iteration technique is used to solve online the continuous time LQR problem without using knowledge about the system detail dynamics. (Chaudhary 2011) considered market segmentation as a vital element of marketing in industrialized countries. They used market segmentation approach in single product inventory system with deteriorating items. Problems studied and solved using Pontryagin's maximun principle. (Adida, 2007) Investigated a continuous time optimal control model for a dynamic pricing and inventory system problem with no backorders. They presented a continuous time solution approach using Pontryagin’s Principle for state-constrained problems. They illustrated the role of capacity and of the dynamic nature of demand in the model.(Yang 2006) Defined the deterioration as obsolescence decay, damage, spoilage, evaporation, , pilferage and loss of marginal value or looses of entity of a product that affect on decreasing usefulness from original one. (Singh 2011) represented a method rely on genetic algorithm to improve the performance of inventory in supply chain management. The proposed method use MATLAB.

2. Mathematical Model and Notations

To build an optimal control model, consider that a firm can manufacture a certain product selling some and stocking the rest in warehouse. Assuming the firm distributed the product to a certain buyer, the firm has set an inventory required level and production required rate. The instantaneous rate of deterioration of the on hand inventory follows the two parameters Weibull distribution and the production is continuous with no shortage allowed. The objective is to minimize total cost. $X_1(t)$; Vendor inventory level at time t. $X_2(t)$; Buyer inventory level at time t. $u(t)$; Production flow rate at time t. $D(t)$; Demand rate at time t. x_{10}; Initial vendor inventory level. x_{20}; Initial buyer inventory level. $\hat{u}$; Production goal rate. h_1 ; Vendor holding cost h_2 ; Buyer holding cost c ;Production unit cost . θ(t) ; Deterioration rate. $\hat{x}_1$; Vendor inventory goal level. $\hat{x}_2$;Buyer inventory goal level.

The interpretation of inventory goal level is that a safety stock that company wants to keep on hand. Similarly the production goal level is interpreted as most efficient level at which it is desired to run the factory. The time of deterioration is a random variable following two parameter Weibull distributions. The probability density function for two parameter Weibull distribution is given by

$$f(t) = ab \quad t^{b-1} e^{-\alpha t^b} \quad t > 0$$

a : Scale parameter $a > 0$

b : Shape parameter $b > 0$

The probability distribution function

$$F(t)=1- e^{-at^b} \qquad t>0$$

The instantaneous rate of deterioration of the on hand inventory is given by

$$\theta(t)= \quad abt^{b-1} \qquad t>0$$

Since our objective to minimize setup and inventory costs the objective function to be expressed as quadratic form:

$$minimize \quad j = \frac{1}{2}\int_0^T \left[h_1 \left(x_1(t)-\hat{x}_1\right)^2 + h_2 \left(x_2(t)-\hat{x}_2\right)^2 + c\left(u_v(t)-u_v\right)^2 \right] dt \tag{1}$$

s.t.

$$\dot{x}_1(t)= u(t) -D(t) -ab\ t^{b-1} \ x_1(t) \qquad x_1(0)=x_1 \tag{2}$$

$$\dot{x}_2(t)= -D(t) -ab\ t^{b-1} \ x_2(t) \qquad x_2(0)=x_2 \tag{3}$$

where: dot denotes differentiation with respect to (w.r.t.) time t.

3. Quadratic Optimal Control

To develop the optimal control model we define the variables $z(t)$, $k(t)$ and $v(t)$ so that

$$z(t) = x_1(t)-\hat{x}_1 \tag{4}$$

$$y(t) = x_2(t)-\hat{x}_2 \tag{5}$$

$$k(t) = u(t)-\hat{u} \tag{6}$$

$$v(t)= \hat{u}(t) - D(t) -ab\ t^{b-1} \ \hat{x}_1 \tag{7}$$

$$m(t)=D(t)-ab\ t^{b-1}\ \hat{x}_2 \tag{8}$$

Substitute the term $ab\ t^{b-1}\ \hat{x}_1$ in equation (2) by adding and subtracting this term. Similarly Substitute the term $ab\ t^{b-1}\ \hat{x}_2$ in equation (3) by adding and subtracting this term we get

$$\dot{z}(t) = -ab\ t^{b-1}\left(x_1(t) - \hat{x}_1\right) - D(t) + u(t) - ab\ t^{b-1}\ \hat{x}_1 \tag{9}$$

$$\dot{y}(t) = -ab\ t^{b-1}\left(x_2(t) - \hat{x}_2(t)\right) - D(t) - ab\ t^{b-1}\ \hat{x}_2 \tag{10}$$

Substitute (4) (6) (7) in equation (9) and (5) (8) in equation (10) we get

$$\dot{z}(t) = -ab\ t^{b-1}\ z(t) + k(t) + v(t) \tag{11}$$

$$\dot{y}(t) = -ab\ t^{b-1}\ y + m(t) \tag{12}$$

This form standard linear quadratic regulator LQR problem with known disturbance defined in (Chaudhary, et al., 2011), (Kou, 1975).

Minimum
$$j = \frac{1}{2}\int_0^T \left(Z^T(t)Q(t)Z(t) + K^T(t)\,R(t)\,k(t)\right) dt \tag{13}$$

s.t

$$\dot{z}(t) = A(t)\,z(t) + B(t)\,k(t) + v(t) \tag{14}$$

where

Q: Error weight positive semi definite matrix R : control weighted matrix positive definite matrix A : state matrix. B : control matrix. Solution using Pontryagin maximum principle (9)(8)

Solution using Pontryagin's maximum principle (Chaudhary, et al., 2011), (Kou, 1975), (Varbie, et al., 2009), (Emamverdi, et al., 2011)

Step 1: Using definition of Hamiltonian (Sethi, et al., 2010)

$$H(x(t), u(t), \lambda(t)) = \frac{1}{2} Z^T(t)Q(t)Z(t) + \frac{1}{2} K^T(t)\,R(t)\,k(t) + \lambda A(t)z(t) + Bu(t) \tag{15}$$

Where λ : Costate vector of n th order

Step 2: Optimal control $\frac{\partial H}{\partial K} = 0$

$$K^*(t) = -R^{-1}(t)\ B(t)\ \lambda(t) \tag{16}$$

Step 3: State and Costate System

$$\dot{Z}(t) = \frac{\partial H}{\partial \lambda} = A(t)\,Z(t) + B(t)\,k(t) + v(t) \tag{17}$$

$$\dot{\lambda}(t) = -\frac{\partial H}{\partial Z} = -Q(t)z(t) - A^T(t)\,\lambda(t) \tag{18}$$

Substitute the optimal control $K^*(t)$ in equations (16) and (17) the state –costate equations can be written in matrix conical form as

$$\begin{bmatrix} \dot{Z}(t) \\ \dot{\lambda}(t) \end{bmatrix} = \begin{bmatrix} A(t) & -B(t)R(t)B^T(t) \\ -Q(t) & -A'(t) \end{bmatrix} \begin{bmatrix} Z(t) \\ \lambda(t) \end{bmatrix} \tag{19}$$

Step 4: Feed back optimal control

Assume a transformation $\lambda(t) = p(t)z(t)$

Where p: Riccati coefficient

$$k^*(t) = -R^{-1}(t)B(t)p(t)Z^*(t) \tag{20}$$

Step 5: Matrix Differential Riccati Equation (DRE)

$$\dot{p}(t) = -p(t)A(t) - A^T p(t) - Q(t) + p(t)B(t)R^{-1}B^T(t)p(t) \tag{21}$$

$$P(T)=F(T) \tag{22}$$

Solution of Differential Riccati Equation (DRE)

$$H_{(\tau)} = -[W_{22} - FW_{12}]^{-1}[W_{21} - FW_{11}] \tag{23}$$

$$H_{(\tau)} = e^{-M(T-t)} * H_{(T)} e^{-M(T-t)} \tag{24}$$

$$P(t) = [W_{21} + W_{22}H(\tau)][W_{11} + W_{12}H(\tau)]^{-1} \tag{25}$$

Where

$$\tau = T - t$$

M: Diagonal matrix contains eigen values of Matrix

$$\begin{bmatrix} A(t) & -B(t)R(t)B'(t) \\ -Q(t) & -A'(t) \end{bmatrix}$$

with positive real parts in right half plane.

W:Matrix of eigenvectors corresponding to diagonal matrix M .

4. Dynamic Programming Method

The discrete state equation (2) can be solved by means of simple recursion procedure. Setting k = 0,1 ,2,3,4,5 in equation (2)

$$K=0: \quad x(m) = x(0) + mu(0) - mD - mx(0) \tag{26}$$

$$K=1: \quad x(2m) = x(m) + mu(m) - mD - mx(m) \tag{27}$$

$$K=2: \quad x(3m) = x(2m) + mu(2m) - mD - mx(2m) \tag{28}$$

$$K=k-1: \quad x(km) = x(k-1) + mu(k-1) - mD - m(x(k-1) \tag{29}$$

Where : m is discretization interval (sampling).

Also replacing the integration in continuous time objective function (1)

$$j_k = \frac{1}{2}\sum_{k=k_0}^{k_f-1} [m\, h_1 (x_1(k) - \hat{x}_1)^2 + m\, h_2 (x_2(k) - \hat{x}_2)^2 + mc (u_v(k) - u_v)^2] \tag{30}$$

The DP forward recursive equation can be expressed as :

$$j(k)=\min(j(k)+j(k-1)) \quad \text{for } k=1,2,3,4,5 \tag{31}$$

5. Numerical Example

In this section numerical example is presented to illustrate the model. Table (1) presents the values of system parameters and initial states which are used in the numerical example.

Table 1. Parameters given

Parameters	value
a	1
b	1
x_{10}	10
x_{20}	15
$\hat{x}_1$	40
$\hat{x}_2$	25
C	\$1
h_1	\$1
h_2	\$1
$\hat{u}$	15
T	1

5.1 Solution by Pontryagin Minimum Principle

Comparing the present plant (11), (12) and the PI (10) of the problem with the corresponding general formulations of the plant (14) and the equation (13), respectively, let us first identify the various quantities as

$$A=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}; \qquad B=\begin{bmatrix}1\\ 0\end{bmatrix}; \qquad Q=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix};$$

$$F=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}; \qquad R=1; \qquad T=1$$

We Substitute these values in equation (25) and plot of Riccati coefficients as function of t as in fig. (1).similarly substitute in equation (20) we get optimum control $k^*(t)$ and substitute in equation (6) to get original optimal control u. Fig. (2). Gives a plot of $u(t)$ as function of time.

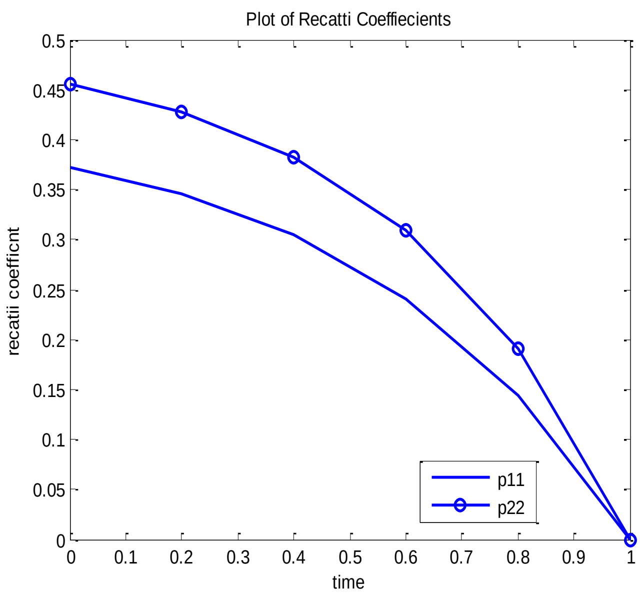

Figure 1. Riccati Coefficients

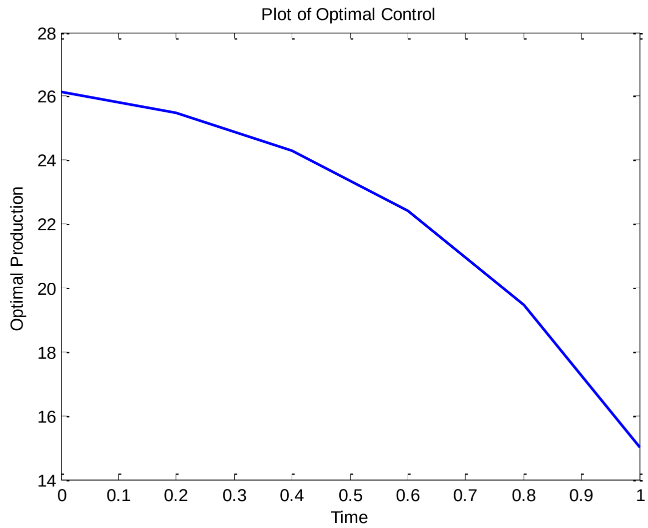

Figure 2. Optimal Control

5.2 Solution by Dynamic Programming

Taking discretization sampling m = 0.2 and assuming the demand rate D(t) is constant and equal 5 .

Substitute in equation (26) starting with x10 = 10 and x20 = 15. This means the problem decomposes into five stages .let the quantization values of control be u = 0, 5, 10, 15, 20, 25, 30. The problem can be solved forward recursion equation (31). The model start with stage 1 (k = 0, 1) and goes forward to stage 2 , stage 3 , stage 4 and ending at stage 5.The computations are shown in table 2 for k =0, 1 The stages 2, 3,4 and 5 can be obtained in similar manner. Because of computations complexity, MATLAB Programming is used to get the solution. The optimal production rate at each stage appears in table 4. Fig. 3 gives the plot of optimal production as function of time. (3)

Table 2. Computation of cost at first stage k=0,1

Current states		Current control	Next states		Cost
X1	X2	U	X1	X2	J01
10	15	0	7	11	273.5
		5	8	11	242
		10	9	11	220.7
		15	10	11	209.6
		20	11	11	208.7
		25	12	11	218
		30	13	11	237.5

Table 3. Computation of cost at second stage k=2

Current states		Current control	Next states		Cost
X1	X2	U	X1	X2	J02
		0	4.6	7.8	450.5
		5	5.6	7.8	431.42
		10	6.6	7.8	417.14
7	11	15	7.6	7.8	408.06
		20	8.6	7.8	404.18
		25	9.6	7.8	405.5
		30	10.6	7.8	412.02
		0	5.4	7.8	413.8
		5	6.4	7.8	394.48
		10	7.4	7.8	380.36
8	11	15	8.4	7.8	371.44
		20	9.4	7.8	367.72
		25	10.4	7.8	369.2
		30	11.4	7.8	375.88
		0	6.2	7.8	387.02
		5	7.2	7.8	367.86
		10	8.2	7.8	353.9
9	11	15	9.2	7.8	345.14
		20	10.2	7.8	341.5
		25	11.2	7.8	343.2
		30	12.2	7.8	350.06
		0	7	7.8	370.58
		5	8	7.8	351.5
		10	9	7.8	337.7
10	11	15	10	7.8	329.18
		20	11	7.8	325.7
		25	12	7.8	327.5
		30	13	7.8	334.58
		0	7.8	7.8	364.468
		5	8.8	7.8	345.628
		10	9.8	7.8	331.988
11	11	15	10.8	7.8	323.548
		20	11.8	7.8	320.308
		25	12.8	7.8	322.268
		30	13.8	7.8	329.428
		0	8.6	7.8	368.68
		5	9.6	7.8	350
		10	10.6	7.8	336.52
12	11	15	11.6	7.8	328.24
		20	12.6	7.8	325.16
		25	13.6	7.8	328.28
		30	14.6	7.8	334.6
		0	9.4	7.8	383.22
		5	10.4	7.8	364.7
		10	11.4	7.8	351.38
13	11	15	12.4	7.8	343.26
		20	13.4	7.8	340.34
		25	14.4	7.8	342.62
		30	15.4	7.8	350.1

The stages 3, 4 and 5 can be obtained in similar manner. Because of table computations are complex, the MATLAB Program is used. The final solutions appear in table (4).Fig (3) gives the plot of optimal production u as function of time.

Table 4. Optimal production at each stage

k	x1	x2	Cost (jk)	u
0	10	15	110	25
1	12	11	108	25
2	13.6	7.8	109.28	25
3	14.88	5.24	112.15	20
4	14.904	3.192	113.04	20

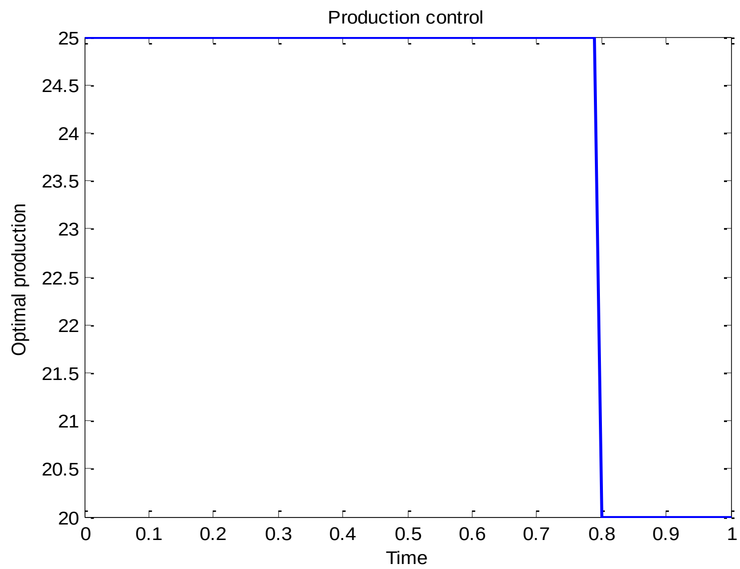

Figure 3. Optimal production

6. Conclusion

This research described the principle of optimality and the Hamilton Jacobi- Bellman (HJB) equation to obtain the optimal production rate for the given problem .Also after the model has been discretized the dynamic programming technique applied to obtain the optimal production rate .The solution of the first method is accurate and exact but the computations are complex even using computer. The solution of the second method is approximate and less complex than first method. The given model may be extended in many ways. For instance transportation cost, order cost, and shortage cost of both buyer and vendor. Also this model can extend to include multiple buyers, multiple vendors and multi-products.

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