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The Packing Spheres Constant for a Class of Separable Orlicz Function Spaces

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Abstract

Few results have been obtained on the packing spheres constant or exact formula for separable Orlicz function spaces (Yang, 2002, P.895-899, Ye, 1987, P.487-493). In this paper, by using the continuity of ideal space norm, we firstly proved that simple function class is dense in L_Φ^* function space. This is a necessary condition of interpolation theorem. Hence, the exact value of packing sphere for a class of sparable Orlicz function spaces (with two kinds of norm) is obtained. Secondly, for the space $L_\Phi^*[0,1]$ discussed in (Yang, 2002, P.895-899), we propose the following conjecture: the $L_\Phi^*[0,1]$ space is actually the $L_p[0,1]$ space, therefore, the results obtained there is actually the proved results in L_p space.

Keywords: N-function, Orlicz function space, Packing spheres constant, Quantitative index function

1. Preliminary material

It is well known that only finite number of sphere can be packed in a finite dimensional space if the spheres have the same radius but uncrossed, no matter how small is the radius. However, for infinite dimensional Banach space X, there exist a constant $\Lambda(X)$ such that infinite number of disjoint spheres can be packed in a unit sphere $B(X)$ if the radius less than $\Lambda(X)$. Whereas, only finite number of disjoint sphere can be packed in sphere $B(X)$ if the radius larger than $\Lambda(X)$. This constant is referred to as packing sphere constant. From the 50s last century, researchers begin to investigate the packing spheres problem in Banach space. In 1970, Kottman (1970) finally determined the range of packing sphere value $\Lambda(X)$ is $[\frac{1}{3}, \frac{1}{2}]$ for general normed linear spaces. In 1932, Orlicz introduced Orlicz space, from then people begin to study the packing sphere problem for this specific Banach space (Ye, 1987, P.487-493). In (Rao, 1997, P.235-251, Wang, 1990, P.197-203, Ye, 1991, P.203-216, Han, 2002, P.1155-1158, Wang, 1987, P.508-513), the authors investigated the packing sphere problem for a class of separable Orlicz function spaces, and obtained the exact packing spheres value; moreover, they also proposed their points of view for the space $L_\Phi^*[0,1]$ studied in (Yang, 2002, P.895-899).

In this paper, by using the continuity of ideal space norm (Matin, 1997), we firstly proved that simple function class is dense in L_Φ^* function space. This is a necessary condition of interpolation theorem. For the functions satisfying the following conditions:

(i). Their index function $F(u) = u\frac{\varphi(u)}{\Phi(u)}$ is not decrease;

(ii). $lim_{u\to 0}\frac{\varphi(u)}{u} = 0$, $lim_{u\to\infty}\frac{\varphi(u)}{u} = \infty$.

From the above two points, we prove that the interpolation function $\Phi_s(u)$ constructed from $\Phi(u)$ is N-function. By using this property, we can construct the interpolation inequality for subspace L_Φ^*, and hence obtain the exact value of packing spheres for Orlicz function subspace satisfying this properties. In the last section, we propose a conjecture for paper (Yang, 2002, P.895-899).

Subsequently, we outline some useful definitions and theorems.

Definition 1.1. The packing sphere constant $\Lambda(X)$ for Banach space is defined by

$$\Lambda(X) = sup\{r > 0 : \exists\{x_i\} \subset B(X), \ni \|x_i\| \le 1 - r, \|x_i - x_j\| \ge 2r, i, j = 1, 2, \ldots, i \ne j\},$$

where $B(X)$ is the unit ball in Banach space X.

Definition 1.2. The Kottman constant for infinite dimensional Banach space X is defined by

$$K(X) = sup\{inf\|x_i - x_j\| : \{x_i\} \subset S(X), i, j = 1, 2, \ldots, i \neq j\}$$

where $S(X)$ is the unit spheres in Banach space X.

Definition 1.3. Mapping Φ: $R \to R^+$ is referred to as N-function, if it satisfies the following conditions,

(i). $\Phi(x)$ is even continuious convex function;

(ii). $\Phi(x) = 0 \Leftrightarrow x = 0$;

(iii). $lim_{x\to 0} \frac{\Phi(x)}{x} = 0$, $lim_{x\to\infty} \frac{\Phi(x)}{x} = \infty$.

Property 1.4 (Wu, 1986). $\Phi(x)$ is a N-function if and only if there exist nonnegative real function defined in $[0, \infty)$ such that

(1). $\varphi(t)$ is right continous and nondecrease;

(2). $\varphi(t) = 0 \Leftrightarrow t = 0$;

(3). $\varphi(\infty) = \infty$ and $\Phi(x) = \int_0^{|x|} \varphi(t)dt$, where $\varphi(x)$ is called the right derivative function of $\Phi(x)$.

Property 1.5. Suppose $\Phi(x)$ is a N-function, $\phi(s)$ is the right inverse function of its right derivative function $\varphi(x)$, then $\Psi(y) = \int_0^{|y|} \phi(s)ds$ is the complementary N-function of $\Phi(x)$.

Property 1.6. $(Rao, 2002)$ Suppose $\Psi(y)$ is a N-function, and $\Psi(y)$ is the complementary N-function of $\Phi(x)$. If there exsit $k > 2$ and $u_0 > 0$ such that

$$\Phi(2u) \leq k\Phi(u) \quad when \quad u \geq u_0$$

then we call $\Phi \in \Delta_2(\infty)$.

Property 1.7. Suppose $\Psi(y)$ is a N-function, (Ω, Σ, μ) is finite complete nonnegative measurable space, then we can define measurable function set $E_\phi = \{u(t) : \forall a > 0, \rho_\Phi(au) < \infty\}$, $L_\phi = \{u(t) : \rho_\Phi(u) < \infty\}$, $L^*_\phi = \{u(t) : \exists a > 0, \rho_\Phi(au) < \infty\}$, where $\rho_\Phi(u) = \int_\Phi \Phi(u)du$.

Property 1.8. $E_\Phi \subset L_\Phi \subset L^*_\Phi$, and $E_\Phi = L_\Phi = L^*_\Phi$ whenever $\Phi \in \Delta_2(\infty)$.

Definition 1.9. For any $u \in L^*_\Phi$, define $\|u\|_\Phi = \sup_{\rho_\Psi(v)\leq 1} |\int_\Phi uvdu|$ and $\|u\|_{(\Phi)} = inf\{k > 0 : \rho_\Phi(\frac{u}{k})du \leq 1\}$, then $\|\cdot\|_\Phi$ and $\|\cdot\|_{(\Phi)}$ are all norms, they are named as Orlicz-norm and Luxemburg-norm respectively. Spaces $(L^*_\Phi, \|\cdot\|_\Phi)$ and $(L^*_\Phi, \|\cdot\|_{(\Phi)})$ are called as Orlica function spaces. We note down briefly as L^*_Φ and $L^*_{(\Phi)}$.

Remark. We note down them as L^*_Φ if no norm is related.

Definition 1.10. Suppose $\Phi(u) = \int_0^{|u|} \varphi(t)dt$ and $\Psi(u) = \int_0^{|u|} \phi(t)dt$ is a pair of complementary N-functions, then we can define the following quantitative index functions:

$A_\Phi = \lim_{t\to\infty} inf \frac{t\varphi(t)}{\Phi(t)}$, $\quad \bar{A}_\Phi = inf_{t>0} \frac{t\varphi(t)}{\Phi(t)}$,

$B_\Phi = \lim_{t\to\infty} sup \frac{t\varphi(t)}{\Phi(t)}$, $\quad \bar{B}_\Phi = sup_{t>0} \frac{t\varphi(t)}{\Phi(t)}$,

$\alpha_\Phi = \lim_{u\to\infty} inf \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}$, $\quad \beta_\Phi = \lim_{u\to\infty} sup \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}$,

$\alpha_\Phi(n) = \lim_{u\to\infty} inf \frac{\Phi^{-1}(u)}{\Phi^{-1}(nu)}$, $\quad \beta_\Phi(n) = \lim_{u\to\infty} sup \frac{\Phi^{-1}(u)}{\Phi^{-1}(nu)}$,

where $n \geq 2$.

Definition 1.11. Suppose (T, Σ, μ) is a measurable space, and S is the set of measurable function defined in this space. Suppose $X \subset S$ is a normed linear space of measurable function, then we call X a semi-ideal space, if $\forall x \in X, y \in S$ that satisfy $|y(s)| \leq |x(s)|$, we have

$$y \in S, \quad and \quad \|y\| \leq \|x\|$$

where $\|\cdot\|$ is a norm defined in X. If X is complete, then we call X an ideal space.

Definition 1.12 (Matin, 1997): For the norm $\|\cdot\|$ defined in the semi-ideal space, if $\|x\| \to 0$ whenever $x \to 0$, then we call norm $\|\cdot\|$ is continous.

Property 1.13 (Wang, 2002, P.9-13). Suppose $\Phi(u)$ and $\Psi(u)$ is a pair of complementary N-functions, then we have

$$n^{\frac{-1}{A_\Phi}} \leq \alpha_\Phi(n) \leq \beta_\Phi(n) \leq n^{\frac{-1}{B_\Phi}} \quad and \quad n\alpha_\Phi(n)\beta_\Psi(n) = 1 = \alpha_\Psi(n)\beta_\Phi(n)$$

for $n \geq 2$.

Property 1.14 (Ren, 1986, P.29-30). For any Orlicz-normed Orlicz function spaces $L_\Phi[0,1]^*$, its packing sphere estimation is given by

$$\frac{1}{1+\beta_\Phi(n)} \le \Lambda_\Phi \le \frac{1}{2}.$$

Property 1.15 (Wu, 1986). For any Orlicz function spaces $L_{(\Phi)}[0,1]^*$, its packing sphere estimation is given by

$$\frac{1}{1+2\alpha_\Phi} \le \Lambda_\Phi \le \frac{1}{2}.$$

Theorem 1.16 (Wu, 1986). The norm of eigenfuction $\chi_\Phi(x)$ defined in Orlicz space is given by

$$\|\chi_\Omega\|_\Phi = \Psi^{-1}(\frac{1}{\mu(\Omega)})\mu(\Omega) \quad and \quad \|\chi_\Omega\|_{(\Phi)} = \frac{1}{\Phi^{-1}(1/\mu(\Omega))}$$

where Ω is a finit nonnegative measurable set.

Theorem 1.17 (Wu, 1983). Suppose Φ and Ψ are complementary N-function, then $\Phi \in \Delta_x(\infty)$ if and only if $\forall l > 1, \exists 0 < \delta < 1\ \forall u \ge u_0$ we have

$$\Phi((1+\delta)u) \le \Phi(u).$$

Theorem 1.18 (Wu, 1986). Orlicz function space L^*_Φ is separable $\Leftrightarrow \Phi \in \Delta_2(\infty)$.

Theorem 1.19 (Yan, 2001, P.1-5). Suppose Φ_1 and Φ_2 are a pair of N-function, then $\Phi(u) = \Phi_1(\Phi_2(u))$ is also a N-function.

Property 1.20 (Yan, 2001, P.1-5). Suppose Φ_0 is a N-function, $\Phi(u) = \int_0^{|u|} \Phi_0(t)dt$, then $B_\Phi = lim_{u\to\infty} sup \frac{u\varphi(u)}{\Phi(u)} = B_{\Phi_0} + 1$.

Theorem 1.21 (Cleaver, 1976, P.325-335, Rao, 1966, P.543-568). Suppose Φ is a N-function, simple function class is dense in Orlicz function space L^*_Φ and $\Phi_0(u) = u^2$, $(u > 0)$. For $0 \le s \le 1$, let

$$\Phi_s^{-1} = (\Phi^{-1})^{1-s}(\Phi_0^{-1})^s$$

with Φ^{-1} is the inverse function of Φ (and correspondingly Φ_0^{-1}). Then for a group of positive numbers $c_1, \dots, c_n$ and any function group $u_1, \dots, u_n$ we have

$$\sum_{i,j=1}^{n} c_i c_j \|u_i - u_j\|_{\Phi_s}^{2/(2-s)} \le 2c^{2(1-s)(2-s)} \sum_{i=1}^{n} c_i \|u_i\|_{\Phi_s}^{2/(2-s)}$$

where $c = \max_{1\le i\le n}(1 - c_i)$.

Theorem 1.22. If X is an infinit dimensional Banach space, then we have

$$\Lambda(X) = \frac{K(X)}{2+K(X)} \quad and \quad \frac{1}{3} \le \Lambda(X) \le \frac{1}{2}.$$

2. The upper bound of packing sphere value and lemmas for interpolation inequalities.

Proposition 2.1. Suppose Φ is a N-function, for $n \ge 2$ we have

$$B_\Phi \le \infty \Leftrightarrow B_\Phi(n) < 1 \Leftrightarrow \Phi \in \Delta_2(\infty) \Leftrightarrow B_\Phi \le \infty$$

which is equivalent to Orlicz function space L^*_Φ is separable.

Proof. (1). When $B_\Phi < \infty$, from Property 1.13 we have $\beta_\Phi(n) \le n^{\frac{-1}{B}} \le 1$.

(2). When $B_\Phi(n) \ge \infty$, if $\Phi \bar{\in} \Delta_2(\infty)$ from Theorem 1.17 we have $\exists (v_i) \uparrow \infty$ such that

$$\Phi((1+\frac{1}{i})v_i) > n^i \Phi(v_i) > n\Phi(v_i)$$

with $(i \ge 1)$. Let $u_i = \Phi(v_i)$, then we have $\frac{1}{1+1/i} < \frac{\Phi^{-1}(u_i)}{\Phi^{-1}(nu_i)} < 1$. So $\lim_{i\to\infty} \frac{\Phi^{-1}(u_i)}{\Phi^{-1}(nu_i)} = 1$, which implies $\beta_\Phi(n) = 1$. Contradiction occurs.

(3). If $\Phi \in \Delta_2(\infty)$, i.e. $\exists\ 2 < k < \infty$, $\exists\ t_0 > 0$, $\forall\ t > t_0$ we have $\Phi(2t) \le k\Phi(t)$. Therefore,

$$t\varphi(t) \le \int_t^{2t} \varphi(u)du = \Phi(2t) \le t\Phi(t),$$

from which we have $\frac{t\Phi(t)}{\Phi(2t)} \leq k < \infty$, i.e. $B_\Phi = \lim_{t\to\infty} \frac{t\varphi(t)}{\Phi(2t)} < \infty$.

(4). From Theorem 1.18, we know Orlicz function space L^*_Φ is separable $\Leftrightarrow \Phi \in \Delta_2(\infty)$.

From the above (1),(2),(3),(4), the Proposition holds.

Proposition 2.2. $\Phi \in \Delta_2(\infty) \Leftrightarrow$ simple function class is dense in L^*_Φ.

Proof. Firstly we prove that if $\Phi \in \Delta_2(\infty)$, then the two norms $\|\cdot\|_\Phi$ and $\|\cdot\|_{(\Phi)}$ are continous in function space L^*_Φ. As two norms are equivalent, we just need to prove that $\|\cdot\|_{(\Phi)}$ is continous, a.e. $\forall u_n \in L^*_{(\Phi)}$, if $u_n(t) \downarrow 0$, a.e. $\|u_n\|_\Phi \to \infty$. Next, we will use reduction to absurdity. Otherwise, $\exists\delta > 0$, $\exists\ \{u_{n_k}\}_{k=1}^\infty$ such that $u_{n_k}(t) \downarrow 0$. a.e. $\|u_{n_k}\|_{(\Phi)} > \delta$ or $\|\frac{u_{n_k}}{\delta}\|_{(\Phi)} > 1$. Hence

$$\rho_\Phi(\frac{u_{n_k}}{\delta}) = \int_\Omega \Phi(\frac{u_{n_k}}{\delta})du > 1.$$

From $\Phi \in \Delta_2(\infty)$, we have $\frac{u_{n_k}}{\delta} \in L^*_\Phi$ when $k > 1$. Here $L^*_\Phi = E_\Phi$. From $u_n(k) \downarrow 0$ we have $\int_\Omega \Phi((\frac{u_{n_k}}{\delta})du \leq \int_\Omega \Phi((\frac{u_{n_k}}{\delta})du \leq \infty$ with $k \geq 1$, and $\lim_{k\to\infty} \Phi(\frac{u_{n_k}}{\delta}) = 0$. From L-convergent Theorem, we have $\lim_{k\to\infty} \int_\Omega \Phi(\frac{u_{n_k}}{\delta}) = 0$ which is contradict with $\rho_\Phi(\frac{u_{n_k}}{\delta}) = \int_\Omega \Phi(\frac{u_{n_k}}{\delta})du > 1$. Thus $\|u_n\|_{(\Phi)} \to 0$.

Subsequently, we will prove that if simple function class is dense in L^*_Φ, then L^*_Φ is seperable, thus from Proposition 2.1 we have $\Phi \in \Delta_2(\infty)$.

Otherwise, $\forall u(t) \in L^*_\Phi$, suppose $u(t) > 0$, a.e. take the trunction function of $u(t)$ as follows,

$$u_n(t) = \begin{cases} u(t), & u(t) \leq n, \\ 0, & u(t) > n. \end{cases}$$

Then we have $u_n(t) \uparrow u(t)$. For nonnegative measurable function $u_n(t)$, there exist simple function series $\{\varphi_{n,k}(t)\}_k$ such that $\varphi_{n,k_n}(t) \uparrow u_n(t)$, a.e. $(k \to \infty)$. Subsequently, we will show that there exist a subseries of $\varphi_{n,k_n}(t) \subset \varphi_{n,k}(t)$ such that $\varphi_{n,k_n}(t) \to u(t)$, a.e. $(k \to \infty)$. For fixed n, $\varphi_{n,k_n}(t) \uparrow u(t)$ a.e. $(k \to \infty)$. We know that there exist $k_n > 0$ such that for almost all t we have $0 < u_n(t) - \varphi_{n,k_n}(t) < \frac{1}{n}$. Therefore, we have

$$0 < u(t) - \varphi_{n,k_n}(t) = u(t) - u_n(t) + u_n(t) - \varphi_{n,k_n}(t) < u(t) - u_n(t) + \frac{1}{n}.$$

As $u_n(t) \uparrow u(t)$ a.e., so we have

$$\|u(t) - \varphi_{n,k_n}(t)\|_{(\Phi)} \to 0.$$

Hence, simple function class is dense in L^*_Φ. □

Lemma 2.3. $B_\Phi < \infty \Leftrightarrow$simple function class is dense in L^*_Φ.

Proof. From Proposition 2.1.7 and Proposition 2.1.12 we can derive the result.

Lemma 2.4. For function space $L^*_\Phi[0, 1]$ normalized by Orlicz norm, its packing sphere value can be estimated by following bound $\max\{\frac{1}{n+\alpha_\Phi(n)}, \frac{1}{1+\beta_\Phi^{-1}(n)}\} \leq \Lambda_\Phi$.

Proof. From Property 1.13 and Property 1.14, we can get the above result.

Lemma 2.4. For function space $L^*_\Phi[0, 1]$ normalized by Luxemburg norm, its packing sphere value can be bounded by $\max\{\frac{1}{n+\alpha_\Phi(n)}, \frac{1}{1+\beta_\Phi^{-1}(n)}, \frac{1}{1+\inf_{n\geq 1} \frac{\Phi^{-1}(nu)}{\Phi^{-1}(u)}}\} \leq \Lambda_{(\Phi)}$.

Proof. (1). $\frac{1}{n+\alpha_\Phi(n)} \leq \Lambda_{(\Phi)}$.

From property 1.15 we have $\frac{1}{2+\alpha_\Phi(2)} \leq \Lambda_{(\Phi)}$; also from the definition of $\alpha_\Phi(n)$ we have $n\alpha_\Phi(n) \geq 2\alpha_\Phi(2)$. So (1) holds.

(2). $\frac{1}{1+\beta_\Phi^{-1}(n)} \leq \Lambda_{(\Phi)}$. We just need to prove the case with $n = 2$.

From the definition of $\beta_{(\Phi)}$, we have $\exists v_n \uparrow \infty$, $\ni \frac{\Phi^{-1}(v_n)}{\Phi^{-1}(2v_n)} \to \beta_\Phi$, $(n \to \infty)$, i.e. $\forall\varepsilon > 0, \exists N \geq 1$ such that $\frac{\Phi^{-1}(v_n)}{\Phi^{-1}(2v_n)} > \beta_\Phi - \frac{\varepsilon}{2}$. Let $u_0 = v_n$, and divid interval $[0, \frac{1}{u_0}]$ into dis-intersect measurable sets $\{E_{k,i}\}_{k=1}^{2^i}$ with $(i \geq 1)$, and $E_{k,i} = [\frac{k-1}{2^i u_0}, \frac{k}{2^i u_0}]$ $1 \geq k \leq 2^i$. Thus we have $\cup_{k=1}^{2^i} E_{k,i} = [0, \frac{1}{u_0}) = E$. Let $R_i = \sum_1^{2^i} (-1)^{k+1}\chi_{E_{k,i}}$ and $u_i = \Phi^{-1}u_0 R_i$ with $i \geq 1$. From Theorem 1.16 we have $\|u_i\|_{(\Phi)} = \Phi^{-1}(u_0)\|\chi_E\|_{(\Phi)} = 1$ and $\|u_i - u_j\|_{(\Phi)} = 2\Phi^{-1}(u_0)\|\chi_E\|_{(\Phi)} = \frac{2\Phi^{-1}(u_0)}{\Phi^{-1}(2u_0)} > 2\beta_\Phi - \varepsilon$ $(i \neq j)$. So Kottman costant $K_\Phi \geq 2\beta_\Phi$. Thus the result is true.

(3). $\frac{1}{1+\inf_{n\geq 1} \frac{\Phi^{-1}(nu)}{\Phi^{-1}(u)}} \leq \Lambda_{(\Phi)}$, we just need to prove that $\frac{1}{1+\inf_{n\geq 1} \frac{\Phi^{-1}(2u)}{\Phi^{-1}(u)}} \leq \Lambda_{(\Phi)}$. For all $u \geq 1$, let $E_{k,i} = [\frac{k-1}{2^i u}, \frac{k}{2^i u}] \subset [0, 1]$, $1 \leq k \leq 2^i$ with $(i \geq 1)$. We also have $\cup_{k=1}^{2^i} E_{k,i} = [0, \frac{1}{u}) = E$. Let $R_i = \sum_1^{2^i} (-1)^{k+1}\chi_{E_{k,i}}$ and $u_i = \Phi^{-1}u_0 R_i$ with $i \geq 1$. From Theorem 1.16 we have $\|u_i\|_{(\Phi)} = 1$ and $\|u_i - u_j\|_{(\Phi)} = \frac{2\Phi^{-1}(u)}{\Phi^{-1}(2u)} > 2\beta_\Phi - \varepsilon$ $(i \neq j)$. So Kottman costant $K_\Phi \geq 2\sup_{u\geq 1} \frac{2\Phi^{-1}(u)}{\Phi^{-1}(2u)}$. Thus the result is true.

In order to construct the Orlicz function space interpolation inequality, we need to prove the following lemma.

Lemma 2.6. Suppose Φ is a N-function, and $\Phi \in \Delta_2(\infty)$, i.e. $B_\Phi = \lim_{u\to\infty} sup \frac{u\varphi(u)}{\Phi(u)} = p < \infty$ with $(u > 0)$. Let $F(u) = \frac{t\varphi(u)}{\Phi(u)}$, $\Phi_0(u) = u^2$ is monotonous nondecrease, then $\lim_{u\to 0} \frac{\varphi(u)}{u} = 0$ and $\lim_{u\to\infty} \frac{\varphi(u)}{u} = \infty$. For $1 \geq l < p < 2$, let $s = \frac{2(p-l)}{p(2-l)}$ (correspondingly For $2 < p < l < \infty$, let $s = \frac{2(l-p)}{p(l-2)}$). Construct function M such that $\Phi_s^{-1} = \Phi^{-1} = [M^{-1}]^{1-s}[\Phi_0^{-1}]^s$. Then interpolation function M is a N-function.

Proof. We just need to prove the case with $1 \leq p < 2$. From the formula of u we have $0 < s < 1$. Let

$$\omega = \Phi^{-1}(u), v = u^{\frac{-s}{2(1-s)}}[\Phi^{-1}(u)]^{\frac{1}{1-s}}. \tag{1}$$

Then from $\Phi^{-1} = [M^{-1}]^{1-s}[\Phi_0^{-1}]^s$, we have $M(v) = u = \Phi(\omega)$. If $\omega = R(v)$ is a N-function, then from Theorem 1.19 we have M is also a N-function. Subsequently, we will prove that $R(v)$ is a N-function. In order to prove this fact, we have to prove

(a). $\lim_{v\to 0} \frac{\varphi(v)}{v} = 0$ and

(b). $\lim_{v\to\infty} \frac{\varphi(v)}{v} = \infty$. Actually, from (1) we have

$$v^{1-s} = \frac{\omega}{[\Phi(\omega)]^{s/2}}, \quad \frac{R(v)}{v} = \frac{\omega}{v} = [\frac{\Phi(\omega)}{\omega^2}]^{\frac{s}{2(1-s)}}. \tag{2}$$

For (a), we have $v^{1-s} = \frac{\omega}{[\Phi(\omega)]^{s/2}} \to 0$ as $v \to 0$. From the property of N-function, we have either $\omega \to 0$ or $\omega \to \infty$. Subsequently, we will show that the later case is impossible. Otherwise, from L'Hospital criterion, we have

$$\lim_{u\to\infty} \frac{ln\varphi(u)}{lnu} = \lim_{u\to\infty} \frac{u\varphi(u)}{\Phi(u)} = p. \tag{3}$$

Hence, $\forall\varepsilon > 0$, $\exists u_0 > 0$, $\forall u > u_0$ we have $\frac{ln\varphi(u)}{lnu} < p + \varepsilon$. So $\Phi(u) < u^{p+\tau}$, $\frac{\Phi(u)}{u^{s/2}} < u^{\frac{2(p-2)}{p-l}+\varepsilon}$. As $\frac{p-2}{p-l} < 0$, so we have $\lim_{\omega\to\infty} \frac{\Phi(\omega)}{\omega^{s/2}} \leq \lim_{\omega\to\infty} \omega^{\frac{2(p-2)}{p-l}} = 0$, which results $v^{1-s} = \frac{\omega}{[\Phi(\omega)]^{s/2}} \to \infty$, contradiction occurs. Thus $\omega \to 0$ as $v \to 0$. From (2) we have $\lim_{v\to\infty} \frac{R(v)}{v} = \lim_{\omega\to 0}[\frac{\Phi(\omega)}{\omega^2}]^{\frac{s}{2(1-s)}} = \lim_{\omega\to 0}[\frac{\varphi(\omega)}{2\omega}]^{\frac{s}{2(1-s)}} = 0$.

For (b). we just need to prove that $\omega \to \infty$ as $v \to \infty$. Otherwise, $\omega \to 0$, let's consider $f(u) = \frac{\Phi(u)}{u^{2/S}}$ with $(u > 0)$. It's derivation is given by $f'(u) = u^{-\frac{2+s}{s}}[\varphi(u)u - \frac{2}{s}\Phi(u)]$. As $\lim_{u\to\infty} F(u) = p < \frac{s}{2}$ with $(1 \leq l < p < 2)$, and $F(u)$ is monotonously non-decrease. So we have $\lim_{u\to 0} F(u) \leq \lim_{u\to\infty} F(u) = p < \frac{s}{2}$. Therefore, $\varphi(u)u - \frac{2}{s}\Phi(u) < 0$. So $f(u)$ is strictly monotounously decrease with $(u > 0)$. Also we have $\lim_{u\to\infty} f(u) = \lim_{u\to\infty} \frac{\Phi(u)}{u^{2/S}} = 0$. Hence $\lim_{\omega\to 0} f(\omega) = \infty$ (as there are only two cases $\lim_{\omega\to 0} f(\omega) = \infty$ or 0). From the above results we have $v^{\frac{2(1-s)}{s}} \to 0$ as $(\omega \to 0)$. This is contradict with $v \to \infty$. Thus the theorem is true.

From Lemma 2.6 and Property 1.20, we have the following results

Corollary 2.7. The right derivation of N-function $\Phi(u)$ is $\varphi(u)$ is also a N-function. If $F(u) = \frac{u\varphi(u)}{\Phi(u)}$ is monotonously non-decrease, and $\lim_{u\to\infty} F(u) = p < \infty$, then $p \geq 2$ and the interpolation function M is a N-function.

3. Interpolation inequalities of Orlicz function space and packing sphere constant

We can construct the following interpolation inequalities.

Theorem 2.8. Suppose Φ is a N-function, and $\Phi \in \Delta_2(\infty)$, i.e. $B_\Phi = \lim_{u\to\infty} \sup \frac{u\varphi(u)}{\Phi(u)} = p < \infty$. ($u > 0$, $\varphi(t)$ is the right derivation of $\Phi(t)$). Let $F(u) = \frac{u\varphi(u)}{\Phi(u)}$, if $F(u)$ is monotonously non-decrease and $\lim_{u\to 0} \frac{\varphi(u)}{u} = 0$, $\lim_{u\to\infty} \frac{\varphi(u)}{u} = \infty$. Then for positive numbers $c_1, \ldots, c_n$ that satisfy $\sum_{i=1}^n c_i = 1$ and function set $u_1, \ldots, u_n \in L_\Phi^*$, we have

$$\sum_{i,j=1}^n c_i c_j \|u_i - u_j\|_\Phi^p \leq 2c^{2-p} \sum_{i=1}^n c_i \|u_i\|_\Phi^p, \qquad 1 \leq p < 2 \tag{4}$$

$$\sum_{i,j=1}^n c_i c_j \|u_i - u_j\|_\Phi^q \leq 2c^{2-q} \sum_{i=1}^n c_i \|u_i\|_\Phi^q, \qquad 2 \leq q < \infty \tag{5}$$

where $c = \max_{1\leq i\leq n}(1 - c_i)$, and p, q are conjugate number.

Proof. For any $1 \leq l < p < 2$, let $s = \frac{2(p-l)}{p(2-l)}$, then $0 < s < 1$.

Let $\Phi_0(u) = u^2$, we can construct function $M(u)$ such that $M^{-1}(u) = u^{-\frac{s}{2(1-s)}}[\frac{1}{\Phi(u)}]^{-1}$. Then $\Phi(u)^{-1} = [M^{-1}(u)]^{1-s}[\Phi_0(u)^{-1}]^s$. Let $\Phi^{-1} = [M^{-1}]^{1-s}[\Phi_0^{-1}]^s$, from Lemma 2.6 we know M is a N-function. From Theorem 1.21 we have $\sum_{i,j=1}^n c_i c_j \|u_i - u_j\|_\Phi^{\frac{s}{2-s}} \leq 2c^{\frac{2(1-s)}{2-s}} \sum_{i=1}^n c_i \|u_i\|_\Phi^{\frac{2}{2-s}}$. Put $s = \frac{2(p-l)}{p(2-l)}$ into the above formula and let $l \to 1$, then (4) holds.

For $2 < p < \infty$, we have $1 < q < 2$. So by the same procedure, we can prove that (5) is right.

Remarks. (i). For $1 \le p < 2$, the condition $\lim_{u\to\infty} F(u) = p$ can weakened as $\lim_{u\to\infty} \sup F(u) = p$. The results are still true.

(ii). The right derivation of N-function $\Phi(u)$, i.e. $\varphi(u)$ is also a N-function. $F(u) = \frac{u\varphi(u)}{\Phi(u)}$ is monotonously non-decrease. $\lim_{u\to\infty} F(u) = p < \infty$. The above results are also right.

(iii). From Proposition 2.2, under the condition $\Phi \in \Delta_2(\infty)$, the results of Theorem 1.21 can be generalized into the function space $L^*_{(\Phi)}$ normalized by Luxemburg norm.

The above results are true in the meaning of two kinds of norm. Therefore, we can compute the exact packing sphere value for the separable Orlicz function space $L^*_{(\Phi)}(\Omega)$, with $\Omega = [0, 1]$ or R^+.

Theorem 2.9. Suppose Φ is a N-function, let $F(u) = \frac{u\varphi(u)}{\Phi(u)}$, $B_\Phi = \lim_{u\to\infty} \sup F(u) = p < \infty$, ($u > 0$, $\varphi(t)$ is the right derivation of $\Phi(t)$). If $F(u)$ is monotonously non-decrease, and $\lim_{u\to 0} \frac{\varphi(u)}{u} = 0$, $\lim_{u\to\infty} \frac{\varphi(u)}{u} = \infty$. Then the packing sphere value of Orlicz function space $L^*_\Phi(\Omega)$ (with $\Omega = [0, 1]$) is given by

$$\Lambda = \begin{cases} \frac{1}{1+2^{1-\frac{1}{p}}} & 1 \le p \le 2, \\ \frac{1}{1+2^{\frac{1}{p}}} & 2 < p < \infty. \end{cases}$$

Proof. Based on Lemma 2.3 and Proposition 2.1, from $B_\Phi < \infty$ we have $L^*_\Phi(\Omega)$ is separable.

From Property 1.13 and Property 1.15 we have $\alpha_\Phi \le \beta_\Phi \le 2^{-\frac{1}{p}} \Rightarrow \frac{1}{1+2^{1-\frac{1}{p}}} \le \frac{1}{1+2\alpha_\Phi} \le \Lambda$.

We will discuss two cases:

(a). $1 \le p < 2$.

From Theorem 2.8, take $c_i = \frac{1}{n}$ $(i = 1, \ldots, n)$, then $\forall u_i(t) \in L^*_\Phi$ we have

$$\sum_{i,j=1}^{n} \frac{1}{n^2} \|u_i - u_j\|_\Phi^p \le 2(1 - \frac{1}{n})^{2-p} \frac{1}{n} \sum_{i=1}^{n} c_i \|u_i\|_\Phi^p. \tag{6}$$

Let us consider Kottman constant $K = \sup\{\inf \|u_i - u_j\| : \|u_i\|_\Phi = 1, i, j = 1, 2, \ldots, n, i \neq j\}$, and take $u_i(t) \in L^*_\Phi$ with $\|u_i(t)\| = 1$ and $\|u_i - u_j\| \ge r$, $(i, j = 1, 2, \ldots, n, i \neq j)$. From (6) we have $\sum_{i,j=1}^{n} \frac{1}{n^2} r^p \le 2(1 - \frac{1}{n})^{2-p} \frac{1}{n} n$ with $i \neq j$. From this inequality we have $r \le 2^{\frac{1}{n}} (1 - \frac{1}{n})^{\frac{1-p}{p}}$. Let $n \to \infty$ then we have $r \le 2^{\frac{1}{p}}$. Thus Kottman constant K satisfies $K \le 2^{\frac{1}{p}}$. Based on Theorem 1.22 we also have $\frac{1}{1+2^{1-\frac{1}{p}}} \ge \Lambda$. Therefore, $\frac{1}{1+2^{1-\frac{1}{p}}} = \Lambda$ holds. Subsequantly let consider the second case that

(b). $2 < p < \infty$.

As $A_\Phi = B_\Phi$, from Property 1.13 we have $\alpha_\Phi = \beta_\Phi = 2^{-\frac{1}{p}}$. Then from Lemma 2.4, Lemma 2.5 and Theorem 1.22 we can derive $\max\{2^{\frac{1}{p}}, 2^{1-\frac{1}{p}}\} \le K$. By repeating the prove process of (a), we have $K \le 2^{1-\frac{1}{p}}$. So the Kottman constant is $K = 2^{\frac{1}{p}}$. Therefore, $\frac{1}{1+2^{\frac{1}{p}}} = \Lambda$ holds.

Remarks.

(1). From the above process we can see that in the case of $1 \le p < 2$, the condition $\lim_{u\to\infty} F(u) = p$ can be weaken as $\lim_{u\to\infty} \sup F(u) = p$. The results are still true. Of course the results are also true for the stronger condition that the right derivation of $\Phi(u)$, i.e. $\varphi(u)$ is a N-function.

(2). For $\Omega = R^+$, we just need to change $B_\Phi = \lim_{u\to\infty} F(u) = p < \infty$ into $\bar{B}_\Phi = \sup_{u>0} F(u) = p < \infty$, all the resuts are ture. The proof is omitted here.

4. A Conjecture on the results in Reference (Yang, 2002, P.895-899)

The relative results in (Yang, 2002, P.895-899) are outlined in the following Definition 4.1 and Theorem 4.2.

Definition 4.1 (Yang, 2002, P.895-899): Suppose Φ is a N-function, if $\lim_{u\to\infty} \frac{\ln \Phi(u)}{\ln u} = p < \infty$, then Φ is called to satisfy $\Phi_\triangle$ condition, we note it briefly as $\Phi \in \Phi_\triangle$.

Theorem 4.2 (Yang, 2002, P.895-899). Suppose $\Phi \in \Phi_\triangle$ is a N-function, $\lim_{u\to\infty} \frac{\ln \Phi(u)}{\ln u} = p > 1$, then the packing sphere constant of Orlicz function space $L^*_\Phi[0, 1]$ is given by

$$\Lambda_\Phi = \begin{cases} \frac{1}{1+2^{1-\frac{1}{p}}} & 1 \le p \le 2, \\ \frac{1}{1+2^{\frac{1}{p}}} & 2 \ge p < \infty. \end{cases}$$

Obviously, the results are true for L_p is space (Rao, 2002). We conjecture that the function space $L^*_\Phi[0, 1]$ constitue by N-function Φ is actually $L_p[0, 1]$. Firstly, we have the following Theorem

Theorem 4.3 Suppose N-function Φ satisfies $1 < \lim_{u\to\infty} \frac{\ln\Phi(u)}{\ln u} = p < \infty$, then for the Orlicz function space

$$L_\Phi^* = \{u(t) : \exists a > 0, \int_0^1 \Phi(au(t))dt < \infty\}$$

constitute by Φ, we have

$$L_{p+\frac{1}{n}} \subset L_\Phi^* \subset L_{p-\frac{1}{n}},$$

for $n \geq 1$ and $L_p = \{u(t) : \int_0^1 |u(t)|^p dt < \infty\}$.

Proof. Firstly, we prove $L_\Phi \subset L_{p-\frac{1}{n}}$.

By L'Hospital criterion, we have $1 < \lim_{u\to\infty} \frac{\ln\Phi(u)}{\ln u} = p < \infty$, so $\lim_{u\to\infty} \frac{u\varphi(u)}{\Phi(u)} = p < \infty$. From Proposition 2.1 we have $\Phi \in \Delta_2(\infty)$. Therefore, from Property 1.8 we have $L_\Phi^* = L_\Phi = \{u(t) : \int_0^1 u(t))dt < \infty\}$. Also, for any $u(t) \in L_\Phi^*$, we have

$$\int_0^1 \Phi(u(t))dt < \infty. \tag{7}$$

From $\lim_{u\to\infty} \frac{\ln\Phi(u)}{\ln u} = p$, we know that for $n > 0$, there exist $0 < k_n < \infty$, $\forall t \in [0, 1]$, if $u(t) > k_n$, then $p+\frac{1}{n} > \frac{\ln\Phi(u)}{\ln u} > p-\frac{1}{n}$, i.e.

$$|u(t)|^{p-\frac{1}{n}} < \Phi(u(t)) < |u(t)|^{p+\frac{1}{n}}. \tag{8}$$

Let $E_1 = \{t \in [0, 1], |u(t)| > k_n\}$, $E_2 = \{t \in [0, 1], |u(t)| \leq k_n\}$, then from (7) and (8) we have $\int_0^1 |u(t)|^{p-\frac{1}{n}} dt < \int_{E_1} |u(t)|^{p-\frac{1}{n}} dt + \int_{E_2} |u(t)|^{p-\frac{1}{n}} dt \leq \int_{E_1} \Phi(u(t))dt + k_n^{p-\frac{1}{n}} u(E_2) \leq \int_0^1 \Phi(u(t))dt + k_n^{p-\frac{1}{n}} < \infty$, i.e. $\int_0^1 |u(t)|^{p-\frac{1}{n}} dt < \infty$ for all $u(t) \in L_{p-\frac{1}{n}}$. Next, we will show that $L_{p+\frac{1}{n}} \subset L_\Phi^*$. For all $u(t) \in L_{p+\frac{1}{n}}$, we have $\int_0^1 |u(t)|^{p+\frac{1}{n}} dt < \infty$. Therefore, from the right inequality of (3) and the monotonous property of Φ we have $\int_0^1 \Phi(u(t))dt = \int_{E_1} \Phi(u(t))dt + \int_{E_2} \Phi(u(t))dt \leq \int_{E_1} |u(t)|^{p+\frac{1}{n}} dt + \int_{E_2} \Phi(k_n)dt \leq \int_{E_1} |u(t)|^{p+\frac{1}{n}} dt + \Phi(k_n) < \infty$, so $u(t) \in L_\Phi^*$. We can make a further conjecture as follows,

Conjecture: The function space L_Φ^* used in Theorem 4.2 is L_p space.

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