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n-Stabilizing Bisets

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Abstract

One generalises the notion of stabilizing bisets from Bouc and Thévenaz (2012) to n-stabilizing bisets. This allows us to find new examples of stabilization for Roquette groups. We first investigate the idea of n-stabilizing bisets. We give a way to construct examples with the notion of idempotent bisets and n-expansive subgroups. Finally, for example, we look at Roquette groups and classify their n-stabilizing bisets.

Keywords: n-stabilizing biset, indecomposable module, Roquette group, n-expansive subgroup

1. Introduction

One purpose in representation theory is to try to describe representations of a finite group from a subgroup or subquotient of order as small as possible. This has been studied in Green's theory of vertices and sources and Harish-Chandra induction for reductive groups (see for instance Dipper & Du, 1997; Bouc, 1996). Another way to do so is to use stabilizing bisets introduced in Bouc and Thévenaz (2012). Indeed, let k be a field, G a finite group, U a (G,G)-biset and L a kG-module, where a (G,G)-biset U is a set which is both a left G-set and a right G-set such that $(gu)h = g(uh)$, for all $g \in G, h \in G$ and $u \in U$. Then U is said to stabilize L if $U(L) := kU \otimes_{kG} L$ is isomorphic to L. If we suppose that L is indecomposable, then one can show that U is of the form $\mathrm{Ind}_A^G \mathrm{Inf}_{A/B}^A \mathrm{Iso}_\phi \mathrm{Def}_{C/D}^C \mathrm{Res}_C^G$ for some subgroups A, B, C, D and an isomorphism $\phi \colon C/D \to A/B$. In particular, this means that L can be obtained by a representation of A/B. Theorem 7.3 of Bouc and Thévenaz (2012) proves the existence of proper stabilizing bisets for simple modules, except when the group is Roquette and the module is faithful. Moreover, it seems impossible to find stabilizing bisets for the majority of Roquette groups. In order to obtain new examples, one generalizes this notion to n-stabilizing bisets, i.e. bisets U such that $U(L) \cong nL$. This forces us to generalize the notions and results of Bouc and Thévenaz (2012).

It is shown in Bouc and Thévenaz (2012) that there is no stabilizing biset for Roquette p-groups. In this article, one shows that this is also true for Roquette groups with a cyclic Fitting subgroup. However, one finds non-trivial examples of n-stabilizing bisets for these groups.

We refer to Section 2 of Bouc and Thévenaz (2012), for the introduction to the notion of induction, inflation, deflation, restriction and isomorphism bisets and the corresponding notation. In particular, throughout this paper IndinfDefres stands for the biset $\mathrm{Ind}_A^G \mathrm{Inf}_{A/B}^A \mathrm{Iso}_\phi \mathrm{Def}_{C/D}^C \mathrm{Res}_C^G$.

We end this introduction with a short description of the organization of the paper, in Section 2 one finds some properties and characterizations of n-stabilization. In Section 3, one looks at n-stabilizing bisets and strong minimality. Then one looks at ways to obtain n-stabilizing bisets. We discuss one way with the help of n-idempotent bisets and characterize them completely. In Section 5 one generalises Section 6 of Bouc and Thévenaz (2012) by introducing the notion of n-expansive subgroups, this is another way to construct examples of n-stabilization. In this section, one also generalizes Section 3 of Bouc and Thévenaz (2012).

Finally, Section 6 is a study of examples. In particular, one treats Roquette p-groups, some simple groups and groups with a cyclic Fitting subgroup. One completely characterizes the n-stabilizing bisets for these examples.

2. *n*-Stabilizing Bisets

In this section one introduces the idea of n-stabilizing bisets. Using the notion of strongly minimality one could generalize Section 3 of Bouc and Thévenaz (2012). Theorem 12 is a generalization of Corollary 3.4 of Bouc and Thévenaz (2012) from the case of stabilization to that of n-stabilization.

Definition 1 (1) *A section* of a group G is a pair (A, B) of subgroups of G such that B is a normal subgroup of A.

(2) Two sections (A, B) and (C, D) of a group G are *linked* if

$$(A \cap C)B = A, (A \cap C)D = C \text{ and } A \cap D = C \cap B.$$

We next quote Lemma 2.5 of Bouc and Thévenaz (2012):

Proposition 2 (Generalized Mackey Formula) *Let (A, B) and (C, D) be two sections of a finite group G. Then there is the following decomposition as a disjoint union of bisets*

$$\mathrm{Defres}^G_{A/B} \,\mathrm{Indinf}^G_{C/D} \cong \bigcup_{x\in[A\backslash G/C]} \mathrm{Btf}(A, B, {}^xC, {}^xD) \,\mathrm{Conj}_x,$$

where

$$\mathrm{Btf}(A, B, C, D) := \mathrm{Indinf}^{A/B}_{(A\cap C)B/(A\cap D)B} \,\mathrm{Iso}_\psi \,\mathrm{Defres}^{C/D}_{(A\cap C)D/(B\cap D)D}$$

is the butterfly biset and ψ is the composite

$$(A \cap C)D/(B \cap C)D \to (A \cap C)/(B \cap C)(A \cap D) \to (A \cap C)B/(A \cap D)B.$$

Definition 3 Let U be a (G, G)-biset, let n be an integer and let L be a kG-module. Then U acts on L as follows

$$U(L) := kU \otimes_{kG} L.$$

$U(L)$ is a kG-module and we say that U is *applied* to L. The biset U is said to *n-stabilize* L if $U(L) \cong nL$. In the case $n = 1$, U is said to *stabilize* L.

Remark 4 We will focus our interest on indecomposable modules. If $U = \cup_{i=1}^r U_i$ is a decomposition of U as disjoint union of transitive bisets and if U n-stabilizes an indecomposable module L then

$$nL \cong U(L) \cong \bigoplus_{i=1}^{r} U_i(L).$$

Therefore by the Krull-Schmidt Theorem one has for every $1 \le i \le r$ that

$$U_i(L) \cong k_i L$$

for some integer k_i. For this reason, we shall assume that the biset U is transitive, hence, by Lemma 2.1 of Bouc and Thévenaz (2012), of the form

$$U = \mathrm{Indinf}^G_{A/B}\mathrm{Iso}_\phi\mathrm{Defres}^G_{C/D}.$$

Example 5 One refers to the last section of Bouc and Thévenaz (2012) for examples with $n = 1$. Here are examples with $n > 1$. Let k be an algebraically closed field of characteristic p and let P be a p-group. Let (A, B) be a section of P, where A and B are normal subgroups of P, and define L as $\mathrm{Ind}^P_A(k)$.

By Green's indecomposability theorem L is indecomposable and then it's easy to see that $U(L) = |P : A|L$ for $U := \mathrm{Indinf}^P_{A/B} \,\mathrm{Defres}^P_{A/B}$. Indeed, $(A, B) = ({}^gA, {}^gB)$ for all g in P because both A and B are normal, therefore using the Generalized Mackey Formula one has

$$\begin{aligned} U(L) &= U(\mathrm{Ind}^P_A(k)) \cong \bigoplus_{g\in[A\backslash P/A]} \mathrm{Indinf}^P_{A/B} \,\mathrm{Btf}(A, B, {}^gA, {}^gB)(k) \\ &\cong \bigoplus_{g\in[A\backslash P/A]} \mathrm{Indinf}^P_{A/B}(k) = |P : A|L. \end{aligned}$$

For example one can apply this to an extraspecial p-group P with $B := Z(P)$ and $A := N_P(\langle x\rangle)$, where x is a non-central element of order p; or also to P the dihedral group D_8 of order 8 with $A = \langle r\rangle$ and $B = \langle r^2\rangle$, where r is the rotation by an angle of $\pi/2$.

Proposition 6 *Let $U := \mathrm{Indinf}^G_{A/B}\mathrm{Iso}_\phi\mathrm{Defres}^G_{C/D}$ be an n-stabilizing biset for a module L. Let $M := \mathrm{Defres}^G_{C/D}(L)$. Then $n = \frac{|G:A|\dim M}{\dim L}$. In particular, one has $n \leq |G|$.*

Proof. By taking the dimension of $U(L) \cong nL$, one has

$$n\dim L = |G : A|\dim\mathrm{Defres}^G_{C/D}(L).$$

Therefore one has $n = \frac{|G:A|\dim M}{\dim L}$. As $\dim M \leq \dim L$, one has $n \leq |G : A| \leq |G|$. □

Definition 7 Let $U = \mathrm{Indinf}^G_{A/B}\mathrm{Iso}_\phi\mathrm{Defres}^G_{C/D}$ be a biset n-stabilizing a kG-module L.

(1) The biset U is said to be *minimal* if, for any transitive biset $U' = \mathrm{Indinf}^G_{A'/B'}\mathrm{Iso}_{\phi'}\mathrm{Defres}^G_{C'/D'}$ n-stabilizing L, we have $|C/D| \leq |C'/D'|$.

(2) The biset U is said to be *strongly minimal* if, for any transitive biset $U' = \mathrm{Indinf}^G_{A'/B'}\mathrm{Iso}_{\phi'}\mathrm{Defres}^G_{C'/D'}$ m-stabilizing L for some integer $m \geq 1$, we have $|C/D| \leq |C'/D'|$.

Lemma 8 *Let $U := \mathrm{Indinf}^G_{A/B}\mathrm{Iso}_\phi\mathrm{Defres}^G_{C/D}$ be an n-stabilizing biset for a non-trivial simple module L. If $|A/B| = p$, where p is the smallest prime dividing $|G|$, then U is strongly minimal.*

Proof. Suppose U is not strongly minimal. Let

$$U' = \mathrm{Indinf}^G_{A'/B'}\,\mathrm{Iso}_{\phi'}\,\mathrm{Defres}^G_{C'/D'}$$

be an m-stabilizing biset such that $|A'/B'| < |A/B| = p$. Then one has $1 = |A'/B'| = |C'/D'|$ and so U can be written as $\mathrm{Ind}^G_{A'}\,\mathrm{Inf}^{A'}_{1'}\,\mathrm{Iso}_{\phi'}\,\mathrm{Def}^{C'}_{1}\,\mathrm{Res}^G_{C'}$. The module $\mathrm{Inf}^{A'}_{1'}\,\mathrm{Iso}_{\phi'}\,\mathrm{Def}^{C'}_{1}\,\mathrm{Res}^G_{C'}(L)$ is isomorphic to copies of the trivial module k and thus $nL = \nu\,\mathrm{Ind}^G_{A'}(k)$ for some integer $\nu \geq 1$. But the trivial kG-module is always a submodule of $\mathrm{Ind}^G_{A'}(k)$, which contradicts the assumption that L is not the trivial module. Therefore such U' cannot exist and U is strongly minimal. □

Theorem 9 *Consider two transitive (G, G)-bisets*

$$U = \mathrm{Indinf}^G_{A/B}\,\mathrm{Iso}_\phi\,\mathrm{Defres}^G_{C/D} \text{ and } U' = \mathrm{Indinf}^G_{A'/B'}\,\mathrm{Iso}_{\phi'}\,\mathrm{Defres}^G_{C'/D'}\,.$$

Let L be an indecomposable kG-module such that $U(L) \cong nL$ and $U'(L) \cong mL$ for $n, m \in \mathbb{N}$. Let $M = \mathrm{Defres}^G_{C/D}(L)$ and suppose U is strongly minimal. Let g be an element of G. Then only two cases are possible:

(i) The module $\mathrm{Btf}(C', D', {}^gA, {}^gB)\,\mathrm{Conj}_g\,\mathrm{Iso}_\phi(M)$ is zero and the section $({}^gA, {}^gB)$ is not linked to $((C' \cap {}^gA)D', (C' \cap {}^gB)D')$.

(ii) The biset $\mathrm{Btf}(C', D', {}^gA, {}^gB)$ is reduced to $\mathrm{Indinf}^{C'/D'}_{(C'\cap {}^gA)D'/(C'\cap {}^gB)D'}\,\mathrm{Iso}_{\beta(g)}$, where $\beta(g)$ is the isomorphism corresponding to the linking between the sections $({}^gA, {}^gB)$ and $((C' \cap {}^gA)D', (C' \cap {}^gB)D')$.

Proof. Applying successively U and U' one obtains

$$\begin{aligned} U'(U(L)) &\cong \bigoplus_{g\in[C'\backslash G/A]} \mathrm{Indinf}^G_{A'/B'}\,\mathrm{Iso}_{\phi'}\,\mathrm{Btf}(C', D', {}^gA, {}^gB)\,\mathrm{Conj}_g\,\mathrm{Iso}_\phi(M) \\ &\cong mnL. \end{aligned}$$

Therefore, by the Krull-Schmidt theorem, one has, for all $g \in [C'\backslash G/A]$,

$$\mathrm{Indinf}^G_{A'/B'}\,\mathrm{Iso}_{\phi'}\,\mathrm{Btf}(C', D', {}^gA, {}^gB)\,\mathrm{Conj}_g\,\mathrm{Iso}_\phi(M) \cong k_g L.$$

In other words, one has a k_g-stabilizing biset for L, for a certain $k_g \in \mathbb{N}$. If $k_g \neq 0$ and because U is strongly minimal, the biset $\mathrm{Btf}(C', D', {}^gA, {}^gB)$ must be reduced to $\mathrm{Indinf}^{C'/D'}_{(C'\cap {}^gA)D'/(C'\cap {}^gB)D'}\,\mathrm{Iso}_{\beta(g)}$, where $\beta(g)$ is the isomorphism corresponding to the linking between the sections $({}^gA, {}^gB)$ and $((C' \cap {}^gA)D', (C' \cap {}^gB)D')$. Indeed, otherwise $\mathrm{Btf}(C', D', {}^gA, {}^gB)$ would go through a subsection of (A, B), which is a contradiction to the fact that U is strongly minimal. If $k_g = 0$, then the module $\mathrm{Btf}(C', D', {}^gA, {}^gB)\,\mathrm{Conj}_g\,\mathrm{Iso}_\phi(M)$ is zero, as the operation $\mathrm{Indinf}^G_{A'/B'}\,\mathrm{Iso}_{\phi'}$

cannot annihilate a module. For such g, the section $({}^gA, {}^gB)$ is not linked to $((C' \cap {}^gA)D', (C' \cap {}^gB)D')$ as otherwise the biset $\mathrm{Btf}(C', D', {}^gA, {}^gB)$ would have been reduced to

$$\mathrm{Indinf}_{(C'\cap {}^gA)D'/(C'\cap {}^gB)D'}^{C'/D'} \mathrm{Iso}_{\beta(g)},$$

but the latter does not annihilate $\mathrm{Conj}_g \mathrm{Iso}_\phi(M)$. □

Remark 10 Let M' be the module $\mathrm{Defres}^G_{C'/D'}(L)$. Using the same notation, we observe that one has

$$\begin{aligned} nM' &= \mathrm{Defres}^G_{C'/D'}(nL) \cong \mathrm{Defres}^G_{C'/D'} \mathrm{Indinf}^G_{A/B} \mathrm{Iso}_\phi \mathrm{Defres}^G_{C/D}(L) \\ &\cong \bigoplus_{g\in[C'\backslash G/A]} \mathrm{Btf}(C', D', {}^gA, {}^gB)\, \mathrm{Conj}_g \mathrm{Iso}_\phi(M) \\ &\cong \bigoplus_{\substack{g\in[C'\backslash G/A] \\ k_g\neq 0}} \mathrm{Indinf}^{C'/D'}_{(C'\cap {}^gA)D'/(C'\cap {}^gB)D'} \mathrm{Iso}_{\beta(g)} \mathrm{Conj}_g \mathrm{Iso}_\phi(M). \end{aligned}$$

Corollary 11 *Using the same notation and hypotheses as in Theorem 9 and suppose that both U and U' are strongly minimal. Let g be an element of G.*

(1) Only two cases are possible:

(i) The module $\mathrm{Btf}(C', D', {}^gA, {}^gB)\, \mathrm{Conj}_g \mathrm{Iso}_\phi(M)$ is zero and the section $({}^gA, {}^gB)$ is not linked to $((C' \cap {}^gA)D', (C' \cap {}^gB)D')$.

(ii) The biset $\mathrm{Btf}(C, D, {}^gA, {}^gB)$ is reduced to $\mathrm{Iso}_{\beta(g)}$, where $\beta(g)$ is the isomorphism corresponding to the linking between the sections $({}^gA, {}^gB)$ and (C', D').

Let $\mathscr{M}$ be the set of elements of $[C'\backslash G/A]$ such that we are in case (ii) and let d be the cardinality of $\mathscr{M}$.

(2) There exists an isomorphism between nM' and $\bigoplus_{g\in\mathscr{M}} \mathrm{Iso}_{\beta(g)} \mathrm{Conj}_g \mathrm{Iso}_\phi(M)$.

(3) One has the following equality $nm = dd'$, where d' is the number of double cosets ChA' such that

$$\mathrm{Indinf}^G_{A/B} \mathrm{Iso}_\phi \mathrm{Btf}(C, D, {}^hA', {}^hB')\, \mathrm{Conj}_h \mathrm{Iso}_{\phi'}(M') \neq \{0\}.$$

Proof. One uses the same argument as in the proof of Theorem 9 but suppose now that U' is strongly minimal. One deduces that $\mathrm{Btf}(C', D', {}^gA, {}^gB)$ is reduced to an isomorphism if $k_g \neq 0$, because U and U' are strongly minimal. This means that, if $k_g \neq 0$,

$$\mathrm{Indinf}^G_{A'/B'} \mathrm{Iso}_{\phi'} \mathrm{Iso}_{\beta(g)} \mathrm{Conj}_g \mathrm{Iso}_\phi(M) \cong k_g L.$$

In particular if $k_g \neq 0$, the dimension on the right hand side does not depend on g, because on the left of the isomorphism it does not. Therefore all non-zero k_g are equal. The isomorphism becomes

$$mnL \cong U'(U(L)) \cong \bigoplus_{\substack{g\in[C'\backslash G/A] \\ k_g\neq 0}} \mathrm{Indinf}^G_{A'/B'} \mathrm{Iso}_{\phi'} \mathrm{Iso}_{\beta(g)} \mathrm{Conj}_g \mathrm{Iso}_\phi(M).$$

By looking at the dimension in this equality, one obtains that

$$mn \dim L = dk_g \dim L$$

where d is the number of double cosets $C'gA$ such that $k_g \neq 0$.

Exchanging the roles of U and U' in the previous argument one has $mn = k'_h d'$, where d' is the number of double cosets ChA' such that $k'_h \neq 0$ and k'_h is such that $\mathrm{Indinf}^G_{A/B} \mathrm{Iso}_\phi \mathrm{Btf}(C, D, {}^hA', {}^hB')\, \mathrm{Conj}_h \mathrm{Iso}_{\phi'}(M')$ is isomorphic to $k'_h L$.

Furthermore, using Remark 10, one has

$$nM' = \bigoplus_{\substack{g\in[C'\backslash G/A] \\ k_g\neq 0}} \mathrm{Iso}_{\beta(g)} \mathrm{Conj}_g \mathrm{Iso}_\phi(M).$$

By looking at the dimension one obtains that $n \dim M' = d \dim M$. Exchanging the roles of U and U' in the previous argument one has $m \dim M = d' \dim M'$. Finally, using these two equations, one obtains that $mn = dd'$ and that $k_g = d'$ and $k'_h = d$, whenever k_g and k'_h are non-zero. □

Theorem 12 *Let* $U = \mathrm{Indinf}^G_{A/B}\,\mathrm{Iso}_\phi\,\mathrm{Defres}^G_{C/D}$ *be a strongly minimal n-stabilizing biset for an indecomposable kG-module L. Let* $M = \mathrm{Defres}^G_{C/D}(L)$*. Then, there exist n double cosets* CgA *such that*

(1) $\mathrm{Btf}(C, D, {}^gA, {}^gB)\,\mathrm{Conj}_g\,\mathrm{Iso}_\phi(M) \neq \{0\}$,

(2) the sections (C, D) *and* $({}^gA, {}^gB)$ *are linked,*

(3) the module M is invariant under $\beta(g)c_g\phi$*, where* $\beta(g)$ *is the isomorphism corresponding to the linking between the sections* (C, D) *and* $({}^gA, {}^gB)$*,*

(4) if $h \in G$ *does not belong to one of these cosets, the section* $({}^hA, {}^hB)$ *is not linked to* (C, D)*.*

Proof. Using part 3 of Corollary 11 with $U' = U$, $m = n$ and $d' = d$, one obtains that $n = d$. Therefore by definition of d, there exist exactly n double cosets CgA such that $\mathrm{Btf}(C, D, {}^gA, {}^gB)\,\mathrm{Conj}_g\,\mathrm{Iso}_\phi(M) \neq \{0\}$. For these double cosets one knows that $\mathrm{Btf}(C, D, {}^gA, {}^gB)$ is reduced to $\mathrm{Iso}_{\beta(g)}$, where $\beta(g)$ is the isomorphism corresponding to the linking between the sections $({}^gA, {}^gB)$ and (C, D). In particular, the sections (C, D) and $({}^gA, {}^gB)$ are linked. If $h \in G$ does not belong to one of these cosets, the section $({}^hA, {}^hB)$ cannot be linked to (C, D), otherwise we would have another non-zero module of the form $\mathrm{Btf}(C, D, {}^hA, {}^hB)\,\mathrm{Conj}_h\,\mathrm{Iso}_\phi(M)$.

Finally one proves (3). By the Krull-Schmidt Theorem we can write M as

$$a_1(M_{11} \oplus \cdots \oplus M_{1f(1)}) \oplus \cdots \oplus a_k(M_{k1} \oplus \cdots \oplus M_{kf(k)}),$$

where the M_{jr_j}'s are indecomposable and pairwise non-isomorphic, $f(j)$ is an integer depending on j and $a_j < a_{j+1}$ for all j. Using the second part of Corollary 11 and the fact that $n = d = |\mathscr{M}|$, one has

$$nM \cong \bigoplus_{g\in\mathscr{M}} \mathrm{Iso}_{\beta(g)}\,\mathrm{Conj}_g\,\mathrm{Iso}_\phi(M) = \bigoplus_{i=1}^{n} \mathrm{Iso}_{\beta(g_i)c_{g_i}\phi}(M),$$

for some $g_1, \ldots, g_n$ in $\mathscr{M}$. Using the decomposition of M one obtains

$$\begin{aligned} nM &\cong na_1(M_{11} \oplus \cdots \oplus M_{1f(1)}) \oplus \cdots \oplus na_k(M_{k1} \oplus \cdots \oplus M_{kf(k)}) \\ &\cong \bigoplus_{i=1}^{n} \mathrm{Iso}_{\beta(g_i)c_{g_i}\phi}(M) \\ &\cong \mathrm{Iso}_{\beta(g_1)c_{g_1}\phi}\left(a_1(M_{11} \oplus \cdots \oplus M_{1f(1)}) \oplus \cdots \oplus a_k(M_{k1} \oplus \cdots \oplus M_{kf(k)})\right) \\ &\oplus \mathrm{Iso}_{\beta(g_2)c_{g_2}\phi}\left(a_1(M_{11} \oplus \cdots \oplus M_{1f(1)}) \oplus \cdots \oplus a_k(M_{k1} \oplus \cdots \oplus M_{kf(k)})\right) \\ &\vdots \\ &\oplus \mathrm{Iso}_{\beta(g_n)c_{g_n}\phi}\left(a_1(M_{11} \oplus \cdots \oplus M_{1f(1)}) \oplus \cdots \oplus a_k(M_{k1} \oplus \cdots \oplus M_{kf(k)})\right). \end{aligned}$$

Note that M_{11} appears in the decomposition of $\mathrm{Iso}_{\beta(g_i)c_{g_i}\phi}(M)$ for all $i = 1, \ldots, n$. Indeed, $\mathrm{Iso}_{\beta(g_i)c_{g_i}\phi}$ sends an indecomposable module to an indecomposable module and if $\mathrm{Iso}_{\beta(g_i)c_{g_i}\phi}(M_{j_1r_{j_1}}) \cong \mathrm{Iso}_{\beta(g_i)c_{g_i}\phi}(M_{j_2r_{j_2}})$ then $M_{j_1r_{j_1}} \cong M_{j_2r_{j_2}}$ by applying $\mathrm{Iso}_{(\beta(g_i)c_{g_i}\phi)^{-1}}$ on both sides. As the M_{jr_j} are all pairwise non-isomorphic this means that there is the same number of indecomposable modules in M than in $\mathrm{Iso}_{\beta(g_i)c_{g_i}\phi}(M)$ and that the indecomposable modules in the decomposition are the same. Denote by m_i the multiplicity of M_{11} in $\mathrm{Iso}_{\beta(g_i)c_{g_i}\phi}(M)$, then $m_i \geq a_1$ for all $i = 1, \ldots, n$, as for all i the module M_{11} corresponds to $\mathrm{Iso}_{\beta(g_i)c_{g_i}\phi}(M_{j_ir_i})$ for some $M_{j_ir_i}$, which means $a_{j_i} \geq a_1$ for all i. Moreover, looking at the two decompositions of nM one has

$$\sum_{i=1}^{n} m_i = na_1$$

and so $m_i = a_1$ for all i. Applying this argument to all the modules M_{1r_1} one obtains that, for all i,

$$\mathrm{Iso}_{\beta(g_i)c_{g_i}\phi}\left(a_1(M_{11} \oplus \cdots \oplus M_{1f(1)})\right) \cong a_1(M_{11} \oplus \cdots \oplus M_{1f(1)}).$$

Using this result, the same argument proves that

$$\mathrm{Iso}_{\beta(g_i)c_{g_i}\phi}\left(a_2(M_{21} \oplus \cdots \oplus M_{2f(1)})\right) \cong a_2(M_{21} \oplus \cdots \oplus M_{2f(1)}).$$

Finally, continuing like this, one has, for all i

$$\begin{aligned}
\mathrm{Iso}_{\beta(g_i)c_{g_i}\phi}(M) &\cong \mathrm{Iso}_{\beta(g_i)c_{g_i}\phi}\left(a_1(M_{11}\oplus\cdots\oplus M_{1f(1)})\oplus\cdots\oplus a_k(M_{k1}\oplus\cdots\oplus M_{kf(k)})\right)\\
&\cong \mathrm{Iso}_{\beta(g_i)c_{g_i}\phi}\left(a_1(M_{11}\oplus\cdots\oplus M_{1f(1)})\right)\oplus\cdots\oplus \mathrm{Iso}_{\beta(g_i)c_{g_i}\phi}\left(a_k(M_{k1}\oplus\cdots\oplus M_{kf(k)})\right)\\
&\cong a_1(M_{11}\oplus\cdots\oplus M_{1f(1)})\oplus\cdots\oplus a_k(M_{k1}\oplus\cdots\oplus M_{kf(k)})\\
&\cong M.
\end{aligned}$$

□

The next three results are generalized forms of respectively Corollary 3.5, Proposition 4.3 and Proposition 8.5 of Bouc and Thévenaz (2012). We omit the proofs here as they are similar to the case $n = 1$. We refer to Monnard (2014) for the proofs.

Corollary 13 *Let $U = \mathrm{Indinf}^G_{A/B}\,\mathrm{Iso}_\phi\,\mathrm{Defres}^G_{C/D}$ be a strongly minimal n-stabilizing biset for an indecomposable kG-module L. Then there exists a section $(\tilde{A}, \tilde{B})$ linked to (C, D) by σ such that L is n-stabilized by*

$$\tilde{U} := \mathrm{Indinf}^G_{\tilde{A}/\tilde{B}}\,\mathrm{Iso}_\sigma\,\mathrm{Defres}^G_{C/D}\,.$$

Proposition 14 *Let $U := \mathrm{Indinf}^G_{A/B}\,\mathrm{Iso}_\phi\,\mathrm{Defres}^G_{C/D}$ be a minimal biset n-stabilizing a module L and let $M := \mathrm{Defres}^G_{C/D}(L)$. Then M is a faithful module.*

Proposition 15 *Let $U = \mathrm{Indinf}^G_{A/B}\,\mathrm{Iso}_\phi\,\mathrm{Defres}^G_{C/D}$ be a (G,G)-biset n-stabilizing a simple kG-module L and let $M = \mathrm{Iso}_\phi\,\mathrm{Defres}^G_{C/D}(L)$. If M is the trivial $k[A/B]$-module then $n = 1$, the kG-module L is trivial and $A = G$.*

Definition 16 Let G be a group and $B \leq G$. *The G-core* of B is the largest normal subgroup of G contained in B, that is, the intersection of all the G-conjugates of B.

Proposition 17 *Let G be a group and L a faithful kG-module such that L is n-stabilized by $\mathrm{Indinf}^G_{A/B}\,\mathrm{Iso}_\phi\,\mathrm{Defres}^G_{C/D}$. Then the G-core of B is trivial.*

Proof. Let M be the module $\mathrm{Iso}_\phi\,\mathrm{Defres}^G_{C/D}(L)$, so nL is $\mathrm{Indinf}^G_{A/B}(M)$, which has the following kernel, $\cap_{g\in G}\,{}^g\mathrm{Ker}(\mathrm{Inf}^A_{A/B}(M))$. Obviously B is contained in $\mathrm{Ker}(\mathrm{Inf}^A_{A/B}(M))$ and so $\cap_{g\in G}\,{}^gB$ is contained in $\cap_{g\in G}\,{}^g\mathrm{Ker}(\mathrm{Inf}^A_{A/B}(M))$. As nL is faithful, the latter is trivial and so too is the G-core of B. □

Proposition 18 *Let G be a group and L a faithful simple kG-module such that L is n-stabilized by $\mathrm{Indinf}^G_{A/B}\,\mathrm{Iso}_\phi\,\mathrm{Defres}^G_{C/D}$. Then the G-core of D is trivial.*

Proof. Let N be the G-core of D. Then

$$\mathrm{Defres}^{G/N}_{C/D}\,\mathrm{Def}^G_{G/N}(L) = \mathrm{Defres}^G_{C/D}(L) \neq \{0\}$$

and thus $\mathrm{Def}^G_{G/N}(L) \neq \{0\}$. But $\mathrm{Def}^G_{G/N}(L)$ is a quotient of L and N acts trivially on it; however, since L is simple and faithful one must have $N = \{1\}$. □

Proposition 19 *Let k be a field and let $U = \mathrm{Indinf}^G_{A/B}\,\mathrm{Iso}_\phi\,\mathrm{Defres}^G_{C/D}$ be a biset n-stabilizing a simple kG-module L. Then $n|A| \geq |N_G(D)|$ and in particular $n|A| \geq |C|$.*

Proof. By the proof of Proposition 8.1 of Bouc and Thévenaz (2012), one has

$$\dim L \leq |G : N_G(D)|\dim \mathrm{Defres}^G_{N_G(D)/D}(L).$$

By Lemma 6, one has $n\dim L = |G : A|\dim \mathrm{Defres}^G_{C/D}(L)$. Moreover, $\dim \mathrm{Defres}^G_{N_G(D)/D}(L)$ is equal to $\dim \mathrm{Defres}^G_{C/D}(L)$ as it only depends on the action of D on L. Therefore

$$\frac{|G : A|\dim \mathrm{Defres}^G_{N_G(D)/D}(L)}{n} \leq |G : N_G(D)|\dim \mathrm{Defres}^G_{N_G(D)/D}(L)$$

and the result follows. □

3. n-Stabilizing Bisets and Strong Minimality

In this section one treats the question of strong minimality and existence of strongly minimal n-stabilizing bisets.

Proposition 20 *Let G be a finite group, U be a n_U-stabilizing biset of the form $\mathrm{Indinf}^G_{A/B}\,V\,\mathrm{Iso}_\phi\,\mathrm{Defres}^G_{C/D}$ for a kG-module L and V a strongly minimal n_V-stabilizing biset for $M := \mathrm{Iso}_\phi\,\mathrm{Defres}^G_{C/D}(L)$. Moreover suppose that M is indecomposable. Then U is strongly minimal.*

Proof. Set $V = \mathrm{Indinf}^{A/B}_{H/J}\,\mathrm{Iso}_\sigma\,\mathrm{Defres}^{A/B}_{S/T}$ and let W be a n_W-stabilizing biset for L. Set $W = \mathrm{Indinf}^{G}_{A'/B'}\,\mathrm{Iso}_{\phi'}\,\mathrm{Defres}^{G}_{C'/D'}$. We have to show that $|H/J| \leq |A'/B'|$. Using these settings, one has

$$\begin{aligned}\mathrm{Iso}_\phi\,\mathrm{Defres}^G_{C/D}\,W\,\mathrm{Indinf}^G_{A/B}\,V(M) &\cong \mathrm{Iso}_\phi\,\mathrm{Defres}^G_{C/D}\,W(n_U L)\\ &\cong n_U n_W\,\mathrm{Iso}_\phi\,\mathrm{Defres}^G_{C/D}(L)\\ &\cong n_U n_W M.\end{aligned}$$

Using the Generalized Mackey Formula, the left hand side becomes

$$\oplus_{g,h}\,\mathrm{Iso}_\phi\,\mathrm{Btf}(C, D, {}^gA', {}^gB')\,\mathrm{Conj}_g\,\mathrm{Iso}_{\phi'}\,\mathrm{Btf}(C', D', {}^hH, {}^hJ)\,\mathrm{Conj}_h\,\mathrm{Iso}_\sigma\,\mathrm{Defres}^{A/B}_{S/T}(M),$$

where the sum is taken over $g \in [C\backslash G/A']$ and $h \in [C'\backslash G/H]$. Because M is indecomposable, this implies that for each summand there exists a certain $k_{g,h}$ such that

$$\mathrm{Iso}_\phi\,\mathrm{Btf}(C, D, {}^gA', {}^gB')\,\mathrm{Conj}_g\,\mathrm{Iso}_{\phi'}\,\mathrm{Btf}(C', D', {}^hH, {}^hJ)\,\mathrm{Conj}_h\,\mathrm{Iso}_\sigma\,\mathrm{Defres}^{A/B}_{S/T}(M) \cong k_{g,h}M.$$

Note that $k_{g,h} \neq 0$ for at least one pair (g, h). The biset V is strongly minimal therefore the biset $\mathrm{Btf}(C', D', {}^hH, {}^hJ)$ has to be reduced to

$$\mathrm{Indinf}^{C'/D'}_{(C'\cap {}^hH)D'/(C'\cap {}^hJ)D'}\,\mathrm{Iso}_\psi,$$

when $k_{g,h} \neq 0$, which means that $({}^hH, {}^hJ)$ is linked to a subsection of (C', D'). In particular $|H/J| \leq |C'/D'| = |A'/B'|$, which proves the strong minimality of U. □

Proposition 21 *Let G be a finite group and let $U := \mathrm{Indinf}^G_{A/B}\,V\,\mathrm{Iso}_\phi\,\mathrm{Defres}^G_{C/D}$ be a strongly minimal n_U-stabilizing biset for an indecomposable kG-module L, where V n_V-stabilizes $M := \mathrm{Iso}_\phi\,\mathrm{Defres}^G_{C/D}(L)$. Then V is strongly minimal.*

Proof. Set $V = \mathrm{Indinf}^{A/B}_{H/J}\,\mathrm{Iso}_\sigma\,\mathrm{Defres}^{A/B}_{S/T}$ and let W be a n_W-stabilizing biset for M. Set $W = \mathrm{Indinf}^{A/B}_{H'/J'}\,\mathrm{Iso}_{\sigma'}\,\mathrm{Defres}^{A/B}_{S'/T'}$, then

$$\begin{aligned}\mathrm{Indinf}^G_{A/B}\,VW\,\mathrm{Iso}_\phi\,\mathrm{Defres}^G_{C/D}(L) &\cong \mathrm{Indinf}^G_{A/B}\,VW(M)\\ &\cong n_W\,\mathrm{Indinf}^G_{A/B}\,V(M)\\ &\cong n_W n_U L.\end{aligned}$$

Using Mackey's Formula, the first term on the left becomes

$$\oplus_g\,\mathrm{Indinf}^G_{H/J}\,\mathrm{Iso}_\sigma\,\mathrm{Btf}(S, T, {}^gH', {}^gJ')\,\mathrm{Conj}_g\,\mathrm{Iso}_{\sigma'}\,\mathrm{Defres}^{A/B}_{S'/T'}\,\mathrm{Iso}_\phi\,\mathrm{Defres}^G_{C/D}(L) \cong n_U n_W L.$$

Because L is indecomposable, this implies that for each summand there exists a certain k_g such that

$$\mathrm{Indinf}^G_{H/J}\,\mathrm{Iso}_\sigma\,\mathrm{Btf}(S, T, {}^gH', {}^gJ')\,\mathrm{Conj}_g\,\mathrm{Iso}_{\sigma'}\,\mathrm{Defres}^{A/B}_{S'/T'}\,\mathrm{Iso}_\phi\,\mathrm{Defres}^G_{C/D}(L) \cong k_g L,$$

and $k_g \neq 0$ for at least one g. By strongly minimality of U the biset $\mathrm{Btf}(S, T, {}^gH', {}^gJ')$ must, at least, be reduced to $\mathrm{Iso}_\psi\,\mathrm{Defres}^{{}^gH'/{}^gJ'}_{(S\cap {}^gH'){}^gJ'/(T\cap {}^gH'){}^gJ'}$, which means that (S, T) is linked to a subsection of $({}^gH', {}^gJ')$. In particular $|H/J| = |S/T| \leq |H'/J'|$, which proves the strongly minimality of V. □

Proposition 22 *Let G be a finite group, $U := \mathrm{Indinf}^G_{A/B}\,\mathrm{Iso}_\phi\,\mathrm{Defres}^G_{C/D}$ and L a kG-module n_U-stabilized by U. Suppose $M := \mathrm{Iso}_\phi\,\mathrm{Defres}^G_{C/D}(L)$ is indecomposable. Then there exists a biset V, n_V-stabilizing M, such that $W := \mathrm{Indinf}^G_{A/B}\,V\,\mathrm{Iso}_\phi\,\mathrm{Defres}^G_{C/D}$ is strongly minimal for L. Moreover V is strongly minimal for M.*

Proof. One proves this by induction hypothesis to $|G|$. If $|G| = 1$, then the trivial biset is strongly minimal. Now suppose the statement is true for groups of order less than $|G|$. If U is strongly minimal then $V = \mathrm{Id}$. Suppose U is not strongly minimal. Moreover suppose $|A/B| < |G|$ and apply the induction on the indecomposable module M with the identity as stabilizing biset. So one obtains a strongly minimal biset V such that $V(M) \cong n_V M$. By Proposition 20 the biset

$$W := \mathrm{Indinf}^G_{A/B}\,V\,\mathrm{Iso}_\phi\,\mathrm{Defres}^G_{C/D}$$

is strongly minimal for L.

It is left to consider the case $|A/B| = |G|$. This implies that $U = \mathrm{Iso}_\phi$, but U is not strongly minimal by assumption, therefore there exists a proper biset V_1, i.e. not reduced to an isomorphism, such that $V_1(L) \cong n_{V_1} L$. Replacing U by V_1 in the argument of the first case, one obtains a strongly minimal n_V-stabilizing biset V for the module L and therefore $W = V \,\mathrm{Iso}_\phi$ is strongly minimal for L. □

Remark 23 Note that W is a $n_U n_V$-stabilizing biset for L and not simply a n_U-stabilizing biset.

Proposition 24 *Let G be a finite group, $U := \mathrm{Indinf}_{A/B}^G \mathrm{Iso}_\phi \mathrm{Defres}_{C/D}^G$ and L an indecomposable kG-module stabilized by U. Then there exists a biset V such that $U' := \mathrm{Indinf}_{A/B}^G V \,\mathrm{Iso}_\phi \mathrm{Defres}_{C/D}^G$ is minimal for L. Moreover V is minimal for $M := \mathrm{Iso}_\phi \mathrm{Defres}_{C/D}^G(L)$.*

Proof. Following exactly the proof of Proposition 22 with $n_U = 1$, the fact that M is indecomposable because $\mathrm{Indinf}_{A/B}^G(M) \cong L$ is and the notion of minimality instead of strongly minimality, one obtains the result. □

Proposition 25 *Let L be a faithful simple kG-module. Suppose that whenever $U(L) \cong L$ for U a minimal biset then U is reduced to an isomorphism. Then, for an arbitrary biset $\mathrm{Indinf}_{A/B}^G \mathrm{Iso}_\phi \mathrm{Defres}_{C/D}^G$ stabilizing L one has $(A, B) = (C, D) = (G, 1)$.*

Proof. By proposition 24 there exist subgroups H and J with J a normal subgroup of H with $B \le H \le A$ and $B \le J \le A$ such that

$$\mathrm{Indinf}_{A/B}^G \mathrm{Indinf}_{H/J}^{A/B} \mathrm{Iso}_\sigma \mathrm{Defres}_{S/T}^{A/B} \mathrm{Iso}_\phi \mathrm{Defres}_{C/D}^G \cong \mathrm{Indinf}_{H/J}^G \mathrm{Iso}_{\sigma\phi} \mathrm{Defres}_{\phi^{-1}(S/T)}^G$$

is minimal for L. As a minimal stabilizing biset one has, by hypothesis, that $J = \{1\}$ and $H = G$ and so in particular $B = \{1\}$ and $A = G$. □

4. *n*-Idempotent Bisets

This generalizes section 5 of Bouc and Thévenaz (2012) on idempotent bisets to n-idempotent bisets for $n > 1$. One gives here a complete classification of such bisets.

Definition 26 Let U be a (G, G)-biset, then U is *an n-idempotent biset* if $U^2 \cong nU$.

Theorem 27 *Let $U = \mathrm{Indinf}_{A/B}^G \mathrm{Iso}_\phi \mathrm{Defres}_{C/D}^G$ be a (G, G)-biset. Then $U^2 \cong nU$ if and only if the following three conditions hold:*

1) There are n (C, A)-double cosets.

2) The sections (C, D) and $({}^gA, {}^gB)$ are linked for all g.

3) For every $g \in G$, there exist $x \in N_G({}^gA, {}^gB)$ and $y \in N_G(C, D)$ such that

$$\phi\beta(g)^{-1} \mathrm{Conj}_g \phi = \mathrm{Conj}_x \phi \,\mathrm{Conj}_y^{-1},$$

where $\beta(g)\colon C/D \to {}^gA/{}^gB$ is the isomorphism induced by the linking.

Proof. The idea of the proof is similar to that of Theorem 5.1 of Bouc and Thévenaz (2012). We refer to Theorem 2.26 of Monnard (2014) for more details. □

As Proposition 5.4 of Bouc and Thévenaz (2012), one obtains the following generalized result:

Proposition 28 *Let U be an n-idempotent (G, G)-biset. For any kG-module L', the kG-module $L := U(L')$ is n-stabilized by U.*

Remark 29 Note that in general L need not be indecomposable.

Example 30 (1) An example can be found in A_5. Let U be $\mathrm{Indinf}_{D_{10}/C_5}^{A_5} \mathrm{Defres}_{D_{10}/C_5}^{A_5}$, where D_{10} denotes $\langle(1, 2, 3, 4, 5), (2, 5)(3, 4)\rangle$ a dihedral group of order 10, and $C_5 = \langle(1, 2, 3, 4, 5)\rangle$, a cyclic group of order 5. An easy calculation, which can be made by GAP (see The GAP Group, 2014) gives two double (D_{10}, D_{10})-cosets in A_5 and the section (D_{10}, C_5) is linked via conjugation to its conjugate. By taking $x = 1 = y$ in the last condition of Theorem 27 one can see that U is a 2-idempotent biset.

(2) If A and B are normal subgroups of G and $U := \mathrm{Indinf}_{A/B}^G \mathrm{Defres}_{A/B}^G$, then U is $|G : A|$-idempotent. Indeed, one has $|G : A|$ (A, A)-double cosets. By normality the sections are trivially linked and by taking $x = y = 1$ the third condition is also fulfilled. This is the case, in particular, of Example 5.

5. n-Expansivity

In this section one introduces a type of subgroup called n-expansive, which will be a useful notion to find n-stabilizing bisets. In particular, Theorem 12 is a generalization of Corollary 3.4 of Bouc and Thévenaz (2012) from the stabilization case to that of n-stabilization.

Definition 31 Let n be an integer. A subgroup T of a group G is called (S,n)-*expansive relatively to* (A,B) if

(1) The pairs (A,B) and (S,T) are sections of G.

(2) The sections (A,B) and (S,T) are linked via ϕ.

(3) The composition of ϕ with the conjugation map, $\phi \circ c_g$, links the sections (A^g, B^g) and (S,T) for exactly n elements g in $[A\backslash G/S]$. For the other elements g in $[A\backslash G/S]$ the S-core of the subgroup $(B^g \cap S)T$ contains T properly.

Remark 32

(1) One will mainly use this notion with $S = N_G(T)$ and $(A,B) = (S,T)$. In this case the subgroup T is simply called n-expansive. If moreover $n = 1$ one says that T is expansive as defined in Chapter 6 of Bouc and Thévenaz (2012).

(2) By assumption (A,B) is linked to (S,T) and therefore the first part of condition (iv) is fulfilled at least for $g = 1$ in $[A\backslash G/S]$.

Lemma 33 *Let (A,B) be a section of a finite group G. Let M be a faithful simple $k[A/B]$-module. Then* $\mathrm{Def}^{A/B}_{A/N}(M) = \{0\}$ *for any non-trivial normal subgroup N/B of A/B.*

Proof. Since M is simple and faithful, the largest quotient of M with trivial action of N/B must be zero and therefore $\mathrm{Def}^{A/B}_{A/N}(M) = \{0\}$. □

Proposition 34 *Let T be (S,n)-expansive relatively to (A,B). Let ϕ be the link between (A,B) and (S,T). Suppose that M is a faithful simple $k[A/B]$-module. Let* $L := \mathrm{Indinf}^G_{S/T}\,\mathrm{Iso}_\phi(M)$. *Then,*

(i) $\mathrm{Defres}^G_{A/B}(L) \cong nM$.

(ii) The biset $U := \mathrm{Indinf}^G_{S/T}\,\mathrm{Iso}_\phi\,\mathrm{Defres}^G_{A/B}$ *n-stabilizes L.*

Proof. We decompose $\mathrm{Defres}^G_{A/B}(L)$ using the Generalized Mackey Formula, see Proposition 2,

$$\begin{aligned}\mathrm{Defres}^G_{A/B}(L) &= \mathrm{Defres}^G_{A/B}\,\mathrm{Indinf}^G_{S/T}\,\mathrm{Iso}_\phi(M)\\ &\cong \bigoplus_{x\in[A\backslash G/S]} \mathrm{Btf}(A,B,{}^xS,{}^xT)\,\mathrm{Conj}_x\,\mathrm{Iso}_\phi(M)\\ &\cong \bigoplus_{x\in[A\backslash G/S]} \mathrm{Conj}_x\,\mathrm{Btf}(A^x,B^x,S,T)\,\mathrm{Iso}_\phi(M).\end{aligned}$$

Now by definition one has

$$\mathrm{Btf}(A^x,B^x,S,T) = \mathrm{Indinf}^{A^x/B^x}_{(A^x\cap S)B^x/(A^x\cap T)B^x}\,\mathrm{Iso}_\psi\,\mathrm{Defres}^{S/T}_{(A^x\cap S)T/(B^x\cap S)T}$$

Since T is (S,n)-expansive the S-core N_x of the subgroup $(B^x\cap S)T$ contains T properly, except for exactly n elements x in $[A\backslash G/S]$. In other words, except for these n elements, N_x/T is a non-trivial subgroup of S/T contained in $(B^x\cap S)T$. As

$$\mathrm{Defres}^{S/T}_{(A^x\cap S)T/(B^x\cap S)T} = \mathrm{Defres}^{S/N_x}_{(A^x\cap S)T/(B^x\cap S)T}\,\mathrm{Def}^{S/T}_{S/N_x}$$

one has, by Lemma 33 applied to $\mathrm{Iso}_\phi(M)$, that

$$\mathrm{Defres}^{S/T}_{(A^x\cap S)T/(B^x\cap S)T}\,\mathrm{Iso}_\phi(M) = \{0\}$$

for all x except n elements. Theses n elements have the property that the composition of ϕ with the conjugation map links the sections (A^x,B^x) and (S,T), which implies that

$$\mathrm{Conj}_x\,\mathrm{Btf}(A^x,B^x,S,T)\,\mathrm{Iso}_\phi(M) \cong M.$$

As this occurs exactly n times, one concludes that

$$\mathrm{Defres}^G_{A/B}(L) \cong nM.$$

The second claim in this theorem follows from the first and the definition of L. □

Example 35 Here is an example of n-expansivity in S_6.

(i) First, consider $T := \langle(1,2,3)\rangle \times \langle(4,5,6),(5,6)\rangle \cong C_3 \times S_3$. Its normalizer S is $T \rtimes \langle(2,3)(4,6)\rangle$. There are four (S,S)-double cosets in S_6. Here is a list of representatives:

$$\{\mathrm{id}, (3,4), (2,4)(3,5), (1,4)(2,5)(3,6)\}.$$

The first two elements satisfy the first part of (iv) in Definition 31 and the last two elements satisfy the second part of that definition. Therefore T is an example of a 2-expansive subgroup in S_6. Setting M to be the sign representation of S/T one obtains an example of a 2-stabilizing biset. However the module $L := \mathrm{Indinf}^{S_6}_{S/T}(M)$ is not an indecomposable module for S_6 over $\mathbb{C}$.

(ii) Now consider $T := \langle(5,6)\rangle \times \langle(1,2)(3,4),(1,3)(2,4),(2,3,4)\rangle \cong C_2 \times A_4$. Its normalizer S is $T \rtimes \langle(3,4)\rangle$. There are three (S,S)-double cosets in S_6. Here is a list of representatives:

$$\{\mathrm{id}, (4,5), (3,5)(4,6)\}.$$

The second one satisfies the second part of Definition 31 and the two others the first part. Therefore T is another example of a 2-expansive subgroup in S_6. Again, setting M to be the sign representation of S/T one obtains an example of a 2-stabilizing biset, but the module $L := \mathrm{Indinf}^{S_6}_{S/T}(M)$ is not indecomposable over $\mathbb{C}$.

6. n-Stabilizing Bisets and Roquette Groups

In Bouc and Thévenaz (2012), Theorem 7.3 states that if k is a field, G a finite group and L a simple kG-module, then there exists an expansive subgroup T of G such that

$$\mathrm{Indinf}^G_{N_G(T)/T}\,\mathrm{Defres}^G_{N_G(T)/T}(L) \cong L.$$

This theorem proves the existence of stabilizing bisets for simple modules. However, it is possible that this biset is trivial, i.e. it is reduced to an isomorphism. The proof of the theorem shows that this could only be the case if G is Roquette and L is faithful. Recall that a finite group G is said to be a *Roquette group* if all its normal abelian subgroups are cyclic.

This raises the question of proving the existence, or non-existence, of stabilizing bisets for Roquette groups and more generally of n-stabilizing bisets. The goal of this section is to study n-stabilization for Roquette groups. Let G be a Roquette group and denote by $F(G)$ the Fitting subgroup of G, which is the product of the normal subgroups $O_p(G)$ for all primes p. As G is Roquette each $O_p(G)$ does not contain a characteristic abelian subgroup that is not cyclic. By Theorem 4.9 of Gorenstein (1980), such groups are known. More precisely, each subgroup $O_p(G)$ is the central product of an extraspecial group with a Roquette p-group. Roquette p-groups are known, see Chapter 5, Section 4 of Gorenstein (1980), so one starts our study with these groups. Then, one continues with groups with a cyclic Fitting subgroup, corresponding to cyclic $O_p(G)$ for every prime p.

6.1 Roquette p-Groups

The case of Roquette p-groups has already been studied in Bouc and Thévenaz (2012). It is shown that if U is a stabilizing biset for a faithful simple module, then U has to be reduced to an isomorphism, see Theorem 9.3. One will discuss the case of n-stabilizing bisets for $n > 1$.

Theorem 36 *Let p be a prime number and let P be a Roquette p-group of order p^{k+1}. Let $U := \mathrm{Indinf}^G_{A/B}\,\mathrm{Iso}_\phi\,\mathrm{Defres}^G_{C/D}$ be a n-stabilizing biset for L where L is a simple faithful $\mathbb{C}P$-module. Then one has $B = D = 1$.*

Proof. First note that by 17 and 18, the P-cores of B and D are trivial. In particular, $B \cap Z(P)$ and $D \cap Z(P)$ have to be trivial, as these intersections are contained in the P-core of, respectively, B and D. It follows from Lemma 9.1 of Bouc and Thévenaz (2012) that B and D are trivial, except possibly if $p = 2$, P is dihedral or semi-dihedral, and B and D are non-central subgroups of order 2. Therefore one has four cases to treat

- B and D are non-central subgroups of order 2,

- B is a non-central subgroup of order 2 and $D = 1$,
- $B = 1$ and D is a non-central subgroup of order 2,
- $B = 1$ and $D = 1$.

One starts with a general remark on the first three cases that occur only if P is dihedral (with $k \geq 3$), or semi-dihedral (with $k \geq 3$). As L is a simple faithful module, by looking at the character tables of $D_{2^{k+1}}$ and $SD_{2^{k+1}}$, one sees that the dimension of L is 2. Also the character of $\mathrm{Res}^P_{C_2\times Z(P)}(L)$, for C_2 a non-central subgroup of order 2, is the following

	1	c	cz	z
$\chi_{\mathrm{Res}^P_{C_2\times Z(P)}(L)}$	2	0	0	-2

where c generates C_2 and z generates $Z(P)$. Thus the module $\chi_{\mathrm{Res}^P_{C_2\times Z(P)}(L)}$ splits in the sum of the following two characters of degree one

1	c	cz	z
1	1	-1	-1
1	-1	1	$-1.$

Therefore, $\mathrm{Defres}^P_{C_2\times Z(P)/C_2}(L)$ is the sign representation.

One proves now that the first three cases are impossible. Consider first the case where B is a non-central subgroup of order 2 without assumption on D. By Lemma 9.1 of Bouc and Thévenaz (2012), one knows that $N_P(B) = B\times Z(P)$. This fact forces us to have $A = N_P(B)$, otherwise the A/B-module $M = \mathrm{Iso}_\phi\,\mathrm{Defres}^P_{C/D}(L)$ would be trivial and by Proposition 15 the module L would be trivial as well, but this contradicts the fact that L is faithful. As A/B is of order 2, the module M is therefore forced to be copies of the sign representation M_1. As L is of dimension 2, either $M = M_1$ or $M = 2M_1$. We would like to know if $\mathrm{Ind}^P_A(\mathrm{Inf}^A_{A/B}(M))$ is a sum of copies of L. To do so one uses the scalar product on characters and Frobenius reciprocity

$$\left\langle L, \mathrm{Ind}^P_A(\mathrm{Inf}^A_{A/B}(M_1))\right\rangle = \left\langle \mathrm{Res}^P_A(L), \mathrm{Inf}^A_{A/B}(M_1)\right\rangle = 1.$$

The latter equality holds because, as described in the general remarks above, $\mathrm{Res}^P_A(L)$ is the sum of two non-isomorphic represention of degree 1. It is easy to check that one of them is $\mathrm{Inf}^A_{A/B}(M_1)$. Thus at most two copies of L are in the decomposition of $\mathrm{Ind}^P_A(\mathrm{Inf}^A_{A/B}(M))$, which has dimension $2^{k-1}\dim M$. As $k \geq 3$ one has

$$\dim \mathrm{Ind}^P_A(\mathrm{Inf}^A_{A/B}(M)) = 2^{k-1}\dim M > \left\langle L, \mathrm{Ind}^P_A(\mathrm{Inf}^A_{A/B}(M))\right\rangle \dim L.$$

Indeed, if $k > 3$, or $k = 3$ but $\dim M = 2$, then $2^{k-1}\dim M > 4 \geq \left\langle L, \mathrm{Ind}^P_A(\mathrm{Inf}^A_{A/B}(M))\right\rangle \dim L$ and if $k = 3$ and $\dim M = 1$ then $2^{k-1}\dim M = 4 > 2 = \left\langle L, \mathrm{Ind}^P_A(\mathrm{Inf}^A_{A/B}(M))\right\rangle \dim L$. So $\mathrm{Ind}^P_A(\mathrm{Inf}^A_{A/B}(M))$ contains other modules, non-isomorphic to L, in its decomposition which implies that it cannot be the sum of n copies of L.

Assume now that $B = 1$ and D is a non-central subgroup of order 2. As above one has $C = N_P(D) = D\times Z(P)$ and M is the sign representation. Moreover the subgroup A is of order 2 as A is isomorphic to C/D. We would like to know if $\mathrm{Ind}^P_A(M)$ is a sum of copies of L. Again using the scalar product one has

$$\left\langle L, \mathrm{Ind}^P_A(M)\right\rangle = \left\langle \mathrm{Res}^P_A(L), M\right\rangle \leq 2.$$

The latter inequality occurs because L is of dimension 2 and therefore the sign representation can only occur twice. In fact, it is easy to see that it is equal to 2 if $A = Z(P)$ and 1 otherwise. In any case one has

$$\dim \mathrm{Ind}^P_A(M) = 2^k > 4 = 2\dim L \geq \left\langle L, \mathrm{Ind}^P_A(M)\right\rangle \dim L.$$

This means again that $\mathrm{Ind}^P_A(M)$ contains other modules, non-isomorphic to L, in its decomposition and so it cannot be the sum of n copies of L.

Finally we are restricted to the last case, namely $B = \{1\} = D$ and the result follows. □

We are therefore reduced to $U := \mathrm{Ind}_A^P \mathrm{Iso}_\phi \mathrm{Res}_C^P$. In this case n must be equal to $|P : A|$ as the restriction does not change the dimension of the module. Now, if we suppose that the n-stabilizing biset is strongly minimal, then this implies that $A = C$ and A is a normal subgroup of P. Indeed, by Corollary 13, one can suppose that $(A, 1)$ and $(C, 1)$ are linked, which implies that $A = C$ and by Theorem 12, there are n double (A, A)-cosets in P and as $n = |P : A|$ this forces A to be a normal subgroup of P.

This is why we focus on that situation and completly describe it in the following theorem.

Theorem 37 *Let p be a prime number and let P be a Roquette p-group of order p^{k+1}. Let A be a normal subgroup of P, $U := \mathrm{Ind}_A^P \mathrm{Iso}_\phi \mathrm{Res}_A^P$ and $n = |P : A|$. Then the following conditions are equivalent*

(1) P is generalized quaternion (with $k \geq 2$), dihedral (with $k \geq 3$), or semi-dihedral (with $k \geq 3$) and A is the maximal cyclic subgroup of order p^k. In particular, n and p are equal to 2.

(2) $U(L) \cong nL$ for all faithful $\mathbb{C}P$-modules L.

(3) $U(L) \cong nL$ for a faithful $\mathbb{C}P$-module L.

Proof. Throughout the proof we denote by M the module $\mathrm{Res}_A^P(L)$. First suppose that the first condition holds, and prove *2*. Let L be an arbitrary faithful $\mathbb{C}P$-module. By Clifford's Theorem, one has $\mathrm{Res}_A^P(L) \cong V \oplus {}^gV$, for V a representation of dimension 1 of A. So

$$\mathrm{Ind}_A^P \mathrm{Iso}_\phi \mathrm{Res}_A^P(L) \cong \mathrm{Ind}_A^P \mathrm{Iso}_\phi(V) \oplus \mathrm{Ind}_A^P \mathrm{Iso}_\phi({}^gV)$$

and using Relations 1.1.3 of Bouc (2010) and the fact that A is normal one has

$$\mathrm{Ind}_A^P \mathrm{Iso}_\phi({}^gV) \cong \mathrm{Ind}_A^P \mathrm{Iso}_\phi(V).$$

Thus, one obtains that $U(L) \cong 2\,\mathrm{Ind}_A^P \mathrm{Iso}_\phi(V)$. Moreover, using Frobenius reciprocity one has $U(L) \cong L \oplus (L \otimes \mathrm{Inf}_{P/A}^P(M_1))$, where M_1 is the sign represention for P/A. So

$$2\,\mathrm{Ind}_A^P \mathrm{Iso}_\phi(V) \cong L \oplus (L \otimes \mathrm{Inf}_{P/A}^P(M_1))$$

and by the Krull Schmidt theorem one deduces that $\mathrm{Ind}_A^P \mathrm{Iso}_\phi(V) \cong L$ and therefore $U(L) \cong 2L$, which is the second condition.

The fact that (2) implies (3) is obvious.

We finally prove that *3* implies *1* by proving the contrapositive. Suppose first that P is a cyclic group. Then by Clifford's Theorem $\mathrm{Res}_A^P(L) = V$ where V is a representation of dimension 1 of A. But then one has

$$\left\langle L, \mathrm{Ind}_A^P \mathrm{Iso}_\phi(V)\right\rangle = \left\langle \mathrm{Res}_A^P(L), \mathrm{Iso}_\phi V\right\rangle \leq 1.$$

Yet, the dimension of $\mathrm{Ind}_A^P(V)$ is $|P : A| > 1$, which is stricly bigger than one and so other modules than L appear in the decomposition of $\mathrm{Ind}_A^P(V)$ which means that it cannot be a sum of copies of L.

Suppose that P is not cyclic. One starts with A a maximal non-cyclic subgroup of P, so that $|P : A| = 2$. Using Frobenius reciprocity one has $U(L) = L \oplus (L \otimes \mathrm{Inf}_{P/A}^P(M_1))$ where M_1 is the sign representation of P/A. In order to have n-stabilization one needs $L \otimes \mathrm{Inf}_{P/A}^P(M_1)$ to be isomorphic to L. In terms of characters one must have $\chi_L(g) = 0$ for all g which are not in A, as these elements act on $\mathrm{Inf}_{P/A}^P(M_1)$ as -1. Looking at the character tables of non-cyclic Roquette p-groups one can check that this does not occur if A is a maximal non-cyclic subgroup of P. So U does not n-stabilize L. As a consequence, one deduces that $\mathrm{Res}_A^P(L)$ is irreducible. Indeed, if not then by Clifford's Theorem one could decompose $\mathrm{Res}_A^P(L)$ as the sum of two conjugate modules and using the same argument as above it would give us an example of 2-stabilization. As $\mathrm{Res}_A^P(L)$ is irreducible, one can actually see that every irreducible A-module can be written in this manner. The reason is that $\mathrm{Res}_A^P(\mathbb{C}P) = \mathbb{C}A \oplus \mathbb{C}A$. Furthermore, by the argument above, we note that this implies that if V is an irreducible A-module, then $\mathrm{Ind}_A^P(V) \cong \mathrm{Ind}_A^P \mathrm{Res}_A^P(L) \cong L \oplus (L \otimes \mathrm{Inf}_{P/A}^P(M_1)) \cong L_1 \oplus L_2$ for L_1 and L_2 two non-isomorphic irreducible $\mathbb{C}P$-modules.

Finally, suppose that P is not cyclic and A is not maximal. Then, there exists a non-cyclic maximal subgroup H containing A and

$$\mathrm{Ind}_A^P(M) \cong \mathrm{Ind}_H^P \mathrm{Ind}_A^H(M).$$

Decompose $\mathrm{Ind}_A^H(M)$ as the sum of irreducible H-modules V_i and using the remark above on the induction on modules from a maximal subgroup, one obtains that

$$\mathrm{Ind}_A^P(M) \cong \mathrm{Ind}_H^P(\oplus_i V_i) \cong \oplus_i(L_{i1} \oplus L_{i2})$$

with, for all i, L_{i1} and L_{i2} two non-isomorphic irreducible P-modules. Thus the module $\mathrm{Ind}_A^P(M)$ cannot be only n copies of a module L. □

6.2 Groups With a Cyclic Fitting Subgroup

In this section one proves that if G is a solvable group such that $F(G) = C_n = \prod_i C_{p_i^{k_i}}$ and U is a stabilizing biset for a simple faithful $\mathbb{C}G$-module, then U has to be reduced to an isomorphism. Then one describes the case of ν-stabilizing bisets as one did for Roquette p-groups, where ν is an integer. In this section, G is assumed to be solvable. Suppose $n = 2^k p_1^{k_1} \dots p_m^{k_m}$ for some distinct odd primes p_i and integers k and k_i, so $C_n = C_{2^k} \times \prod_{i=1}^m C_{p_i^{k_i}}$. First note that it is a well known fact that $C_G(F(G)) \leq F(G)$ and therefore $G/F(G)$ injects into $\mathrm{Out}(F(G))$. Thus one has the following exact sequence

$$\{1\} \to C_n \to G \to S \to \{1\}$$

where S is a subgroup of $\mathrm{Aut}(C_n)$. The map $\iota\colon C_n \to G$ is the inclusion map. The map $\pi\colon G \to S$ sends an element g to the conjugation map c_g. Suppose moreover that S is a subgroup of $C_2 \times \prod_i C_{p_i-1}$ where C_2 is either generated by $\beta_1\colon g \mapsto g^{-1}$ or $\beta_2\colon g \mapsto g^{-1+2^{k-1}}$ where g is a generator of C_{2^k} with $k > 2$, or $S \leq \prod_i C_{p_i-1}$ if $k \leq 2$. This added condition is to ensure that G is Roquette, see Theorem 3.7 of Monnard (2014). We start with a number of general lemmas.

Lemma 38 *Let G be an extension of S by C_n as above. Let D be a subgroup of G such that $D \cap C_n = \{1\}$, then $N_{C_n}(D) = C_{C_n}(D) = C_{C_n}(\pi(D))$.*

Proof. For the first equality, let $x \in N_{C_n}(D)$. Then, for all $d \in D$ one has $xdx^{-1} \in D$. But $xdx^{-1} = x\,{}^d x^{-1} d$ which belongs to D if and only if $x\,{}^d x^{-1} = 1$ that is $x = {}^d x$. This implies that x is an element of $C_{C_n}(D)$. The other inclusion is trivial.

For the second equality, note that the action of D on C_n is the same as the action of $\pi(D)$ on C_n by definition of the map π. □

Lemma 39 *Let C_{2^k} be a cyclic group of order 2^k and C_2 its subgroup of order 2. Denote by T_+ and T_- the trivial and the sign $\mathbb{C}$-representation of dimension 1 of C_2. Then the module $\mathrm{Ind}_{C_2}^{C_{2^k}}(T_+)$ decomposes as the sum of all non-faithful representations of C_{2^k} and the module $\mathrm{Ind}_{C_2}^{C_{2^k}}(T_-)$ decomposes as the sum of all faithful representations of C_{2^k}.*

Proof. Observe that

$$\mathrm{Ind}_{C_2}^{C_{2^k}}(T_-) \oplus \mathrm{Ind}_{C_2}^{C_{2^k}}(T_+) = \mathrm{Ind}_{C_2}^{C_{2^k}}(T_- \oplus T_+) = \mathrm{Ind}_{C_2}^{C_{2^k}}(\mathbb{C}C_2) = \mathbb{C}C_{2^k}.$$

But $\mathbb{C}C_{2^k}$ decomposes as the sum of all simple $\mathbb{C}C_{2^k}$-modules. Using the Krull-Schmidt Theorem and the fact that $\mathrm{Ind}_{C_2}^{C_{2^k}}(T_+)$ is not faithful as C_2 is in its kernel, one can conclude that $\mathrm{Ind}_{C_2}^{C_{2^k}}(T_+)$ decomposes as the sum of all non-faithful representations of C_{2^k}. Therefore the module $\mathrm{Ind}_{C_2}^{C_{2^k}}(T_-)$ has to decompose as the sum of all faithful representations of C_{2^k}. □

Theorem 40 *Let G be a Roquette group with $F(G) = C_n$. Let $U := \mathrm{Indinf}_{A/B}^G \mathrm{Iso}_\phi \mathrm{Defres}_{C/D}^G$ be a ν-stabilizing biset for L, where L is a simple faithful $\mathbb{C}G$-module. Then $B = \{1\}$ and A contains $C_2 C_{p_1} \dots C_{p_m}$.*

Proof. The idea of this proof is to restrict the module L to certain well-chosen subgroups using once Clifford's Theorem and then Mackey's Formula as νL can be written as $U(L)$. Then one utilizes the fact that these two decompositions should be isomorphic.

By Proposition 17, one knows that B has a trivial G-core. Therefore $B \cap C_n = \{1\}$. Denote by $\tilde{M}$ the A-module $\mathrm{Inf}_{A/B}^A \mathrm{Iso}_\phi \mathrm{Defres}_{C/D}^G(L)$ and by H the product $C_2 C_{p_1} \dots C_{p_m}$. Using Clifford's Theorem one has

$$\mathrm{Res}_H^G(\nu L) \cong \nu\,\mathrm{Res}_H^G(L) \cong \nu \oplus_{g \in G/I} \mu\,{}^g V$$

where V is a simple H-module and $I := \{g \in G \mid {}^g V \cong V\}$. As L is faithful the module $\mathrm{Res}_H^G(L)$ is also faithful and so is V, because $\ker({}^g V) = {}^g\ker(V) = \ker(V)$, as the subgroups of H are characteristic. Now by Mackey's Formula one has

$$\mathrm{Res}^G_H(\nu L) = \mathrm{Res}^G_H(\mathrm{Ind}^G_A(\tilde{M})) \cong \bigoplus_{g\in[H\backslash G/A]} \mathrm{Ind}^H_{H\cap {}^gA}\, {}^g(\mathrm{Res}^A_{H\cap A}(\tilde{M})).$$

Let Q be a complement of $H \cap A$ in H. Such a complement exists because $H \cap A \leq C_2C_{p_1} \dots C_{p_m}$ and so $Q = C_{|H|/|H\cap A|}$. Now one extends $\mathrm{Res}^A_{H\cap A}(\tilde{M})$ to an H-module N by saying that Q acts trivially on N. Therefore one has $\mathrm{Res}^H_{H\cap A}(N) = \mathrm{Res}^A_{H\cap A}(\tilde{M})$. Using this in the previous equation one has:

$$\begin{aligned}
\mathrm{Res}^G_H(\nu L) &\cong \bigoplus_{g\in[H\backslash G/A]} \mathrm{Ind}^H_{H\cap {}^gA}\, {}^g(\mathrm{Res}^A_{H\cap A}(\tilde{M}))\\
&\cong \bigoplus_{g\in[H\backslash G/A]} \mathrm{Ind}^H_{H\cap {}^gA}\, {}^g(\mathrm{Res}^H_{H\cap A}(N))\\
&\cong \mathrm{Ind}^H_{H\cap A}\mathrm{Res}^H_{H\cap A}(N) \oplus \bigoplus_{\substack{g\in[H\backslash G/A],\\ g\neq 1}} \mathrm{Ind}^H_{H\cap {}^gA}\, {}^g(\mathrm{Res}^H_{H\cap A}(N))\\
&\cong N \oplus (N\otimes \mathrm{Ir}_2) \oplus \cdots \oplus (N \otimes \mathrm{Ir}_f) \oplus \bigoplus_{\substack{g\in[H\backslash G/A],\\ g\neq 1}} \mathrm{Ind}^H_{H\cap {}^gA}\, {}^g(\mathrm{Res}^H_{H\cap A}(N)),
\end{aligned}$$

where $\{\mathrm{Ir}_j\}$ is a set of isomorphism classes of simple $\mathbb{C}[H/H\cap A]$-modules for $1 \leq j \leq f$, with $f = |H : H \cap A|$. The kernel of N is Q but, as mentioned before, $\mathrm{Res}^G_H(L)$ is a sum of faithful modules, therefore Q is trivial and so $H \cap A = H$. This in turn implies that $H \leq A$ and therefore normalizes B, because B is normal in A. This implies that B acts trivially on H by Lemma 38. Therefore B is either trivial or $\pi(B)$ is generated by β_1 or β_2, where π denotes the homomorphism from G to S. Suppose the latter holds, so $k > 2$. By Clifford's Theorem

$$\nu\,\mathrm{Res}^G_{C_{2^k}}(L) = \nu \bigoplus_{g\in G/I_1} m_1\, {}^gL_1,$$

where L_1 is a simple C_{2^k}-module and $I_1 := \{g \in G \mid {}^gL_1 \cong L_1\}$. By definition C_n is a subgroup of I_1. As $\prod_i C_{p_i-1}$ acts trivially on C_{2^k}, it is a subgroup of I_1/C_n and so the order of G/I_1 is at most 2. This implies that there are at most 2 non-isomorphic modules appearing in $\mathrm{Res}^G_{C_{2^k}}(L)$.

Next we note that

$$C_2 = H \cap C_{2^k} \leq A \cap C_{2^k} \leq N_G(B) \cap C_{2^k} = N_{C_{2^k}}(B) = C_2$$

where the last equality holds because either for β_1 or β_2 one has $C_{2^k}(\langle\beta_i\rangle) = \{c \in C_{2^k} \mid c^2 = 1\} = C_2$. Using this remark and Mackey's Formula we restrict L to C_{2^k}:

$$\begin{aligned}
\mathrm{Res}^G_{C_{2^k}}(\nu L) &\cong \bigoplus_{g\in[C_{2^k}\backslash G/A]} \mathrm{Ind}^{C_{2^k}}_{C_{2^k}\cap {}^gA}\, {}^g(\mathrm{Res}^A_{C_{2^k}\cap A}(\tilde{M}))\\
&\cong \mathrm{Ind}^{C_{2^k}}_{C_2}\mathrm{Res}^A_{C_2}(\tilde{M}) \oplus \bigoplus_{\substack{g\in[C_{2^k}\backslash G/A]\\ g\neq 1}} \mathrm{Ind}^{C_{2^k}}_{C_{2^k}\cap {}^gA}\, {}^g(\mathrm{Res}^A_{C_{2^k}\cap A}(\tilde{M})).
\end{aligned}$$

Now $\mathrm{Res}^A_{C_2}(\tilde{M})$ decomposes as a sum of representations that are either the trivial or the sign representation, but the trivial cannot occur. Indeed suppose the trivial representation T_+ appears in the decomposition of $\mathrm{Res}^A_{C_2}(\tilde{M})$. Then $\mathrm{Ind}^{C_{2^k}}_{C_2}(T_+)$ is not a faithful representation as C_2 is in its kernel, contrary to the fact that $\mathrm{Res}^G_{C_{2^k}}(L)$ is faithful. Therefore $\mathrm{Res}^A_{C_2}(\tilde{M})$ is a sum of copies of the sign representation T_- and $\mathrm{Ind}^{C_{2^k}}_{C_2}\mathrm{Res}^A_{C_2}(\tilde{M}) = \oplus\,\mathrm{Ind}^{C_{2^k}}_{C_2}(T_-)$. But $\mathrm{Ind}^{C_{2^k}}_{C_2}(T_-)$ decomposes as the sum of all faithful representations of C_{2^k} by Lemma 39 and there are 2^{k-1} such non-isomorphic representations. So the module $\mathrm{Res}^G_{C_{2^k}}(L)$ decomposes with at least 2^{k-1} non-isomorphic representations. As $k > 2$ one has $2^{k-1} > 2$ and so a contradiction is obtained with the decomposition using Clifford's Theorem. Therefore the only possibility is that $B = \{1\}$. □

Theorem 41 *Let G be a Roquette group with $F(G) = C_n$. Let $U := \mathrm{Indinf}^G_{A/B}\mathrm{Iso}_\phi\mathrm{Defres}^G_{C/D}$ be a stabilizing biset for L, where L is a simple faithful $\mathbb{C}G$-module. Then one has $(A, B) = (C, D) = (G, 1)$.*

Proof. By Proposition 25 it is sufficient to look at minimal stabilizing bisets. If U is minimal, one knows that if $B = \{1\}$ then $A = G$ by Proposition 8.4 of Bouc and Thévenaz (2012), but Theorem 40 shows that $B = \{1\}$ and so the results follows. □

One continues our investigation of ν-stabilizing bisets for $\nu > 1$. Next we reduce our study to strongly minimal bisets.

Theorem 42 *Let G be a Roquette group with $F(G) = C_n$. Let $U := \mathrm{Ind}_A^G \mathrm{Iso}_\phi \mathrm{Defres}_{C/D}^G$ be a strongly minimal ν-stabilizing biset for L where L is a simple faithful $\mathbb{C}G$-module. Then $D = \{1\}$ and $A = C$ is a normal subgroup of G.*

Proof. First recall that by Proposition 18, we know that D has a trivial G-core. Therefore $D \cap C_n = \{1\}$. By Corollary 13, one can suppose that $(A, 1)$ and (C, D) are linked, which implies that $A \cap C = A$ and so $A \leq C$, therefore A normalizes D. As A contains $C_2 C_{p_1} \ldots C_{p_m}$ by Theorem 40, this implies that D acts trivially on $C_2 C_{p_1} \ldots C_{p_m}$ by Lemma 38. Therefore D is either trivial or $\pi(D)$ is generated by β_1 or β_2, where π denotes the homomorphism from G to S. As in the proof of Theorem 40, one restricts L to C_{2^k} using first Clifford's Theorem and secondly Mackey's Formula to obtain with exactly the same arguments that $D = \{1\}$. The key ingredient is that $A \cap C_{2^k}$ is again equal to C_2 as A normalizes D.

Finally, as the sections are linked and $D = \{1\}$ one obtains that $A = C$. Moreover, by Theorem 12, there are ν double (A, A)-cosets in G, but also $\nu = |G : A|$, which forces A to be a normal subgroup of G. □

One finishes our study by completely describing the remaining case.

Theorem 43 *Let G be a Roquette group with $F(G) = C_n$. Let A be a normal subgroup of G, $U := \mathrm{Ind}_A^G \mathrm{Res}_A^G$ and $\nu = |G : A|$. Then the following conditions are equivalent*

(1) A contains $F(G)$.

(2) $U(L) \cong \nu L$ for all faithful $\mathbb{C}G$-modules L.

(3) $U(L) \cong \nu L$ for a faithful $\mathbb{C}G$-module L.

Proof. Suppose first that A contains $F(G)$, we will then prove that $U := \mathrm{Ind}_A^G \mathrm{Res}_A^G$ is a $|G : A|$-stabilizing biset for an arbitrary faithful $\mathbb{C}G$-module L. First note that L can be written as $\mathrm{Ind}_{F(G)}^G(\xi)$, where ξ is a primitive nth root of unity. Indeed, every irreducible $\mathbb{C}G$-module comes from a summand of an induction from $F(G)$, but the module $\mathrm{Ind}_{F(G)}^G(\xi)$ is irreducible as the conjugate representations of ξ by the action of $G/F(G)$ are not isomorphic. The condition of primitivity on the root is to ensure the faithfulness of the induced module. Furthermore, as A contains $F(G)$, then $L \cong \mathrm{Ind}_A^G(V)$ where $V := \mathrm{Ind}_{F(G)}^A(\xi)$. The A-module V is irreducible because $\mathrm{Ind}_{F(G)}^G(\xi)$ is. Therefore, using Mackey's Formula, one has

$$\begin{aligned} U(L) &= \mathrm{Ind}_A^G \mathrm{Res}_A^G(L) \cong \mathrm{Ind}_A^G \mathrm{Res}_A^G \mathrm{Ind}_A^G(V) \cong \bigoplus_{g \in G/A} \mathrm{Ind}_A^G({}^gV) \\ &= |G : A| \mathrm{Ind}_A^G(V) \cong |G : A| L, \end{aligned}$$

where the isomorphism between the first and the second line holds because A is normal. As L was arbitrarily chosen, this holds for any faithful $\mathbb{C}G$-module L.

The fact that (2) implies (3) is obvious.

We prove now that (3) implies *1* by proving the contrapositive. Let A be a normal subgroup of G such that $A \cap F(G)$ is not equal to $F(G)$. Recall that by Theorem 40, one knows that A contains $C_2 C_{p_1} \ldots C_{p_m}$, so this intersection is non-trivial. One shows that it is not possible to ν-stabilize L for all faithful $\mathbb{C}G$-modules L. One knows that $L \cong \mathrm{Ind}_{F(G)}^G(\xi)$ where ξ is a primitive nth root of unity. Then, by Mackey's Formula, one has

$$\begin{aligned} U(L) &\cong \mathrm{Ind}_A^G \mathrm{Res}_A^G \mathrm{Ind}_{F(G)}^G(\xi) \cong \bigoplus_{g \in [A\backslash G/F(G)]} \mathrm{Ind}_A^G \mathrm{Ind}_{A \cap F(G)}^A \mathrm{Res}_{A \cap F(G)}^{F(G)}({}^g\xi) \\ &\cong \bigoplus_{g \in [A\backslash G/F(G)]} \mathrm{Ind}_{A \cap F(G)}^G \mathrm{Res}_{A \cap F(G)}^{F(G)}({}^g\xi) \\ &\cong |A\backslash G/F(G)| \mathrm{Ind}_{A \cap F(G)}^G \mathrm{Res}_{A \cap F(G)}^{F(G)}(\xi) \\ &\cong |A\backslash G/F(G)| \mathrm{Ind}_{F(G)}^G \mathrm{Ind}_{A \cap F(G)}^{F(G)} \mathrm{Res}_{A \cap F(G)}^{F(G)}(\xi). \end{aligned}$$

Using Frobenius reciprocity one has $\mathrm{Ind}_{A \cap F(G)}^{F(G)} \mathrm{Res}_{A \cap F(G)}^{F(G)}(\xi) \cong \oplus_j \xi \otimes \mathrm{Ir}_j$ where $\{\mathrm{Ir}_j\}$ is a set of isomorphism classes of simple $\mathbb{C}[F(G)/(F(G) \cap A)]$-modules. The sum is not reduced to one module as $A \cap F(G) \neq F(G)$ by assumption.

This means that

$$U(L) \cong \bigoplus_j |A\backslash G/F(G)| \operatorname{Ind}_{F(G)}^G(\xi \otimes \mathrm{Ir}_j).$$

Thus our purpose is to show that $\operatorname{Ind}_{F(G)}^G(\xi \otimes \mathrm{Ir}_j)$ is not isomorphic to $L = \operatorname{Ind}_{F(G)}^G(\xi)$ for at least one representation Ir_j. To do so, one proves that $\xi \otimes \mathrm{Ir}$ is not conjugate, by an element of $G/F(G)$ to ξ, where Ir denotes a non-trivial $\mathbb{C}[F(G)/(F(G) \cap A)]$-module. We specify which Ir is taken later on.

Let p be a prime dividing $|F(G) : A \cap F(G)|$ and let i be its highest power dividing $|F(G) : A \cap F(G)|$. Choose p such that p^i is strictly smaller that p^k, where k is the highest power of p such that p^k divides n. As $F(G)$ is cyclic, one decomposes Ir as the tensor product of a representation θ of C_{p^i} and a representation θ^c of its complement in $F(G)/(F(G) \cap A)$, i.e. $\mathrm{Ir} = \theta \otimes \theta^c$. Note that θ is a p^ith root of unity. In the same fashion $\xi = \xi_1 \otimes \xi_2$, where ξ_1 is a p^kth root of unity and ξ_2 is a representation for C_{n/p^k}. Then one has

$$\xi \otimes \mathrm{Ir} \cong \xi_1 \otimes \theta \otimes \xi_2 \otimes \theta^c.$$

One now sets Ir such that $\theta = \xi_1^{p^{k-i}}$ and then one has $\xi_1 \otimes \theta = \xi_1^{1+p^{k-i}}$. Because of the assumption on S made at the beginning of the section, this representation cannot be conjugate to the representation ξ_1 by an element of $G/F(G)$. Indeed, such an element would have order a divisor of p^i, as such an element must be of the following form

$$\alpha : \xi_1 \mapsto \xi_1^{1+p^{k-i}}.$$

Moreover, it is easy to check that $\alpha^\delta(\xi_1) = \xi_1^{1+\delta p^{k-i}}$ and so $\alpha^{p^i} = \mathrm{id}$. So $\xi \otimes \mathrm{Ir}$ is not conjugate to ξ. Finally, one has proved that $\operatorname{Ind}_{F(G)}^G(\xi \otimes \mathrm{Ir}) \ncong L = \operatorname{Ind}_{F(G)}^G(\xi)$ and therefore other modules than L appear in the decomposition of $U(L)$. □

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