Journal of Mathematics Research; Vol. 6, No. 3; 2014
ISSN 1916-9795 E-ISSN 1916-9809
Published by Canadian Center of Science and Education

On Generalization of Helices in the Galilean and the Pseudo-Galilean Space

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Received: May 20, 2014 Accepted: June 19, 2014 Online Published: July 7, 2014

doi:10.5539/jmr.v6n3p39 URL: http://dx.doi.org/10.5539/jmr.v6n3p39

The research is financed by MSF project 016-0372785-0892

Abstract

In this paper a generalization of helices in the three-dimensional Galilean and the pseudo-Galilean space is proposed. The equiform general helices, which represent a generalization of "ordinary" helices, are defined and characterized. Particularly, all obtained results can be transferred to other Cayley-Klein spaces, including Euclidean.

Keywords: Galilean space, pseudo-Galilean space, equiform differential geometry, helices

1. Introduction

Among all space curves, helices have special emplacement regarding their properties and applicability in other sciences. Helices can be seen in DNA molecules, bacterial flagella, structure of proteins, carbon nanotubes etc. Because of this they deserve especial attention in Euclidean as well as in other geometries. Besides the Euclidean geometry, a palette of new geometries has been developed over the last two centuries and some properties of curves and surfaces are more emphasized in newly developed non-Euclidean geometries than in the Euclidean.

Among these non-Euclidean geometries there are also the Galilean and the pseudo-Galilean geometry which represent ambient geometries in which we investigate generalization of helices. Although ambient spaces are the Galilean and the pseudo-Galilean space, instead of an isometry group the corresponding action group will be its subgroup, the equiform group.

We found motivation for this work in (Bektaş, 2004; Öğrenmiş, Bektaş, & Ergüt, 2006) and (Öğrenmiş, Ergüt, & Bektaş, 2007), where the authors considered characterizations of general helices in the pseudo-Galilean, double isotropic and Galilean space. Examining a similar problem in the equiform geometry of the Galilean space, we found a new family of curves which represent a generalization of "ordinary" circular helices and partially "ordinary" general helices. These curves will be called equiform helices.

The main goal of this article is to define, describe and characterize equiform general helices.

Although we use the Galilean and the pseudo-Galilean as ambient spaces, similar generalization is possible in other Cayley-Klein spaces, including the simple and double isotropic space as well as Euclidean space.

First, we define equiform general helices and explain their relation to "ordinary" circular and general helices. Next, we give a characterization of general helices common for all Cayley-Klein spaces, including Euclidean. Further, we obtain characterizations of circular and general helices in an equiform geometry of the Galilean and the pseudo-Galilean space, which represent a generalization of characterizations of "ordinary" circular helices given in (Bektaş, 2004). Finally, we investigate several types of equiform general helices in the Galilean and the pseudo-Galilean space and furnish the theory by graphs of newly discovered curves.

This paper is written from the pseudo-Galilean point of view but with minor changes the obtained results can be transferred to the Galilean space. These changes will be emphasized directly in the text or in remarks at the end of each section.

2. Preliminaries

In 1872, Felix Klein in his Erlangen program proposed how to classify and characterize geometries on the basis of projective geometry and group theory. He showed that the Euclidean and non-Euclidean geometries could be considered as spaces that are invariant under a given group of transformations. The geometry motivated by this approach is called a ***Cayley-Klein geometry***. Actually, the formal definition of Cayley-Klein geometry is pair (G, H) where G is a Lie group and H is a closed Lie subgroup of G such that the (left) coset G/H is connected. G/H is called the space of the geometry or simply Cayley-Klein geometry.

Let us recall the basic facts about the three-dimensional pseudo-Galilean G_3^1 and the Galilean geometry G_3. The geometry of the pseudo-Galilean (resp. Galilean) space has been firstly explained in (Divjak, 1997) (resp. (Röschel, 1984)). The curves and some special surfaces in G_3^1 and G_3 are considered in (Divjak, 1998, 2008) and (Divjak & Milin-Šipuš, 2003). The notions and symbols from those papers will be used in this paper.

The ***pseudo-Galilean*** (resp. ***Galilean***) ***geometry*** is a real Cayley-Klein geometry with projective signature $(0, 0, +, -)$ (resp. $(0, 0, +, +)$), according to (Mólnar, 1997). The absolute of the pseudo-Galilean (resp. Galilean) geometry is an ordered triple $\{\omega, f, I\}$ where ω is the ideal (absolute) plane ($x_0 = 0$), f is a line in ω ($x_0 = x_1 = 0$) and I is the fixed hyperbolic $((0 : 0 : x_2 : x_3) \overset{I}{\longmapsto} (0 : 0 : x_3 : x_2))$ (resp. elliptic $((0 : 0 : x_2 : x_3) \overset{I}{\longmapsto} (0 : 0 : x_3 : -x_2))$) involution of the points of f.

Remark 1 The absolute of the three-dimensional ***simple isotropic space*** $I_3^{(1)}$ consists of a plane and a pair of two conjugate lines in this plain. The absolute of the ***double isotropic space*** $I_3^{(2)}$ consists of a plane, a line in this plane and a point on this line. More about the isotropic spaces can be found in (Sachs, 1986).

Projective transformations

$$\bar{x}_i = a_0^i x_0 + a_1^i x_1 + a_2^i x_2 + a_3^i x_3 \qquad i = 0, 1, 2, 3$$

which preserve the absolute in the three-dimensional pseudo-Galilean space G_3^1 form the eight parameter group, called the similarity group H_8. After replacing the homogeneous coordinates with the inhomogeneous affine coordinates ($x_0 : x_1 : x_2 : x_3 = 1 : x : y : z,\ \ x_0 \neq 0$), the similarity group H_8 of the pseudo-Galilean space G_3^1 has the following form

$$\begin{aligned} \bar{x} &= a + b \cdot x, \\ \bar{y} &= c + d \cdot x + r \cdot \cosh\varphi \cdot y + r \cdot \sinh\varphi \cdot z, \\ \bar{z} &= e + f \cdot x + r \cdot \sinh\varphi \cdot y + r \cdot \cosh\varphi \cdot z, \end{aligned} \tag{1}$$

where a, b, c, d, e, f, r and φ are real numbers. In particular, for $b = r = 1$, the group (1) becomes the ***group of isometries*** (proper motions), $B_6 \subset H_8$, of the pseudo-Galilean space G_3^1. The motion group leaves the absolute invariant and defines the other invariants of this geometry, e.g. distance.

According to the motion group in the pseudo-Galilean space, there are non-isotropic vectors $\mathbf{x}(x, y, z)$ (for which holds $x \neq 0$) and four types of isotropic vectors: space-like ($x = 0,\ y^2 - z^2 > 0$), time-like ($x = 0\ y^2 - z^2 < 0$) and two types of light-like vectors ($x = 0,\ y = \pm z$).

Remark 2 The similarity group and the motion group of the Galilean space G_3 have a similar form, except that hyperbolic sine and hyperbolic cosine are replaced by sine and cosine. In the Galilean space there are just two types of vectors, non-isotropic and isotropic. We do not distinguish classes of vectors among isotropic vectors in G_3.

A plane of the form $x = const.$ in the pseudo-Galilean (resp. Galilean) space is called ***pseudo-Euclidean plane*** (resp. ***Euclidean***), since its induced geometry is pseudo-Euclidean (resp. Euclidean). Otherwise it is called ***isotropic plane***.

A curve $\mathbf{r}(t) = (x(t), y(t), z(t))$ is ***admissible*** in G^3 and G_1^3 if it has no inflection points ($\dot{r}(t) \times \ddot{r}(t) \neq \mathbf{0}$) and no isotropic tangents ($\dot{x}(t) \neq 0$). Additionally in G_3^1 such a curve is admissible if it has no tangents or normals whose projections on the absolute plane would be light-like vectors ($\dot{y}(t) \neq \pm\dot{z}(t)$). An admissible curve in G^3 (and G_1^3) is an analogue of a regular curve in Euclidean space.

For an admissible curve $\mathbf{r}$: $I \to G_3^1(G_3)$, $I \subseteq \mathbb{R}$ parameterized by the arc length s with differential form $ds = dx$, given by

$$\mathbf{r}(x) = (x, y(x), z(x)), \tag{2}$$

curvature $\kappa(x)$ and ***torsion*** $\tau(x)$ are defined by

$$\begin{aligned} \kappa(x) &= \sqrt{|y''(x)^2 - \delta \cdot z''(x)^2|}, \quad (3)\\ \tau(x) &= \frac{y''(x)z'''(x) - \delta \cdot y'''(x)z''(x)}{\kappa^2(x)}, \quad (4)\end{aligned}$$

where the constant δ takes value $+1$ in the pseudo-Galilean space and -1 in the Galilean space. Note that an admissible curve has non-zero curvature.

The associated trihedron is given by

$$\begin{aligned} \mathbf{t}(x) &= \mathbf{r}'(x) = (1, y'(x), z'(x)),\\ \mathbf{n}(x) &= \frac{1}{\kappa(x)}\mathbf{r}''(x) = \frac{1}{\kappa(x)}(0, y''(x), z''(x)), \quad (5)\\ \mathbf{b}(x) &= \frac{1}{\kappa(x)}(0, \delta \cdot \varepsilon z''(x), \varepsilon y''(x)),\end{aligned}$$

where ε is $+1$ in the Galilean space and $\varepsilon = +1$ or -1 in the pseudo-Galilean space, chosen by criterion $\det(\mathbf{t}, \mathbf{n}, \mathbf{b}) = 1$, which means

$$|y''^2(x) - z''^2(x)| = \varepsilon(y''^2(x) - z''^2(x)).$$

For derivatives of the tangent $\mathbf{t}$, normal $\mathbf{n}$ and binormal $\mathbf{b}$ vector field, the following ***Frenet formulas*** in the pseudo-Galilean (Divjak, 1997) and the Galilean space (Röschel, 1984) hold respectively

$$G_3^1 \ldots \begin{cases} \mathbf{t}' = \kappa \cdot \mathbf{n},\\ \mathbf{n}' = \tau \cdot \mathbf{b},\\ \mathbf{b}' = \tau \cdot \mathbf{n}, \end{cases} \qquad G_3 \ldots \begin{cases} \mathbf{t}' = \kappa \cdot \mathbf{n},\\ \mathbf{n}' = \tau \cdot \mathbf{b},\\ \mathbf{b}' = -\tau \cdot \mathbf{n}. \end{cases} \quad (6)$$

From (5) and (6), we derive an important relation that is true in both spaces

$$\mathbf{r}'''(x) = \kappa'(x) \cdot \mathbf{n}(x) + \kappa(x) \cdot \tau(x) \cdot \mathbf{b}(x). \quad (7)$$

Let us recall the basic facts about ***equiform geometry***. The equiform differential geometry of curves in the simple isotropic I_3^1 and the double isotropic space I_3^2 has been described in (Pavković, 1986), and in the pseudo-Galilean space G_3^1 and the Galilean space G_3 in (Erjavec & Divjak, 2008) and (Pavković & Kamenarović, 1987), respectively.

If we assume $b = r\ (\neq 1)$ in (1), then the eight parameter similarity group become the seven parameter equiform group. An analogous group does not exist in the Euclidean geometry because in Euclidean space there are no different kinds of planes as in the Galilean or the pseudo-Galilean space.

In the equiform geometry a few specific terms will be introduced. So, for an admissible curve $\mathbf{r}\colon I \to G_3^1(G_3)$, the ***equiform parameter*** is defined by

$$\sigma := \int \frac{ds}{\rho},$$

where $\rho = \frac{1}{\kappa}$ is the radius of curvature of the curve $\mathbf{r}$. Furthermore, the tangent, normal and binormal vector field in the equiform geometry are defined by

$$\mathbf{T} = \frac{d\mathbf{r}}{d\sigma} = \frac{d\mathbf{r}}{ds} \cdot \frac{ds}{d\sigma} = \rho \cdot \frac{d\mathbf{r}}{ds} = \rho \cdot \mathbf{t}, \quad \mathbf{N} = \rho \cdot \mathbf{n}, \quad \mathbf{B} = \rho \cdot \mathbf{b},$$

respectively. We differentiate these vector fields and introduce the functions $\mathcal{K}, \mathcal{T}\colon I \to \mathbb{R}$, called the ***equiform curvature*** and the ***equiform torsion***, respectively, by

$$\begin{aligned} \mathcal{K} &:= \frac{d\rho}{ds} = \dot{\rho},\\ \mathcal{T} &:= \rho\tau = \frac{\tau}{\kappa}. \end{aligned} \quad (8)$$

Further, in (Erjavec & Divjak, 2008) we obtained the formulas analogous to the Frenet's in the equiform geometry of the pseudo-Galilean and the Galilean space

$$\begin{aligned}\nabla_T \mathbf{T} &= \mathcal{K}\cdot\mathbf{T}+\mathbf{N},\\ \nabla_T \mathbf{N} &= \mathcal{K}\cdot\mathbf{N}+\mathcal{T}\cdot\mathbf{B},\\ \nabla_T \mathbf{B} &= \delta\cdot\mathcal{T}\cdot\mathbf{N}+\mathcal{K}\cdot\mathbf{B},\end{aligned} \tag{9}$$

where ∇_T denotes covariant differentiation in direction of the equiform tangent vector field $\mathbf{T}$ defined by

$$\nabla_{T(\sigma)}\mathbf{X}(\sigma) = \frac{d}{d\sigma}\left(\mathbf{X}(\mathbf{r}(\sigma))\right)$$

for an arbitrary vector field $\mathbf{X}$. The notation ∇_T is motivated with a similar notation introduced in (Ikawa, 1985) and used in (Bektaş, 2004) and (Öğrenmiş et al., 2006, 2007). The constant δ takes value $+1$ in the pseudo-Galilean space and -1 in the Galilean space.

Remark 3 The equiform parameter $\sigma = \int \kappa(s)ds$ for closed curves is called ***total curvature*** and plays important role in global differential geometry of the Euclidean space (for more see Fechnel, 1951 and Kuhnel, 2005). Also, the function $\frac{\tau}{\kappa}$ has been already known as ***conical curvature*** and has interesting geometrical interpretation (see Pottman, 2001).

3. Equiform General Helices

In this section we define equiform general helices and explain their relation with "ordinary" circular and general helices. By "ordinary" helices, we imply helices in "ordinary" (not equiform) geometry. Although definitions of "ordinary" helices vary from space to space, in all Cayley-Klein spaces they are characterized by a constant conical curvature ($\frac{\tau}{\kappa} = const.$).

Although in this paper we define equiform helices in the Galilean and the pseudo-Galilean space, the same definition can be applied in all Cayley-Klein spaces, including Euclidean. Further, we give a characterization of equiform general helices common for all Cayley-Klein spaces, which will be used later as a recipe for generating concrete curves.

Definition 1 An admissible curve $\mathbf{r}$: $I \to G_3^1(G_3)$ such that its equiform torsion $\mathcal{T} \neq 0$ or its equiform curvature $\mathcal{K} = 0$ whenever $\mathcal{T} = 0$, and

$$\frac{\mathcal{K}}{\mathcal{T}} = const$$

is called ***equiform general helix***.

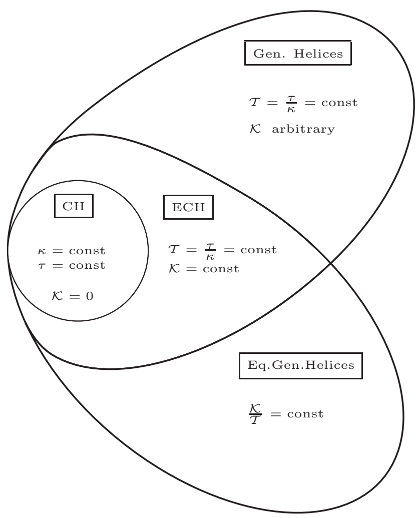

Figure 1. Equiform helices

Particularly, if $\mathcal{K} = const.$ and $\mathcal{T} = const.$, the curve is called ***equiform circular helix***.

Note that from the characterization of "ordinary" circular helices ($\kappa = const.$ and $\tau = const.$) directly follows that they are equiform circular helices ($\mathcal{T} = \frac{\tau}{\kappa} = const.$, $\mathcal{K} = const.$). Furthermore, there are two generalizations of equiform circular helices (see Figure 1). The first one gives the well-known "ordinary" general helices ($\mathcal{T} = \frac{\tau}{\kappa} = const.$, $\mathcal{K}$ arbitrary) and the second one leads to equiform general helices ($\frac{\mathcal{K}}{\mathcal{T}} = const.$). It is easy to show that there are no other curves, except for equiform circular helices, which are at the same time "ordinary" general helices and equiform general helices.

Remark 4 For an admissible curve in the equiform geometry of the pseudo-Galilean space it is not possible that the equiform curvature is equal to the equiform torsion. The request $\mathcal{K} = \pm\mathcal{T}$ implies the fact that projections of tangents or normals on the absolute plane are light-like vectors what is not allowed by the definition of an admissible curves in G_3^1.

Proposition 1 *Let* $\mathbf{r}\colon I \to G_3^1(G_3)$ *be an admissible curve parameterized by the arc length parameter s,* κ *and* $\tau \neq 0$ *its curvature and torsion, functions of class* C^1 *and* C^0*, respectively. The curve is an equiform general helix if and only if*

$$\kappa = d \cdot e^{-c\int \tau ds}, \tag{10}$$

where $c, d \in \mathbb{R}$.

Proof. From the definition of an equiform general helix $\frac{\mathcal{K}}{\mathcal{T}} = c, c \in \mathbb{R}\backslash\{0\}$ and the definition of the equiform curvature and the equiform torsion (8), we have $\frac{\rho}{\dot{\rho}} = c \cdot \tau$. After integration we obtain the requested formula.

On the other hand, from (10) it follows

$$\ln\rho = \ln\frac{1}{d} + \int c\tau ds.$$

If we differentiate this relation and use the notation from the definition (8), we finally obtain $\mathcal{K} = c \cdot \mathcal{T}$. □

Remark 5 By definition, an admissible curve in "ordinary" geometry of G_3^1 and G_3 is called ***general helix*** if there is a fixed non-isotropic vector, called the axis of the helix, such that the angle between tangent vectors and the axis is constant. Also, as we know, an admissible curve is a general helix if and only if its conical curvature is constant.

In the equiform geometry of G_3^1 and G_3, we do not know a similar geometric interpretation which could replace the existing definition. However, we know that equiform transformations preserve angles between lines and foresee that the definition from the "ordinary" geometry with some adjustments can be transferred in the equiform geometry.

4. Characterization of Equiform Helices in G_3^1 and G_3

In this section we give a characterization of equiform general helices with respect to the Frenet frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ in the equiform geometry of G_3^1 and G_3. Further, we give a characterization of equiform circular helices which represent a generalization of the characterization of "ordinary" circular helices given in (Öğrenmiş et al., 2007). Finally, we characterize equiform circular helices using only the normal vector.

Theorem 1 *Let* $\mathbf{r}\colon I \to G_3^1(G_3)$ *be an admissible curve and* $\mathcal{K} \neq 0$ *of class* C^2*,* $\mathcal{T}$ *of class* C^1 *its equiform curvature and equiform torsion, respectively. The curve is an equiform general helix if and only if*

$$\nabla_T\nabla_T\nabla_T\mathbf{T} - a_1 \cdot \nabla_T\mathbf{T} = a_2 \cdot \nabla_T\mathbf{N} + a_3 \cdot \nabla_T\mathbf{B}, \tag{11}$$

where

$$\begin{aligned} a_1 &= \frac{\mathcal{K}''}{\mathcal{K}} + 3\mathcal{K}' + \mathcal{K}^2, \\ a_2 &= 2\mathcal{K} - \mathcal{K}''\vartheta - \delta\mathcal{T}\mathcal{T}'\vartheta, \\ a_3 &= \mathcal{T} + \mathcal{K}'\mathcal{T}\vartheta + \frac{\mathcal{K}''}{\mathcal{K}}\mathcal{T}\vartheta, \end{aligned}$$

and

$$\vartheta = \frac{1}{\mathcal{K}^2 - \delta\mathcal{T}^2}.$$

The constant δ *takes value* $+1$ *in the pseudo-Galilean space and* -1 *in the Galilean space.*

Proof. Suppose that $\mathbf{r}$ is an equiform general helix with respect to the Frenet frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$. Since $\mathbf{r}$ is an equiform general helix, we have

$$\frac{\mathcal{K}}{\mathcal{T}} = const. \qquad \text{and hence} \qquad \mathcal{K}' \cdot \mathcal{T} = \mathcal{K} \cdot \mathcal{T}'.$$

From the Frenet formulas (9) we can express vector fields $\mathbf{T}, \mathbf{N}, \mathbf{B}$ as follows

$$\begin{aligned} \mathbf{T} &= \frac{1}{\mathcal{K}} \cdot \nabla_T \mathbf{T} - \vartheta \cdot \nabla_T \mathbf{N} + \frac{\mathcal{T}}{\mathcal{K}} \cdot \vartheta \cdot \nabla_T \mathbf{B}, \\ \mathbf{N} &= \mathcal{K} \cdot \vartheta \cdot \nabla_T \mathbf{N} - \mathcal{T} \cdot \vartheta \cdot \nabla_T \mathbf{B}, \\ \mathbf{B} &= \mathcal{K} \cdot \vartheta \cdot \nabla_T \mathbf{B} - \delta \mathcal{T} \cdot \vartheta \cdot \nabla_T \mathbf{N}, \end{aligned} \tag{12}$$

where

$$\vartheta = \frac{1}{\mathcal{K}^2 - \delta \mathcal{T}^2}.$$

If we differentiate the first equation of the Frenet formulas (9) twice, we obtain

$$\nabla_T^3 \mathbf{T} = (\mathcal{K}'' + 2\mathcal{K}\mathcal{K}') \cdot \mathbf{T} + (\mathcal{K}' + \mathcal{K}^2) \cdot \nabla_T \mathbf{T} + 2\mathcal{K}' \cdot \mathbf{N} + 2\mathcal{K} \cdot \nabla_T \mathbf{N} + \mathcal{T}' \cdot \mathbf{B} + \mathcal{T} \cdot \nabla_T \mathbf{B}. \tag{13}$$

Further, if we insert the equalities (12) in the Equation (13), after reducing we obtain the requested relation (11).

Conversely, let us assume that the Equation (11) holds. We have to prove that the curve is an equiform general helix. Differentiating covariantly the second equation of (9) and using (12), we obtain

$$\nabla_T^2 \mathbf{N} = (\mathcal{K}\mathcal{K}'\vartheta + \mathcal{K} - \delta \mathcal{T}\mathcal{T}'\vartheta) \cdot \nabla_T \mathbf{N} + (\mathcal{T} - \mathcal{K}'\mathcal{T}\vartheta + \mathcal{K}\mathcal{T}'\vartheta) \cdot \nabla_T \mathbf{B}. \tag{14}$$

If we express $\mathbf{N}$ from the first of the Frenet Equations (9) and differentiate it, we have

$$\nabla_T \mathbf{N} = \nabla_T^2 \mathbf{T} - (\mathcal{K}' + \mathcal{K}^2) \cdot \mathbf{T} - \mathcal{K} \cdot \mathbf{N}.$$

Differentiating again and using (12) we obtain

$$\nabla_T^2 \mathbf{N} = \nabla_T^3 \mathbf{T} - \left(\frac{\mathcal{K}''}{\mathcal{K}} + 3\mathcal{K}' + \mathcal{K}^2\right) \cdot \nabla_T \mathbf{T} + (\mathcal{K}''\vartheta + \mathcal{K}\mathcal{K}'\vartheta - \mathcal{K}) \cdot \nabla_T \mathbf{N} - \left(\mathcal{K}'\mathcal{T}\vartheta + \frac{\mathcal{K}''}{\mathcal{K}}\mathcal{T}\vartheta\right) \cdot \nabla_T \mathbf{B}.$$

So, if we equalize the right side of the last relation and the relation (14), using the Equation (11) we get

$$(-\mathcal{K}'\mathcal{T}\vartheta + \mathcal{K}\mathcal{T}'\vartheta)\nabla_T \mathbf{B} = 0,$$

and therefore

$$\left(\frac{\mathcal{K}}{\mathcal{T}}\right)' = 0 \qquad \Rightarrow \qquad \frac{\mathcal{K}}{\mathcal{T}} = const.$$

If we assume that the equiform curvature and the equiform torsion are constant, then from the previous theorem we obtain a characterization of equiform circular helices. □

Corollary 1 *Let* $\mathbf{r}\colon I \to G_3^1$ (G_3) *be an admissible curve and* $\mathcal{K} \neq 0$ *of class* C^2*,* $\mathcal{T}$ *of class* C^1 *its equiform curvature and equiform torsion, respectively. The curve is an equiform circular helix if and only if*

$$\nabla_T \nabla_T \nabla_T \mathbf{T} - \mathcal{K}^2 \cdot \nabla_T \mathbf{T} = 2\mathcal{K} \cdot \nabla_T \mathbf{N} + \mathcal{T} \cdot \nabla_T \mathbf{B}. \tag{15}$$

Proof. Assuming that $\mathcal{K} = const.$ and $\mathcal{T} = const.$, the statement follows directly from Theorem 1. □

Remark 6 In Corollary 3.1 (Öğrenmiş et al., 2007) the authors stated that a curve in the G_3 is a circular helix if and only if

$$\nabla_T \nabla_T \nabla_T \mathbf{T} + \tau^2 \cdot \nabla_T \mathbf{T} = 0,$$

where $\mathbf{T}$ is a tangent vector and τ is a torsion of the curve. Assuming that $\kappa = const.$ and $\tau = const.$, which means $\mathcal{K} = 0$ and $\mathcal{T} = const.$, the Equation (15) becomes

$$\nabla_T \nabla_T \nabla_T \mathbf{T} - \mathcal{T} \cdot \nabla_T \mathbf{B} = 0.$$

Thus, after taking into account the third and the first of the Frenet formulas (9), we obtain

$$\nabla_T \nabla_T \nabla_T \mathbf{T} + \mathcal{T}^2 \cdot \nabla_T \mathbf{T} = 0,$$

where $\mathbf{T}$ is an equiform tangent vector field and $\mathcal{T}$ is the equiform torsion. In this manner the characterization (15) represents a generalization of the characterization given in (Öğrenmiş et al., 2007).

At the end of this section, we give a characterization of equiform general helices only using the normal vector field. This characterization is interesting because starting from it we can obtain a generalization of Corollary 3.1 given in (Bektaş, 2005).

Proposition 2 *Let* $\mathbf{r}\colon I \to G_3^1(G_3)$ *be an admissible curve and* $\mathcal{K} \neq 0$ *of class* C^2*,* $\mathcal{T} \neq 0$ *of class* C^1 *its equiform curvature and equiform torsion, respectively. The curve is an equiform general helix if and only if*

$$\nabla_T^2\mathbf{N} - \left(2\mathcal{K} + \frac{\mathcal{T}'}{\mathcal{T}}\right)\cdot \nabla_T\mathbf{N} + (\mathcal{K}^2 - \delta\mathcal{T}^2)\cdot\mathbf{N} = 0, \tag{16}$$

where the constant δ *takes value* $+1$ *in the pseudo-Galilean space and* -1 *in the Galilean space.*

Proof. Since $\mathbf{r}$ is an equiform general helix, we have

$$\frac{\mathcal{K}}{\mathcal{T}} = const. \qquad \text{and hence} \qquad \mathcal{K}'\cdot\mathcal{T} = \mathcal{K}\cdot\mathcal{T}'.$$

Using the second and the third equation of (9) we can express the vector fields $\mathbf{B}$ and $\nabla_T\mathbf{B}$ as follows:

$$\begin{aligned} \mathbf{B} &= \frac{1}{\mathcal{T}}\cdot\nabla_T\mathbf{N} - \frac{\mathcal{K}}{\mathcal{T}}\cdot\mathbf{N}, \\ \nabla_T\mathbf{B} &= \frac{\mathcal{K}}{\mathcal{T}}\cdot\nabla_T\mathbf{N} - \frac{(\mathcal{K}^2 - \delta\mathcal{T}^2)}{\mathcal{T}}\cdot\mathbf{N}, \end{aligned} \tag{17}$$

If we differentiate the second of the Frenet Equations (9), after taking into account (17) and $\mathcal{K}'\cdot\mathcal{T} = \mathcal{K}\cdot\mathcal{T}'$, we obtain relation (16).

Conversely, let us assume that the Equation (16) holds. If we differentiate the second of the Frenet Equations (9) and insert the relations (17) we have

$$\nabla_T^2\mathbf{N} - \left(2\mathcal{K} + \frac{\mathcal{T}'}{\mathcal{T}}\right)\cdot\nabla_T\mathbf{N} - \left(\mathcal{K}' - \frac{\mathcal{K}\mathcal{T}'}{\mathcal{T}} - \mathcal{K}^2 + \delta\mathcal{T}^2\right)\cdot\mathbf{N} = 0. \tag{18}$$

After comparing the relation (18) to the Equation (16), we obtain $-\mathcal{K}'\mathcal{T} + \mathcal{K}\mathcal{T}' = 0$, and finally $\frac{\mathcal{K}}{\mathcal{T}} = const.$ □

5. Several Types of Equiform General Helices

In this section we will find parametric equations and show graphs of relevant equiform general helices.

First, we recall the equations and graphs of equiform circular helices characterized by constant equiform curvature ($\mathcal{K} = a$) and constant equiform torsion ($\mathcal{T} = b$). The equiform circular helices in G_3^1 and G_3 are considered in detail in (Erjavec & Divjak, 2008) and (Pavković & Kamenarović, 1987), respectively. The parametric equations of an equiform circular helix in G_3^1 are given by

$$\mathbf{r}(t) = \Big(\frac{1}{a}e^{\frac{a}{b}t},\ \frac{a^2}{b(b^2-a^2)}e^{\frac{a}{b}t}(b\cosh t - a\sinh t),\ \frac{a^2}{b(b^2-a^2)}e^{\frac{a}{b}t}(b\sinh t - a\cosh t)\Big),$$

and in G_3 by

$$\mathbf{r}(t) = \Big(\frac{1}{a}e^{\frac{a}{b}t},\ \frac{a^2}{b(b^2+a^2)}e^{\frac{a}{b}t}(a\sin t - b\cos t),\ \frac{-a^2}{b(b^2+a^2)}e^{\frac{a}{b}t}(a\cos t + b\sin t)\Big).$$

Due to the fact that they are at the same time "ordinary" general helices, they lie on a cone of revolution in a corresponding space (see Figure 2).

Furthermore, we will describe the procedure of finding parametric equations of equiform general helices common for all considered cases. If we suppose that the coordinate functions of the curve are $x = t$, $y = y(t)$, $z = z(t)$, then the relation (7), by use of (5), can be written in the following way

$$(0, y''', z''') = \frac{\kappa'}{\kappa}\cdot(0, y'', z'') + \tau\cdot(0, \varepsilon z'', \varepsilon y''). \tag{19}$$

Thus the computations of the coordinate functions y and z are reduced to solving the following symmetric system of ordinary differential equations

$$\begin{aligned} y''' &= \frac{\kappa'}{\kappa}\cdot y'' + \tau\cdot\varepsilon z'', \\ z''' &= \frac{\kappa'}{\kappa}\cdot z'' + \tau\cdot\varepsilon y''. \end{aligned} \tag{20}$$

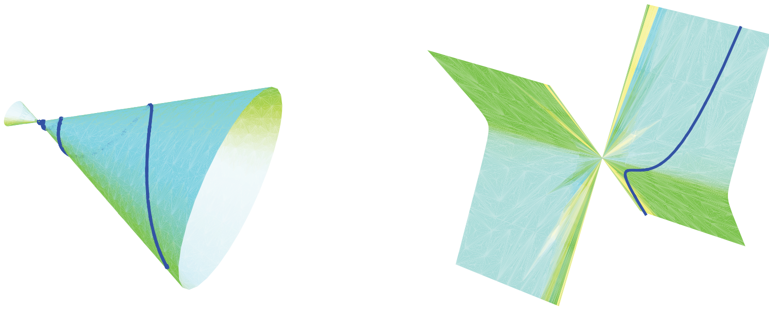

Figure 2. Equiform circular helices in G_3 and G_3^1

Starting from natural equations of a concrete curve, reducing the order ($u = \frac{d^2y}{dt^2}, v = \frac{d^2z}{dt^2}$), and eliminating v and $\frac{dv}{dt}$ from this system, we get a homogenous differential equation which after solving and double integration gives us the coordinate function y of the curve.

In a similar way, eliminating y'' and y''' from (20), we get a homogenous differential equation for v and finally the coordinate function z.

5.1 Equiform General Helices of Type A

We want to find parametric equations of equiform general helices in the pseudo-Galilean space characterized by constant torsion. If we suppose that $\tau = b = const.$, using the Proposition 1 we find the natural equations of the curve

$$\kappa = e^{-at}, \qquad \tau = b = const., \tag{21}$$

where $a \in \mathbb{R}$ and $b \in \mathbb{R}\backslash\{0\}$. If we follow the procedure described above, we obtain the following symmetric system of ordinary differential equations

$$y''' = -a \cdot y'' + \varepsilon \cdot b \cdot z'', \qquad z''' = -a \cdot z'' + \varepsilon \cdot b \cdot y'', \tag{22}$$

which gives the following differential equation of the second order with constant coefficients

$$u'' + 2a \cdot u' + (a^2 - b^2) \cdot u = 0. \tag{23}$$

Solving this equation we obtain

$$u(t) = c_1 \cdot e^{-at} \cosh bt + c_2 \cdot e^{-at} \sinh bt,$$

which for $c_1 = 1$ and $c_2 = 0$, after double integration, gives

$$y(t) = \frac{e^{-at}}{(a^2 - b^2)^2}(2ab \sinh bt + (a^2 + b^2) \cosh bt).$$

Similarly, we find the coordinate function z and hence the parametric equations of the curve.

According to Theorem 4.1 (Erjavec & Divjak, 2008) there are two curves in G_3^1 which satisfies the conditions in (21). The parametric representations of the first curve is given by

$$\mathbf{r}(t) = \left(t,\ \frac{e^{-at}}{(a^2 - b^2)^2}((a^2 + b^2) \cosh bt + 2ab \sinh bt),\ \frac{e^{-at}}{(a^2 - b^2)^2}(2ab \cosh bt + (a^2 + b^2) \sinh bt)\right),$$

and the parametric equations of the second curve are obtained replacing the y and z coordinate of the first curve.

In the Galilean space, the equation analogues to (19) for the case $\kappa = e^{-at}$, $\tau = b \neq 0$, $a, b \in \mathbb{R}$, has the following form

$$(0, y''', z''') = \frac{\kappa'}{\kappa} \cdot (0, y'', z'') + \tau \cdot (0, -z'', y''). \tag{24}$$

From there we obtain the following system of ordinary differential equations

$$y''' = -a \cdot y'' - b \cdot z'', \quad z''' = -a \cdot z'' + b \cdot y'', \tag{25}$$

Further, we obtain the following ordinary differential equation of the second order with constant coefficients

$$u'' + 2a \cdot u' + (a^2 + b^2) \cdot u = 0, \tag{26}$$

and finally the parametric equations

$$\mathbf{r}(t) = \left(t,\ \frac{e^{-at}}{(a^2+b^2)^2}((a^2-b^2)\cos bt - 2ab\sin bt),\ \frac{e^{-at}}{(a^2+b^2)^2}(2ab\cos bt + (a^2-b^2)\sin bt)\right).$$

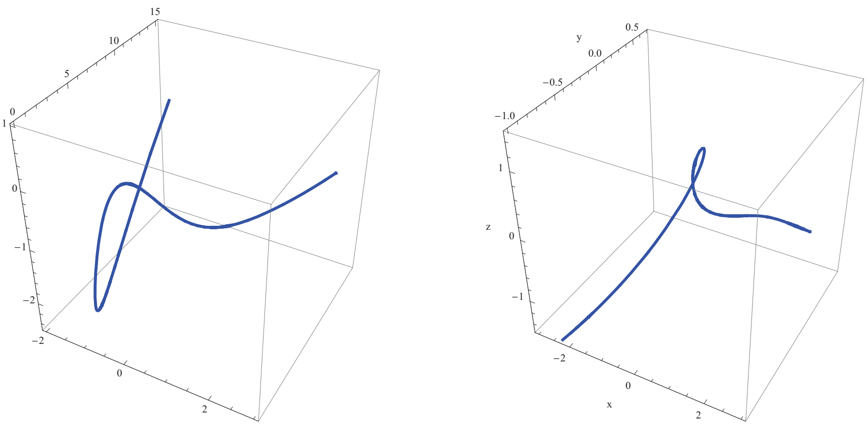

Figure 3. Equiform general helices of type A in G_3^1 and G_3

Remark 7 All figures of equiform general helices are made by *Wolfram Mathematica* using constant values $a = 1$ and $b = 2$.

Remark 8 As we mentioned before, the equiform circular helices lie on a cone of revolution in a corresponding space. In a similar way, it is easy to show that the considered equiform general helices lie on the surface given by

$$y^2(x) \mp z^2(x) = \frac{e^{-2ax}}{(a^2 \mp b^2)^2},$$

where the upper sign holds in the pseudo-Galilean space and the lower one in the Galilean space. Since these surfaces represent a kind of generalization of "ordinary" cones, we could call them ***equiform cones***.

5.2 Equiform General Helices of Type B

In the pseudo-Galilean and the Galilean space for $\kappa = \frac{1}{t^a}, \tau = \frac{b}{t}$, $a, b \in \mathbb{R}$, we obtain the following systems of ordinary differential equations

$$G_3^1 \ldots y''' = -\frac{a}{t} \cdot y'' + \varepsilon \cdot \frac{b}{t} \cdot z'', \quad z''' = -\frac{a}{t} \cdot z'' + \varepsilon \cdot \frac{b}{t} \cdot y'',$$

$$G_3 \ldots y''' = -\frac{a}{t} \cdot y'' - \frac{b}{t} \cdot z'', \quad z''' = -\frac{a}{t} \cdot z'' + \frac{b}{t} \cdot y''.$$

It gives us the following Euler's differential equation of the second order

$$t^2 \cdot u'' + (2a + 1) \cdot t \cdot u' + (a^2 - \delta \cdot b^2) \cdot u = 0, \tag{27}$$

where $u(t) = y''(t)$ and the constant δ is $+1$ in the pseudo-Galilean space and -1 in the Galilean space.

The corresponding parametric equations in G_3^1 and G_3 are

$$\begin{aligned} x(t) &= t, \\ y(t) &= \frac{t^{2-a}((2-3a+a^2+b^2)\cosh(b\ln t) + (-3+2a)b\sinh(b\ln t))}{(4-4a+a^2-b^2)(1-2a+a^2-b^2)}, \\ z(t) &= \frac{t^{2-a}((-3+2a)b\cosh(b\ln t) + (2-3a+a^2+b^2)\sinh(b\ln t))}{(4-4a+a^2-b^2)(1-2a+a^2-b^2)}. \end{aligned} \tag{28}$$

$$\begin{aligned} x(t) &= t, \\ y(t) &= \frac{t^{2-a}((2-3a+a^2-b^2)\cos(b\ln t)+(3-2a)b\sin(b\ln t))}{(4-4a+a^2+b^2)(1-2a+a^2+b^2)}, \\ z(t) &= \frac{t^{2-a}((-3+2a)b\cos(b\ln t)+(2-3a+a^2-b^2)\sin(b\ln t))}{(4-4a+a^2+b^2)(1-2a+a^2+b^2)}, \end{aligned} \tag{29}$$

respectively.

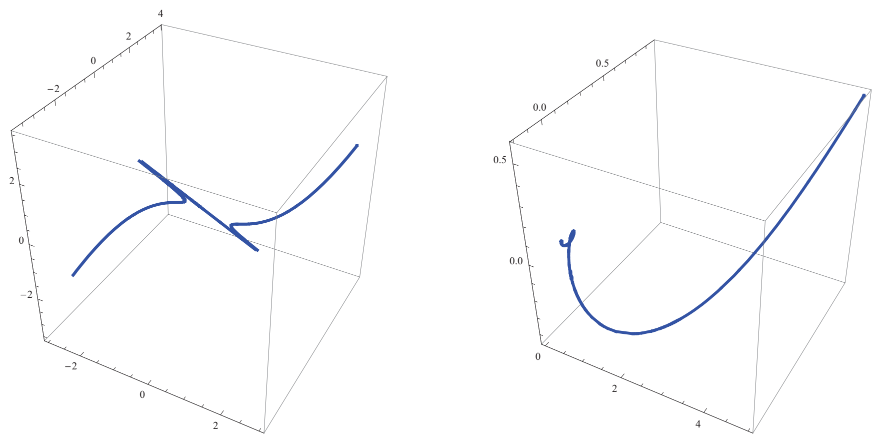

Figure 4. Equiform general helices of type B in G_3^1 and G_3

5.3 Equiform General Helices of Type C

In the pseudo-Galilean and the Galilean space for $\kappa = e^{-\frac{a}{2}t^2}, \tau = b \cdot t$, $a, b \in \mathbb{R}$, we obtain the following systems of ordinary differential equations

$$G_3^1 \ldots y''' = -a \cdot t \cdot y'' + \varepsilon \cdot b \cdot t \cdot z'', \quad z''' = -a \cdot t \cdot z'' + \varepsilon \cdot b \cdot t \cdot y'',$$

$$G_3 \ldots y''' = -a \cdot t \cdot y'' - b \cdot t \cdot z'', \quad z''' = -a \cdot t \cdot z'' + b \cdot t \cdot y''.$$

It gives us the following differential equation of the second order

$$t \cdot u'' + (2a \cdot t^2 - 1) \cdot u' + (a^2 - \delta \cdot b^2) \cdot t^3 \cdot u = 0, \tag{30}$$

where $u(t) = y''(t)$ and the constant δ is $+1$ in the pseudo-Galilean space and -1 in the Galilean space.

The corresponding parametric equations in G_3^1 are

$$\begin{aligned} x(t) &= t, \\ y(t) &= \frac{\sqrt{\frac{\pi}{2}}}{2(a^2-b^2)}\Bigg(\sqrt{\frac{2}{\pi}}(a+b)\cdot e^{-\frac{1}{2}(a-b)t^2} + \sqrt{\frac{2}{\pi}}(a-b)\cdot e^{-\frac{1}{2}(a+b)t^2} + \\ &\quad + \sqrt{a-b}\cdot(a+b)\cdot t\cdot \mathbf{erf}\Bigg(\sqrt{\frac{a-b}{2}}\cdot t\Bigg) + \sqrt{a+b}\cdot(a-b)\cdot t\cdot \mathbf{erf}\Bigg(\sqrt{\frac{a+b}{2}}\cdot t\Bigg)\Bigg) \\ z(t) &= \frac{\sqrt{\frac{\pi}{2}}}{2(a^2-b^2)}\Bigg(\sqrt{\frac{2}{\pi}}(a+b)\cdot e^{-\frac{1}{2}(a-b)t^2} - \sqrt{\frac{2}{\pi}}(a-b)\cdot e^{-\frac{1}{2}(a+b)t^2} + \\ &\quad + \sqrt{a-b}\cdot(a+b)\cdot t\cdot \mathbf{erf}\Bigg(\sqrt{\frac{a-b}{2}}\cdot t\Bigg) - \sqrt{a+b}\cdot(a-b)\cdot t\cdot \mathbf{erf}\Bigg(\sqrt{\frac{a+b}{2}}\cdot t\Bigg)\Bigg), \end{aligned}$$

where the function **erf** is the Gauss error function defined by $\mathbf{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}dt$.

The parametric equations in G_3 are

$$
\begin{aligned}
x(t) &= t,\\
y(t) &= \frac{1}{4(a^2+b^2)} \cdot e^{-\frac{1}{2}(a+ib)t^2}\Big(2\Big(i\cdot b\cdot(-1+e^{ibt^2}) + a\cdot(1+e^{ibt^2})\Big) +\\
&\quad + \sqrt{a+ib}\cdot(a-ib)\cdot e^{\frac{1}{2}(a+ib)t^2}\cdot \sqrt{2\pi}\cdot t\cdot \mathbf{erf}\Big(\sqrt{\frac{a+ib}{2}}\cdot t\Big) +\\
&\quad + \sqrt{a-ib}\cdot(-ia+b)\cdot e^{\frac{1}{2}(a+ib)t^2}\cdot \sqrt{2\pi}\cdot t\cdot \mathbf{erfi}\Big(\frac{(ia+b)t}{\sqrt{2(a-ib)}}\cdot t\Big)\Big)\\
z(t) &= \frac{1}{4(a^2+b^2)} \cdot e^{-\frac{1}{2}(a+ib)t^2}\Big(2\Big(-i\cdot a\cdot(-1+e^{ibt^2}) + b\cdot(1+e^{ibt^2})\Big) +\\
&\quad + \sqrt{a+ib}\cdot(ia+b)\cdot e^{\frac{1}{2}(a+ib)t^2}\cdot \sqrt{2\pi}\cdot t\cdot \mathbf{erf}\Big(\sqrt{\frac{a+ib}{2}}\cdot t\Big) +\\
&\quad - \sqrt{a-ib}\cdot(a+ib)\cdot e^{\frac{1}{2}(a+ib)t^2}\cdot \sqrt{2\pi}\cdot t\cdot \mathbf{erfi}\Big(\frac{(ia+b)t}{\sqrt{2(a-ib)}}\cdot t\Big)\Big)
\end{aligned}
$$

where the function **erf** is the Gauss error function defined above and **erfi** is the imaginary error function defined as $\mathbf{erfi}(z) = -i\cdot\mathbf{erf}(i\cdot z)$.

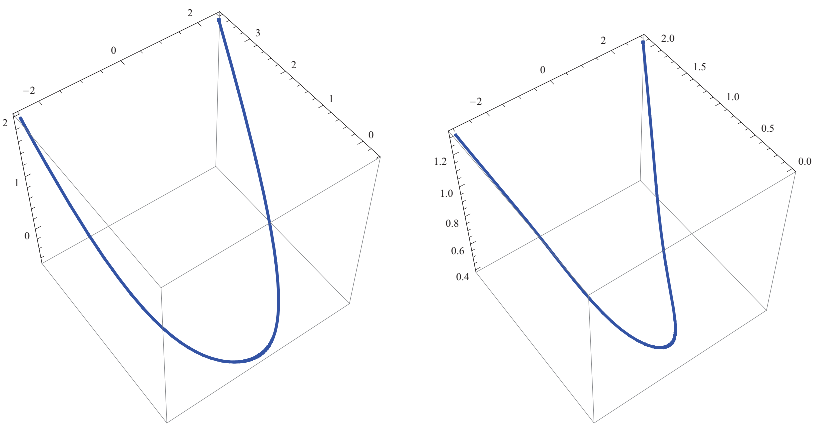

Figure 5. Equiform general helices of type C in G_3^1 and G_3

In the last section we have examined three types of equiform general helices (type A: $\tau = const$, type B: $\tau = \frac{b}{t}$, type C:$\tau = b\cdot t$). All other examples of equiform general helices could be classified as an extra type. Therefore, under the equiform general helices of type D we can assume all other curves whose natural equations fulfill the request given in Proposition 1 and at the same time their corresponding system of differential equations has no exact solutions or the solutions are too complicated.

References

Bektaş, M. (2004). On characterizations of general helices for ruled surfaces in the pseudo-Galilean space G_3^1. *J. Math. Kyoto Univ., 44*(3), 523-528.

Bektaş, M. (2005). The characterizations of general helices in the 3-dimensional pseudo-Galilean space. *Soochow J. Math., 31*(3), 441-447.

Divjak, B. (1997). *Geometrija pseudogalilejevih prostora*. Ph. D. Thesis, University of Zagreb.

Divjak, B. (1998). Curves in pseudo-Galilean geometry. *Annales Univ. Sci. Budapest., 41*, 117-128.

Divjak, B. (2008). Some special surfaces in the pseudo-Galilean space. *Acta Math. Hungar., 118*(3), 209-226. http://dx.doi.org/10.1007/s10474-007-6171-x

Divjak, B., & Milin-Šipuš, Ž. (2003). Minding's isometries of ruled surfaces in Galilean and pseudo-Galilean space. *J. Geom., 77*, 35-47.

Erjavec, Z., & Divjak, B. (2008). The equiform differential geometry of curves in the pseudo-Galilean space. *Math. Commun., 13*, 321-332.

Fechnel, W. (1951). On the differential geometry of closed space curves. *Bull. Amer. Math. Soc., 57*, 44-54.

Ikawa, T. (1985). On curves and submanifolds in an indefinite-Riemannian manifold. *Tsukuba J. Math., 9*(2), 353-371.

Kuhnel, W. (2005). *Differential geometry: Curves, Surfaces, Manifolds* (2nd ed.). American Mathematical Society.

Mólnar, E. (1997). The projective interpretation of the eight 3-dimensional homogeneous geometries. *Beiträge Algebra Geom., 38*(2), 261-288.

Öğrenmiş, A. O., Bektaş, M., & Ergüt, M. (2006). On helices in the doubly isotropic space $I_3^{(2)}$. *Int. Math. Forum, 1*(13), 623-627.

Öğrenmiş, A. O., Ergüt, M., & Bektaş, M. (2007). On the helices in the galilean Space G_3. *Iranian J. of Science and Technology, 31*(A2), 177-181.

Pavković, B. J. (1986). Equiform geometry of curves in the isotropic spaces $I_3^{(1)}$ and $I_3^{(2)}$. *Rad JAZU*, 39-44.

Pavković, B. J., & Kamenarović, I. (1987). The equiform differential geometry of curves in the Galilean space G_3. *Glasnik mat., 22*(42), 449-457.

Pottman, H., & Wallner, J. (2001). *Computational Line Geometry*. Berlin Heidelberg: Springer-Verlag. http://dx.doi.org/10.1007/978-3-642-04018-4

Röschel, O. (1984). *Die Geometrie des Galileischen Raumes*. Habilitationsschrift, Institut für Math. und Angew. Geometrie, Leoben.

Sachs, H. (1986). *Lehrbuch der isotropen Geometrie*. Wiesbaden: Vieweg-Verlag.